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ON THE INTERSECTION OF THE SETS OF THE RIGHT AND LEFT INTERNAL APPROXIMATE DERIVATIVES

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NOTATION AND INTRODUCTION

We denote by R the set of real numbers and put $\bar{R} = R \cup \{-\infty, +\infty\}$. If $x \in R$ and $K \subset R$, we denote by $\varrho(x, K)$ the distance between x and K and by μK the outer Lebesgue measure of K. We set

$$D^+(K, x) = \limsup_{h \to 0+} \frac{1}{h} \mu((x, x + h) \cap K),$$

$$D_+(K, x) = \liminf_{h \to 0+} \frac{1}{h} \mu((x, x + h) \cap K).$$

Similarly we define the densities $D^-(K, x)$, $D_-(K, x)$.

In this article, whenever $f : R \to R$ is a function, we set

$$g(x, y) = \frac{f(x) - f(y)}{x - y}.$$

Let $f : R \to R$ be a function and $x \in R$. By the set $\mathscr{D}^+ f(x)$ of the right internal derivatives we mean the set of all $y \in \bar{R}$ such that x is a point of accumulation of the set $\{t : x < t, g(x, t) \in V\}$ for any neighbourhood V of y. By the set $\mathscr{D}^+_{ap} f(x)$ of the right internal approximate derivatives we mean the set of all $y \in \bar{R}$ such that $D^+(\{t : g(x, t) \in V\}, x) > 0$ for any neighbourhood V of y. The sets $\mathscr{D}^- f(x)$, $\mathscr{D}^-_{ap} f(x)$ are defined by symmetry.

Dini's derivatives and approximate Dini's derivatives are denoted by $D^+ f(x)$, $D_+ f(x)$, $D^- f(x)$, $D_- f(x)$ and $D^+_{ap} f(x)$, $D_{+ap} f(x)$, $D^-_{ap} f(x)$, $D_{-ap} f(x)$, respectively. Evidently $D^+ f(x) = \max \mathscr{D}^+ f(x)$, $D^+_{ap} f(x) = \max \mathscr{D}^+_{ap} f(x)$ etc.

It is well-known that for an arbitrary function $f : R \to R$, the set of all points x such that $\mathscr{D}^+ f(x) \cap \mathscr{D}^- f(x) = \emptyset$ is countable. This theorem is proved in [3], p. 150, by Blumberg's method which is described in [1].

The purpose of the present article is to investigate analogues of the mentioned theorem for the sets of the right and left internal approximate derivatives. It follows from the note in [4], p. 297, that the set $\{x : \mathscr{D}^+_{ap} f(x) \cap \mathscr{D}^-_{ap} f(x) = \emptyset\}$ is of measure zero for an arbitrary function f. This assertion may be improved if f is a continuous function. From Theorem 2 [2] it follows, if we use the notation from [2], that if $f : H \to R$ is an arbitrary continuous function and θ_1, θ_2 two directions, then $\{x : C_e(f, x, \theta_1) \cap C_e(f, x, \theta_2) = \emptyset\}$ is a set of the first category. This assertion immediately implies by Blumberg's method that $\{x : \mathscr{D}^+_{ap} f(x) \cap \mathscr{D}^-_{ap} f(x) = \emptyset\}$ is a set of the first category for an arbitrary continuous function f.

The problem arises for which function f the set $\{x : \mathscr{D}^+_{ap} f(x) \cap \mathscr{D}^-_{ap} f(x) = \emptyset\}$ must be countable. The main aim of the present article is to prove that this set is countable for any Lipschitzian function and, on the other hand, there exists a continuous function for which this set is uncountable.

I.

Theorem 1. *There exists a continuous function f defined on $(0, 1)$ such that the set $\{x : D^-_{ap} f(x) < D_{+ap} f(x)\}$ and hence also the set $\{x : \mathscr{D}^+_{ap} f(x) \cap \mathscr{D}^-_{ap} f(x) = \emptyset\}$ is uncountable.*

Proof. For positive integers k, m, $1 \leqq k$, $1 \leqq m \leqq 2^{k-1}$ we define open intervals $I_{k,m}$ by induction:

(i) $I_{1,1} = (\frac{1}{3}, \frac{2}{3})$.

(ii) If we have defined $I_{k,m}$ for $1 \leqq k \leqq n$, $1 \leqq m \leqq 2^{k-1}$, then there are 2^n closed intervals $J_{n,1}, \ldots, J_{n,2^n}$ such that $J_{n,k} \cap J_{n,m} = \emptyset$ for $k \neq m$ and the union of these intervals and the intervals $I_{k,m}$ already defined is the whole interval $\langle 0, 1 \rangle$. We define $I_{n+1,j} \subset J_{n,j}$ for $1 \leqq j \leqq 2^n$ such that $I_{n+1,j}$ and $J_{n,j}$ have the same centre and $\mu I_{n+1,j} = ((n+1)^2/(n+1)^2 + 2)\, \mu J_{n,j}$.

Put $G = \bigcup I_{k,j}$ and $D = \langle 0, 1 \rangle - G$. Clearly D is a perfect set. It is defined so that the following proposition holds:

(A) Let $x \in D$ be a bilateral point of accumulation of D. Then there exists a sequence $\{(c_n, d_n)\}_{n=1}^{\infty}$ of intervals contiguous to D such that

$$d_n < c_{n+1} < x\,, \quad \lim \frac{|x - d_n|}{|c_n - d_n|} = 0 \quad \text{and} \quad D_-\big(\bigcup_{n=1}^{\infty} (c_n, d_n), x\big) = 1.$$

We shall prove (A). Clearly the lengths of the intervals $I_{n,j}$, $J_{n,j}$ depend only on n; denote them r_n, s_n, respectively. It is easy to see that $\lim r_n = 0$, $\lim s_n = 0$, $s_n/r_n = 1/n^2$ and $\mu D = 0$. For $n > 1$ denote by j_n the positive integer such that $x \in J_{n-1,j_n}$ and put $R_n = I_{n,j_n}$. Then evidently $\varrho(x, R_n)/\mu R_n \leqq 1/n^2$. Put $G_n = \bigcup_{j=1}^{2^{n-1}} I_{n,j}$,

$Z_n = G_n - R_n$ for $n > 1$ and $Z = \bigcup_{n=2}^{\infty} Z_n$. We shall prove that $D(Z, x) = 0$. For $h > 0$ such that $(x - h, x + h) \subset (0, 1)$ it is evident that the interval $(x - h, x + h)$ contains at least as many intervals of the type $I_{k,j}$, $k < n$, as many intervals which form Z_n it intersects. For $k < n$ we have

$$\frac{\mu(I_{n,j})}{\mu(I_{k,j'})} \leqq \frac{r_n}{r_{n-1}} < \frac{s_{n-1}}{r_{n-1}} = \frac{1}{(n-1)^2}$$

and hence $\mu((x - h, x + h) \cap Z_n)/2h \leqq 1/(n-1)^2$ for $n > 1$. If $h < r_n$ then evidently $(x - h, x + h) \cap Z_n = \emptyset$. Therefore

$$\mu((x - h, x + h) \cap Z)/2h = \sum_{n=2}^{\infty} \mu((x - h, x + h) \cap Z_n)/2h \leqq \sum_{n>1, h \geqq r_n} 1/(n-1)^2 ,$$

consequently $D(Z, x) = 0$. Since $G = Z \cup \bigcup_{n=2}^{\infty} R_n \cup I_{1,1}$, the subsequence of the sequence $\{R_n\}_{n=2}^{\infty}$ of intervals which lie to the left from the points x has clearly all the properties of the sequence (c_n, d_n).

Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be a sequence of all intervals contiguous to D. Put $B_n = (a_n, a_n + (b_n - a_n)/2n)$ and $B = \bigcup_{n=1}^{\infty} B_n$. Evidently $D^-(B, x) = 0$ for $x \in D$. We shall prove that the set of all points $x \in D$ such that $D^+(B, x) > 0$ is uncountable. Put $C_n = (a_n - (b_n - a_n)/2n, a_n)$, $T_k = \bigcup_{n=k}^{\infty} C_n$. Each set $P_k = D \cap T_k$ is dense and open in D. Therefore the set $P = \bigcap_{k=1}^{\infty} P_k$ is not a set of the first category in D and hence P is uncountable. If $x \in P$ then x lies in an infinite number of C_n and therefore $D^+(B, x) \geqq \frac{1}{2} > 0$. We put

$$\begin{aligned}
f(x) &= 0 && \text{for } x \in D ,\\
f(x) &= 0 && \text{for } x \in \langle a_n, a_n + (b_n - a_n)/4n \rangle ,\\
f(x) &= 8n(x - a_n - (b_n - a_n)/4n) && \text{for } x \in \langle a_n + (b_n - a_n)/4n, a_n + (b_n - a_n)/2n \rangle ,\\
f(x) &= 2(b_n - a_n) && \text{for } x \in \langle a_n + (b_n - a_n)/2n, b_n - (b_n - a_n)/4n \rangle ,\\
f(x) &= 8n(b_n - x) && \text{for } x \in \langle b_n - (b_n - a_n)/4n, b_n \rangle .
\end{aligned}$$

Clearly f is a continuous function on $\langle 0, 1 \rangle$ and $D_{+\mathrm{ap}} f(x) \geqq 0$ for $x \in P$. Let $z \in P$ be a bilateral point of accumulation of D. Let $\{(c_k, d_k)\}_{k=1}^{\infty}$ be a sequence from Proposition (A) for $x = z$ and put $S = \bigcup_{k=1}^{\infty} (c_k, d_k)$. Put $M = \{y : y < z \text{ and } g(z, y) > -1\}$. Define the sequence $\{n_k\}_{k=1}^{\infty}$ by the condition $(c_k, d_k) = (a_{n_k}, b_{n_k})$. From (A) it follows that there exists k_0 such that $d_k - c_k > z - d_k$ for $k > k_0$. Let $k > k_0$ and $y \in \langle c_k + (d_k - c_k)/2n_k, d_k - (d_k - c_k)/4n_k \rangle$. Then

$$g(z, y) = \frac{f(z) - f(y)}{z - y} = \frac{-2(d_k - c_k)}{z - y} \leqq \frac{-2(d_k - c_k)}{2(d_k - c_k)} = -1 .$$

Let

$$y \in \langle d_k - (d_k - c_k)/4n_k, d_k - (z - d_k)/(8n_k - 1) \rangle .$$

Then $d_k - y \geqq (z - d_k)/(8n_k - 1)$ and therefore

$$g(z, y) = \frac{-8n_k(d_k - y)}{z - y} = \frac{-8n_k(d_k - y)}{(z - d_k) + (d_k - y)} =$$
$$= -8n_k + \frac{8n_k(z - d_k)}{(z - d_k) + (d_k - y)} \leqq -1 .$$

Hence we have

$$(c_k, d_k) \cap M \subset ((c_k, c_k + (d_k - c_k)/2n_k) \cup (d_k - (z - d_k)/(8n_k - 1), d_k)) .$$

Since evidently $\lim_{k \to \infty} n_k = \infty$, we have $D^-(S \cap M, z) = 0$. As $D_-(S, z) = 1$, we have $D^-(M, z) = 0$ and consequently $D^-_{ap} f(z) \leqq -1$. Hence $D^-_{ap} f(z) < D_{+ap} f(z)$ on an uncountable set.

II.

Lemma 1. *Let $I \subset R$ be an open interval with the endpoints a, b and let f be a real function defined on I. Let $B \subset R$ such that $\varrho(g(b, a), B) \geqq \varepsilon > 0$ and let $K > 0$. Then the measure of the set S of all points $z \in I$ such that $g(z, b) \in B$, $|g(z, a)| \leqq K$ and $|g(z, b)| \leqq K$ is at most $2K|b - a|/(\varepsilon + 2K)$.*

Proof. Put $J = I \cap (a - (\varepsilon|b - a|/(\varepsilon + 2K)), a + (\varepsilon|b - a|/(\varepsilon + 2K)))$. We shall prove that $J \cap S = \emptyset$. Assume that there exists $z \in J \cap S$. Then

$$|g(b, a) - g(z, b)| = \left| \frac{f(b) - f(z)}{b - a} + \frac{f(z) - f(a)}{b - a} - \frac{f(z) - f(b)}{z - b} \right| =$$
$$= \left| \frac{f(z) - f(a)}{z - a} \frac{z - a}{b - a} + \frac{f(b) - f(z)}{z - b} \frac{z - a}{b - a} \right| \leqq \frac{2K \, . \, \varepsilon}{\varepsilon + 2K} < \varepsilon .$$

Therefore $g(z, b) \notin B$ and this is a contradiction. Hence $J \cap S = \emptyset$ and the assertion of Lemma follows from the identity

$$\frac{2K|b - a|}{\varepsilon + 2K} = |b - a| - \frac{\varepsilon|b - a|}{\varepsilon + 2K} .$$

Theorem 2. *Let $f : R \to R$ be an arbitrary function. Then the set A of all points $x \in R$ such that the sets $\mathscr{D}^+_{ap} f(x)$, $\mathscr{D}^-_{ap} f(x)$ are bounded and $\mathscr{D}^+_{ap} f(x) \cap \mathscr{D}^-_{ap} f(x) = \emptyset$ is countable.*

Proof. Let $\mathscr{P}$ be the system of all $P = (M, N)$ such that both $M \subset R$ and $N \subset R$ consist of a finite number of open intervals with rational endpoints and $\overline{M} \cap \overline{N} = \emptyset$. We denote by A_P the set of all $x \in R$ such that $\mathscr{D}^+_{ap} f(x) \subset M$ and $\mathscr{D}^-_{ap} f(x) \subset N$ for $P = (M, N) \in \mathscr{P}$. Then clearly $A = \bigcup\{A_P : P \in \mathscr{P}\}$.

For each $P = (M, N) \in \mathscr{P}$ we define $\varepsilon = \varepsilon(P) = \varrho(M, N)/2$ and $K = K(P) = \sup\{|x|, x \in M \cup N\}$. If n is a positive integer, we denote by $A_{P,n}$ the set of all $x \in R$ such that

$$(1/h) \,.\, \mu\{y \in (x, x + h) : g(x, y) \notin M\} < \varepsilon/(2\varepsilon + 4K)$$

and

$$(1/h) \,.\, \mu\{y \in (x - h, x) : g(x, y) \notin N\} < \varepsilon/(2\varepsilon + 4K)$$

for $h < 1/n$. Evidently $A_P \subset \bigcup_{n=1}^{\infty} A_{P,n}$ for each $P \in \mathscr{P}$. Therefore

$$A \subset \bigcup\{A_{P,n} : P \in \mathscr{P},\ n \text{ positive integer}\}\,. \tag{1}$$

Now suppose that $a \in A_{P,n}$, $b \in A_{P,n}$ and $|b - a| < 1/n$, for a positive integer n and $P = (M, N) \in \mathscr{P}$. Clearly either $\varrho(g(b, a), M) \geqq \varepsilon$ or $\varrho(g(b, a), N) \geqq \varepsilon$. Without any loss of generality we may suppose $\varrho(g(b, a), N) \geqq \varepsilon$ and $b > a$. By Lemma 1 we have that the measure of the set S of all points $z \in (a, b)$ such that $g(z, b) \in N$, $|g(z, a)| \leqq K$ and $|g(z, b)| \leqq K$ is at most $2K|b - a|/(\varepsilon + 2K)$. By the definition of $A_{P,n}$ we have that both the measure of the set $G = \{z \in (a, b) : g(z, b) \notin N\}$ and the measure of the set $H = \{z \in (a, b) : g(z, a) \notin M\}$ are less than $\varepsilon|b - a|/(2\varepsilon + 4K)$. Since $S \cup H \cup G = (a, b)$, we have

$$b - a < (b - a)\left(\frac{2K}{\varepsilon + 2K} + \frac{2\varepsilon}{2\varepsilon + 4K}\right) = b - a$$

and this is a contradiction. Therefore each $A_{P,n}$ is countable and by (1) the set A is countable as well.

Corollary 1. *For any Lipschitzian function $f : R \to R$ the set*

$$\{x : \mathscr{D}^+_{ap} f(x) \cap \mathscr{D}^-_{ap} f(x) = \emptyset\}$$

is countable.

Corollary 2. *For an arbitrary function $f : R \to R$ the set of all points at which the right and left approximate derivatives exist, are finite and not equal to each other, is countable.*

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