Jiří Kopáček

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Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 3, 419–436

Persistent URL: http://dml.cz/dmlcz/101184

Czechoslovak Mathematical Journal, 23 (98) 1973, Praha

ON L_p-ESTIMATES FOR HYPERBOLIC SYSTEMS

JIŘÍ KOPÁČEK, Praha

(Received April 19, 1972)

1. Introduction. In this paper we shall give the proofs of the following theorems anounced in [1]:

Theorem 1. *Let the system*

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + Bu \tag{1}$$

where A_j, B are $N \times N$ constant matrices, satisfy the condition

(H) *There exist positive constants C_1, C_2, C_3, T such that*

$$|\psi(t, y)| = |e^{(iA(y)+B)t}| \leqq C_1 + C_2|y|^{C_3}$$

for all $y \in R_n$, $t \in \langle 0, T \rangle$.

If for some $p \in \langle 1, +\infty \rangle$, $p \neq 2$ and for some $C > 0$ the inequality

$$\|u_\varphi(t, x)\|_{L_p} \leqq C\|\varphi(x)\|_{L_p} \tag{2}$$

holds for all $t \in \langle 0, T \rangle$, $\varphi \in \mathcal{S}$, where $u_\varphi(t, x) = F^{-1}(e^{(iA(y)+B)t} F\varphi)$ is the solution of (1) with the initial condition $u_\varphi(0, x) = \varphi(x)$ and $\mathcal{S}$ is the set of infinitely differentiable vector functions φ on R_n with finite pseudonorms $\sup_{x \in R_n} |x|^k |D^\alpha \varphi(x)|$, $\alpha = \alpha_1, \ldots, \alpha_n$, $k, \alpha_1, \ldots, \alpha_n$ arbitrary nonnegative integers, then

1) *the matrix $A(y) =_{\mathrm{Df}} \sum_{j=1}^{n} y_j A_j$ has for all $y \in R_n$ only real eigenvalues and can be diagonalized by a similarity transformation $T^{-1}(y)\, A(y)\, T(y)$ for all $y \in R_n$.*

2) $A_i A_j = A_j A_i$, $i, j = 1, 2, \ldots, n$.

3) *A_i, $i = 1, 2, \ldots, n$ can be diagonalized by the same similarity transformation.*

Theorem 2. *Let the system* (1) *satisfy* (H) *and, for some* $k \geqq 1$ *integer and some* $p \in (1, +\infty)$, *let the inequality*

$$(2') \qquad \|u_\varphi(t, x)\|_{L_p} \leqq C\|\varphi(x)\|_{W_p^k}$$

hold for all $t \in \langle 0, T\rangle$, $\varphi \in \mathcal{S}$, *where* W_p^k *are Sobolev spaces on* R_n, C *a constant. Then the inequality* (2) *is valid and if* $p \neq 2$ *then the assertions* 1)–3) *of the previous theorem hold.*

Theorem 3. *Let the system* (1) *satisfy* (H) *with* $C_2 = C_3 = 0$. *Then for* $p \in \langle 1, +\infty\rangle$ *there exist constants* $\tilde{C}_1$, $\tilde{C}_2$, *depending on* p, n, N *but not on* B, *such that for its solutions the following estimation*

$$(3) \qquad \|u_\varphi(t, x)\|_{L_p} \leqq \tilde{C}_1 e^{\tilde{C}_2|B||t|} \sum_{s=0}^{k} |t|^s \|\varphi\|_{\mathcal{L}_p{}^s}$$

holds for all $t \in R_1$, $\varphi \in \mathcal{S}$, *where* k *is the smallest integer satisfying*

$$k \geqq \begin{cases} [n/2] + 1 & \text{for} \quad p = +\infty \\ n|1/p - \frac{1}{2}| & \text{for} \quad p \in \langle 1, +\infty) \,. \end{cases}$$

Theorem 1 was published for the first time by the author in [2] without the detailed proof. Its proof made use of a matrix theorem from [3]. The latter theorem holds but its proof in [3] is not correct. In [4] P. BRENNER proved a more general result than Theorem 1. For completeness we give here the complete proof of Theorem 1 together with the correct proof of the matrix theorem from [3] (see Lemma 6 below). Theorem 3 generalizes the result of L. A. MURAVEJ [5]. In [4] P. BRENNER proved a slightly weaker result than Theorem 3 but for a very large class of systems.

2. Notation. $R_n = \{x\} = \{x_1, \ldots, x_n\}$ is the linear space of n-tuples of real numbers with

$$(x, y) = \sum_{i=1}^{n} x_i y_i \,, \quad |x| = (x, x)^{1/2} \,,$$

$x, y \in R_n$. C_n is the linear space of n-tuples of complex numbers with

$$(v, w) = \sum_{i=1}^{n} v_i, \bar{w}_i \,, \quad |v| = (v, v)^{1/2} \,.$$

For $N \times N$ matrix A we denote by

$$|A| = \sup_{v \in C_n, v \neq 0} |Av|/|v| \,.$$

E is the set of complex infinitely differentiable functions on R_n, $\mathcal{E}$ the set of vector functions $f = f_1, f_2, \ldots, f_N$ with components in E, S is the subset of E consisting of functions satisfying for all k and $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ (k. α_i nonnegative integers)

$$\sup_{x \in R_n} |x|^k \, |D^\alpha f(x)| < +\infty\,, \quad |\alpha| \sum_{i=1}^{n} \alpha_i\,, \quad D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}\,.$$

$\mathcal{S}$ is the analogous subset of $\mathcal{E}$. D and $\mathcal{D}$ are subsets of S and $\mathcal{S}$, respectively, of functions with a compact support in R_n. The topology in S and $\mathcal{S}$ is defined by the system of the above mentioned seminorms. S', $\mathcal{S}'$ are dual spaces of S and $\mathcal{S}$, respectively. If M is a domain in R_n, then $D(M)$, $\mathcal{D}(M)$ are subsets of D and $\mathcal{D}$, respectively, with supp $\varphi \subset M$. For a scalar or vector function f defined on M we denote $\|f\|_{L_p(M)} = (\int_M |f|^p \, dx)^{1/p}$ for $p \in \langle 1, +\infty)$, $\|f\|_{L_\infty(M)} = \operatorname{supess}_{x \in M} |f(x)|$ for $p = +\infty$, $\|f\|_{L_p{}^k(M)} = (\sum_{|\alpha|=k} \|D^\alpha f\|^p_{L_p(M)})^{1/p}$, $\|f\|_{L_\infty{}^k} = \sum_{|\alpha|=k} |D^\alpha f|_{L_\infty(M)}$, $\|f\|_{W_p{}^k(M)} = (\sum_{l=0}^{k} \|f\|_{L_p{}^l(M)})^{1/p}$, $\|f\|_{W_\infty{}^k(M)} = \sum_{l=0}^{k} \|f\|_{L_\infty{}^l(M)}$ with k nonnegative integer. $W_p^k(M)$, $\mathcal{W}_p^k(M)$ ($p \in \langle 1, +\infty\rangle$, k nonnegative integer) are usual Sobolev spaces with the above mentioned norms. For $M = R_n$ we shall write W_p^k, $\mathcal{W}_p^k$ instead of $W_p^k(R_n)$, $\mathcal{W}_p^k(R_n)$. For $p \in \langle 1, +\infty)$ these spaces are the closures of S, $\mathcal{S}$ in the corresponding norms. For $p \in (1, +\infty)$, $1/q + 1/p = 1$ denote by W_p^{-k}. $\mathcal{W}_p^{-k}$ the dual spaces to W_q^k, $\mathcal{W}_q^k$, k-nonnegative integer. For $u \in \mathcal{S}$, $Fu = \hat{u}(\xi) = (2\pi))^{-n/2} \int_{R_n} e^{-i(x,\xi)} u(x) \, dx$, $F^{-1} u(\xi) = \check{u}(\xi) = (2\pi)^{-n/2} \int_{R_n} e^{i(x,\xi)} u(x) \, dx$. For $x^0 \in R_n$, $\varrho > 0$, $K(x^0, \varrho) = \{x; x \in R_n, |x_i - x_i^0| < \varrho\}$, $B(x^0, \varrho) = \{x \in R_n, |x - x^0| < \varrho\}$.

3. Preliminaries. Remark. The condition (H) is satisfied if one of the following conditions holds:

(H_s) (strong hyperbolicity) The eigenvalues of $A(y)$ are all real and distinct for $|y| = 1$, $y \in R_n$.

(H_p) (hyperbolicity in the sense of Petrovski) For all $|y| = 1$, $y \in R_n$, $A(y)$ has only real eigenvalues and can be diagonalized by a similarity transformation $T^{-1}(y)\, A(y)\, T(y)$, where $T(y) \leqq$ const, $T^{-1}(y) \leqq$ const for $|y| = 1$, $y \in R_n$.

(H_c) $A(y)$ has only real eigenvalues for $y \in R_n$, $A_i B = B A_i$, $i = 1, 2, \ldots, n$.

The first two conditions are well known, the sufficiency of (H_c) for $B = 0$ is proved in [6] (pp. 80, 93) and for general B it follows from $e^{(iA(y)+B)t} = e^{iA(y)t} \cdot e^{Bt}$.

Lemma 1. *The condition* (H) *implies*

$$|D_y^\alpha \, e^{t(iA(y)+B)}| \leqq C_1(\alpha) + C_2(\alpha) \, |y|^{C_3(\alpha)}$$

for $t \in \langle 0, T\rangle$, $y \in R_n$ *and all* $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$.

Proof. Since $g(t, y) = e^{(iA(y)+B)t}$ satisfies the system

$$\frac{d}{dt} g(t, y) = (i\,A(y) + B)\, g(t, y)$$

and $g(0, y) = I$, hence $\partial g/\partial y_k$ is the solution of

$$\frac{d}{dt}\left(\frac{\partial g}{\partial y_k}\right) = (i\,A(y) + B)\frac{\partial g}{\partial y_k} + (i\,A_k g)$$

with $(\partial g/\partial y_k)(0, x) = 0$. Then

$$\frac{\partial g}{\partial y_k}(t, y) = \int_0^t g(t - \tau, y)\,(iA_k\, g(t, y))\, d\tau$$

and thus

$$\left|\frac{\partial g}{\partial y_k}(t, y)\right| \leqq \int_0^t |g(t - \tau, y)|\,|A_k|\,|g(\tau, y)|\, d\tau \leqq |A_k|\, T(C_1 + C_2|y|\, C_3)$$

for $t \in (0, T\rangle$, $y \in R_n$. Higher derivatives may be estimated analogously.

Lemma 2. *If the condition* (H) *is satisfied for some* $T > 0$, *then it is satisfied for arbitrary* $\tilde{T} > 0$.

Proof. This follows easily from $e^{(iA(y)+B)(\sigma+\tau)} = e^{(iA(y)+B)\sigma} \cdot e^{(iA(y)+B)\tau}$ for $\sigma, \tau \geqq 0$.

Corollary. *For* $\varphi \in \mathscr{S}$, $u_\varphi(t, x) = F^{-1} e^{(iA(y)+B)t} F\varphi$ *is an infinitely differentiable solution of* (1) *in* $t \geqq 0$, $x \in R_n$ *with* $u_\varphi(0, x) = \varphi(x)$ *and* $u_\varphi(t, .) \in \mathscr{S}$ *for all* $t \geqq 0$.

Proof. This follows from $F\mathscr{S} = \mathscr{S}$, $F^{-1}\mathscr{S} = \mathscr{S}$ and Lemmas 1, 2.

Lemma 3. *Let* $h(x) \in E$, $|h(x)| \leqq C_1(1 + |x|^k)$ *for some* $k \geqq 0$. *If*

(4) $$\sup_{v \in S, v \neq 0} \|F^{-1}h\, Fv\|_{L_p}/\|v\|_{L_p} = C < +\infty$$

for some $p \in \langle 1, +\infty\rangle$, *then* $\sup_{x \in R_n} |h(x)| \leqq C$.

(For a more general result see [7], Corollary 1.3.)

Proof. a) $p = 2$. Then $\|F^{-1}h\, Fv\|_{L_2} = \|h\, Fv\|_{L_2} \leqq C\|v\|_{L_2} = C\|Fv\|_{L_2}$ and thus for every $w \in S$, $\|hw\|_{L_2} \leqq C\|w\|_{L_2}$. Let $\psi(x) \in D$, $\psi \geqq 0$, $\psi = 0$ for $|x| \geqq 1$, $\int_{R_n} \psi^2(x)\, dx = 1$. Let $x_0 \in R_n$. Taking in the last inequality $w(x) = \psi_m(\tilde{x}) \equiv m^{n/2} \cdot \psi(m\tilde{x})$, $\tilde{x} = x - x_0$, we get $\|h\psi_m\|^2_{L_2} \leqq C\|\psi_m\|^2_{L_2}$. Since $\|\psi_m\|^2_{L} := 1$, $\|h\psi_m\|^2_{L_2} \to |h(x_0)|^2$ for $m \to \infty$, we obtain $|h(x_0)| \leqq C$.

b) $p \neq 2$. Then it is sufficient to prove that (4) holds with 2 instead of p with the same constant C. First we prove that (4) holds with q instead of p $(1/q + 1/p = 1)$.

$$\|F^{-1}h\, Fv\|_{L_q} = \sup_{\substack{\|w\|_{L_p}=1 \\ w\in S}} \left| \int_{R_n} (F^{-1}h\, Fv, w)\, dx \right| =$$

$$= \sup_{\substack{\|w\|_{L_p}=1 \\ w\in S}} \left| \int_{R_n} (h\, Fv, Fw)\, dx \right| = \sup_{\substack{\|w\|_{L_p}=1 \\ w\in S}} \left| \int_{R_n} (\bar{h}\, \overline{Fv}, \overline{Fw})\, dx \right| =$$

$$= \sup_{\substack{\|\tilde{w}\|_{L_p}=1 \\ \tilde{w}\in S}} \left| \int_{R_n} (\tilde{v}, F^{-1}h\, F\tilde{w})\, dx \right| \leqq C\|\tilde{v}\|_{L_q} = C\|v\|_{L_q},$$

where $\tilde{v}(x) = (-1)^n \bar{v}(-x)$ $(\overline{Fv} = F\tilde{v})$. Applying to the operator S_0: $S_0\varphi = F^{-1}h\, F\varphi$ from S to S the Riesz - Thorin convexity theorem ([8], p. 144) we obtain the desired result in L_2-norms and hence the lemma follows.

Remark. The Riesz - Thorin theorem is formulated in [8] for operators defined on simple functions. For $p \in (1, +\infty)$ no difficulties arise because of the density of S in L_p, L_q. If $p = 1$, $q = +\infty$ one can proceed as follows: By continuity the operator S_0 may be extended to all functions in L_1, particularly to simple functions. It remains to show that this extended operator satisfies $\|S_0\varphi\|_{L_\infty} \leqq C\|\varphi\|_{L_\infty}$ for every simple φ, if originally this estimate held for $\varphi \in S$. Let φ be a simple function. Then the mollifiers $J_m\varphi \in S$ with the radius $1/m$ tend to φ in L_1 when $m \to \infty$, $\|J_m\varphi\|_{L_\infty} \leqq \|\varphi\|_{L_\infty}$, $S_0 J_m\varphi \to S_0\varphi$ in L_1. We can suppose (extracting an appropriate subsequence) that $S_0 J_m\varphi \to S_0\varphi$ almost everywhere. Then from $\|S_0 J_m\varphi\|_{L_\infty} \leqq \leqq C\|J_m\varphi\|_{L_\infty} \leqq C\|\varphi\|_{L_\infty}$ we obtain for almost all $x \in R_n$: $|S_0\, \varphi(x)| \leqq C\|\varphi\|_{L_\infty}$. This justifies the application of the Riesz - Thorin theorem.

Lemma 4. *If* (H) *and* (2) *hold, then*

1) $|e^{iA(y)}| \leqq \text{const}$ *for all* $y \in R_n$,

2) $A(y)$ *has only real eigenvalues and is diagonalizable for all* $y \in R_n$.

3) $\|F^{-1}\, e^{iA(y)t}\, F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p}$ *for all* $\varphi \in \mathscr{S}$, $t \in R_1$, *C being the same as in* (2).

The constant in 1) *depends on C from* (2) *and on N.*

Proof. Let $h_{ij}(t, y)$ be an element of $e^{(iA(y)+B)t}$. By (H) and (2) $h_{ij}(t, y)$ satisfies for every $t \in \langle 0, T\rangle$, the conditions of the preceding lemma with C from (2). (Putting $\varphi = (\varphi_1, \dots, \varphi_N)$, $\varphi_i = \tilde{\varphi}\delta_{ij}$ with $\tilde{\varphi} \in S$.) Then for all $t \in \langle 0, T\rangle$, $y \in R_n$, $|h_{ij}(t, y)| \leqq \leqq C$. Thus $|e^{(iA(y)+B)t}| \leqq N^{1/2}C$ for all $t \in \langle 0, T\rangle$, $y \in R_n$. For $T > t > 0$ we have

$$N^{1/2}C \geqq \sup_{y\in R_n} |e^{(iA(y)+B)t}| = \sup_{\eta\in R_n} |e^{iA(\eta)+tB}|.$$

Letting $t \to 0+$ we obtain 1). This implies 2), because $g(t, y) = e^{iA(y)t}$ is the fundamental matrix of the system $\mathrm{d}u/\mathrm{d}t = (i\,A(y))\,u$ satisfying $g(0, y) = I$, which is bounded for all $t \in R_1$ if and only if 2) holds.

Using the substitution $yt = \eta$, $t \in (0, T\rangle$, we get easily

$$(F^{-1}\,\mathrm{e}^{(iA(y)+B)t}\,F\varphi)\,(x) = (F^{-1}\,\mathrm{e}^{iA(y)+Bt}\,F\varphi_t)\,(x/t)$$

where $\varphi_t(x) = \varphi(tx)$. This implies $\|F^{-1}\,\mathrm{e}^{iA(y)+Bt}\,F\varphi_t\|_{L_p} < C\|\varphi_t\|_{L_p}$ for all $\varphi \in \mathscr{S}$, $t \in (0, T)$, $\varphi_t(x) = \varphi(tx)$. When $\varphi(x)$ varies over all $\mathscr{S}$ then $\varphi_t(x)$ does so. Thus we have

$$\text{(4a)} \qquad \|F^{-1}\,\mathrm{e}^{iA(y)+iB}\,F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p}$$

for all $\varphi \in \mathscr{S}$, $t \in (0, T\rangle$. By Lemma 3 it is $|\mathrm{e}^{iA(y)+tB}| \leqq N^{1/2}C$ for $t \in (0, T\rangle$, $y \in R_n$. This and $F\varphi \in \mathscr{S}$ implies (by the Lebesgue theorem) $F^{-1}\,\mathrm{e}^{iA(y)+Bt}\,F\varphi \to F^{-1}\,\mathrm{e}^{iA(y)}\,F\varphi$ in R_n, as $t \to 0+$. Then it follows from (4a) by the Fatou lemma that $\|F^{-1}\,\mathrm{e}^{iA(y)}\,F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p}$. Similarly as above we get from this estimate the desired estimate 3). Simple modifications for $p = +\infty$ are left to the reader.

Lemma 5. *Let the $N \times N$ matrices $A_1, A_2, \ldots, A_n$ be such that $A(y) = \sum_{i=1}^{n} y_i A_i$ has only real eigenvalues and is diagonalizable for all y in some $B(y^1, \varrho_1)$, $y^1 \in R^n$, $\varrho_1 > 0$. Then there exist $B(y^0, \varrho_0)$, $y^0 \in R_n$, $\varrho_0 > 0$, natural numbers $n_1, n_2, \ldots, n_k$, $\sum_{i=1}^{k} n_i = N$ and real infinitely differentiable in $B(y^0, \varrho_0)$ functions $\lambda_1(y) < \lambda_2(y) \ldots < < \lambda_k(y)$ such that $\lambda_i(y)$ is the eigenvalue of $A(y)$ of multiplicity n_i for $y \in B(y^0, \varrho_0)$, $i = 1, 2, \ldots, k$. For each $\lambda_i(y)$ the corresponding eigenvector $v_i(y)$ may be taken infinitely differentiable in $B(y^0, \varrho_0)$.*

This lemma is an easy consequence of the following

Assertion. *Let the polynomial $P(\lambda; y) \equiv \sum_{k=0}^{N} a_k(y)\,\lambda^k$ with $a_N \equiv 1$, $a_k(y)$ infinitely differentiable real functions of y in $B(y^0, \varrho_0)$ have in $B(y^0, \varrho_0)$ only real roots. Then there exist $B(\tilde{y}^0, \tilde{\varrho}_0) \subset B(y^0, \varrho_0)$, $\tilde{\varrho}_0 > 0$, natural numbers $n_1, n_2, \ldots, n_k$, $\sum_{i=1}^{k} n_i = N$ and infinitely differentiable functions on $B(\tilde{y}^0_1, \tilde{\varrho}_0)$ $\lambda_1(y) < \lambda_2(y) \ldots < < \lambda_k(y)$ such that $\lambda_i(y)$ is the root of $P(\lambda; y)$ of multiplicity n_i for all $y \in B(\tilde{y}^0, \tilde{\varrho}_0)$, $i = 1, 2, \ldots, k$.*

This assertion may be easily proved by induction, using the implicit function theorem.

3. A matrix lemma. Lemma 6. *Let A, B be $N \times N$ matrices with complex elements, satisfying*

1) *A and B have only real eigenvalues,*

2) *$\alpha A + \beta B$ is diagonalizable for all real α, β,*

3) *A, B have the property L: the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_N, \mu_1, \mu_2, \ldots, \mu_N$ *of A and B respectively may be arranged in such a way that* $\alpha\lambda_1 + \beta\mu_1, \alpha\lambda_2 + \beta\mu_2, \ldots$ $\ldots, \alpha\lambda_N + \beta\mu_N$ *are all eigenvalues of* $\alpha A + \beta B$ *for all* α, β *real.*

Then A, B are simultaneously diagonalizable and $AB = BA$.

Proof. If we prove that $\alpha A + \beta B$ is diagonalizable for all α, β complex, we get the lemma by a theorem of Motzkin - Taussky ([9], Theorem 4).

First we remark that 3) is satisfied for all complex α, β. Supposing that the eigenvalues of A and B are arranged in such a way that 3) holds, we can divide the set of pairs $(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_N, \mu_N)$ into k ($k \leqq N$) groups in the following manner: $(\lambda_i, \mu_i), (\lambda_j, \mu_j)$ belong to the same group if and only if $\lambda_i = \lambda_j$, $\mu_i = \mu_j$. We may suppose that the first k pairs $(\lambda_1, \mu_1), (\lambda_2, \mu_2), \ldots, (\lambda_k, \mu_k)$ are different. Denote by ϱ_j the number of elements in the group containing (λ_j, μ_j), $j = 1, 2, \ldots, k$.

Obviously $\sum_{j=1}^{k} \varrho_j = N$. Then for every α complex, $\alpha\lambda_j + \mu_j$, $j = 1, 2, \ldots, k$ is the eigenvalue of $R(\alpha) = \alpha A + B$ of multiplicity $\geqq \varrho_j$. This multiplicity may be $> \varrho_j$ only for $\alpha = \alpha_{jl} = (\mu_j - \mu_l)/(\lambda_l - \lambda_j)$ with $\lambda_l \neq \lambda_j$, $l = 1, 2, \ldots, k$, $l \neq j$. Note that α_{jl} are real. It is sufficient to prove that $R(\alpha)$ is diagonalizable for all α complex. Since the rank of $(\alpha\lambda_j + \mu_j) I - R(\alpha)$ is $\leqq n - \varrho_j$ for real α (by 2)), it is $\leqq n - \varrho_j$ for all complex α (because the minors of this matrix are polynomials in α). Thus the dimension of the corresponding eigensubspace $N_j(\alpha)$ is $\geqq \varrho_j$. For $\alpha \neq \alpha_{jl}$ we have $\dim N_j(\alpha) \leqq$ multiplicity of the eigenvalue $\alpha\lambda_j + \mu_j = \varrho_j$. Thus $\dim N_j(\alpha) = \varrho_j$ and $R(\alpha)$ is diagonalizable for $\alpha \neq \alpha_{jl}$. For $\alpha = \alpha_{jl}$, $R(\alpha)$ is diagonalizable by 2), because α_{jl} is real. The lemma is proved.

4. Proof of Theorem 1. We begin with some lemmas.

Lemma 7. ([10]) *Let* $p \in \langle 1, +\infty \rangle$, $v \in \mathscr{S}$, $|v| \geqq C_0 > 0$ *in* $B = B(y^0, \varrho)$, $\varrho > 0$, $y^0 \in R_n$. *Then there exist a constant* C_1 *and* $B' = B(y^0, \varrho')$, $0 < \varrho' \leqq \varrho$ *such that*

$$C \|F^{-1} v h\|_{L_p} \geqq \|F^{-1} h\|_{L_p}$$

for all $h \in D(B')$.

Proof. There exist k, $1 \leqq k \leqq N$ and $\tilde{\varrho}$, $0 < \tilde{\varrho} \leqq \varrho$ such that $v_k \neq 0$ in $B(y^0, \tilde{\varrho})$ and hence $1/v_k$ is infinitely differentiable in $B(y^0, \tilde{\varrho})$. Putting $w = 1/v_k \, \sigma$ where $\sigma \in S$,

$$\sigma = \begin{cases} 1 & \text{in} \quad B(y^0, \tfrac{1}{2}\tilde{\varrho}), \\ 0 & \text{in} \quad B(y^0, \tfrac{3}{4}\tilde{\varrho}), \end{cases}$$

then $v_k w = 1$ in $B(y^0, \tilde{\varrho}/2)$. Denote $\varrho' = \tilde{\varrho}/2$, $B' = B(y^0, \varrho')$. For $h \in D(B')$ we have

$$\|F^{-1} h\|_{L_p} = \|F^{-1}(h w v_k)\|_{L_p} \leqq \|F^{-1}(v w h)\|_{L_p} \leqq \|F^{-1} w\|_{L_1} \cdot \|F^{-1} v h\|_{L_p} = $$
$$= C \|F^{-1} v h\|_{L_p}$$

with $C = \|F^{-1} w\|_{L_1}$.

Lemma 8. *Let $f \in E$, $|D^\beta f(y)| \leqq C_\beta(1 + |y|^{l_\beta})$ for arbitrary β, $y \in R_n$. If for some $p \in \langle 1, +\infty \rangle$ and all $g \in S$ with $Fg \in D$ we have the estimate*

$$(5) \qquad \|F^{-1} f\, Fg\|_{L_p} \leqq C\|g\|_{L_p},$$

C = constant, then this estimate holds for all $g \in S$.

Proof. It is known (see e.g. [11] pp. 29–31) that for $g \in S$ there exist $g_n \in S$, $Fg_n \in D$, $n = 1, 2, \ldots$ such that $g_n \to g$, $Fg_n \to Fg$ in S. By the assumptions on f we have $f\,Fg_n \to f\,Fg$ and then $F^{-1}f\,Fg_n \to F^{-1}f\,Fg$ in S and hence in L_p. Using (5) for g_n and tending $n \to \infty$, we obtain (5) for g.

The following lemma is essential.

Lemma 9. ([7]) *Let for a real number a the estimate*

$$(6) \qquad \|F^{-1}\, \mathrm{e}^{iay^2}\, F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p}$$

hold for every $\varphi \in S(E_1)$ and some $p \in \langle 1, +\infty \rangle$, $p \neq 2$. Then $a = 0$.

Proof. Putting $U_\varphi(t, x) = F^{-1}\, \mathrm{e}^{iay^2t}\, F\varphi$ for $t \neq 0$, $a \neq 0$, we have

$$U_\varphi(t, x) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \mathrm{e}^{ixy}\, \mathrm{e}^{iay^2t} (F\varphi)(y)\, \mathrm{d}y =$$

$$= -(2\pi)^{-1/2} \int_{-\infty}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}y} [(F\varphi)(y)] \int_0^y \mathrm{e}^{i(at\eta^2 + x\eta)}\, \mathrm{d}\eta\,,$$

$$\int_0^y \mathrm{e}^{i(at\eta^2 + x\eta)}\, \mathrm{d}\eta = \mathrm{e}^{-iat(x^2/4)}\, |t|^{-1/2} \int_{x\sqrt{|t|}/2}^{(y+x/2)\sqrt{|t|}} \mathrm{e}^{ia\zeta^2 \operatorname{sgn} t}\, \mathrm{d}\zeta\,.$$

Since

$$\int_0^N \sin a\zeta^2\, \mathrm{d}\zeta = \int_{-N}^0 \sin a\zeta^2\, \mathrm{d}\zeta = (\operatorname{sgn} a) \frac{1}{2\sqrt{|a|}} \int_0^{N^2|a|} \sin\theta/\sqrt{\theta}\, \mathrm{d}\theta$$

has a finite limit for $N \to +\infty$ and similarly

$$\int_0^N \cos a\zeta^2\, \mathrm{d}\zeta = \int_{-N}^0 \cos a\zeta^2\, \mathrm{d}\zeta\,,$$

we obtain for all $x, y \in R_1$, $t \neq 0$, $a \neq 0$

$$\left|\int_0^y \mathrm{e}^{i(at\eta^2 + x\eta)}\, \mathrm{d}\eta\right| \leqq C(a)\,.\,1/\sqrt{|t|}$$

and then

$$|U_\varphi(t, x)| \leqq |t|^{-1/2}\, \tilde{C}(a) \int_{-\infty}^{+\infty} \left|\frac{\mathrm{d}}{\mathrm{d}y}(F\varphi)(y)\right| \mathrm{d}y \leqq C_1(a, \varphi)\, |t|^{-1/2}\,.$$

By the Parsevall identity,

$$\|U_\varphi(t, x)\|_{L_2} = \|\varphi\|_{L_2}\,.$$

From (6) it follows

(7) $$\|F^{-1}\, e^{iay^2t}\, F\psi\|_{L_p} \leqq C\|\psi\|_{L_p}$$

for $t \in R_1$, $\psi \in S$ (for $t > 0$ by the substitution $y\sqrt{t} = z$, for $t < 0$ using

$$\overline{(F^{-1}\, e^{-iay^2t}\, F\psi)}\,(x) = (F^{-1}\, e^{iay^2t}\, F\tilde{\psi})\,(-x)$$

where $\tilde{\psi}(x) = (-1)^n\, \overline{\psi}(-x)$.

Let $p > 2$. Putting in (7) $\psi = U_\varphi(-t, x)$, we obtain

$$F\psi = e^{-iay^2t}\, F\varphi\,, \quad F^{-1}\, e^{iay^2t}\, F\psi = \varphi\,,$$

which implies

$$\|\varphi\|_{L_p} \leqq C\|U_\varphi(-t, x)\|_{L_p} = C\left(\int_{-\infty}^{+\infty} |U_\varphi|^2\, |U_\varphi|^{p-2}\, dx\right)^{1/p} \leqq \tilde{C}(a, \varphi)\, |t|^{1/p-1/2}\,.$$

For $t \to +\infty$ the right hand side tends to zero, which is a contradiction since φ was arbitrary $\in S$.

For $p < 2$ we have

$$\|U_\varphi\|_{L_2} = \left(\int_{-\infty}^{+\infty} |U_\varphi|^p\, |U_\varphi|^{2-p}\, dx\right)^{1/2} \leqq (C(a, \varphi)\, |t|^{-1/2})^{1-p/2}\,.\, \|U_\varphi\|_{L_p}^{p/2}\,.$$

If (6) (and hence (7)) holds, then

$$\|\varphi\|_{L_2} = \|U_\varphi\|_{L_2} \leqq \tilde{C}|t|^{(p-2)/4}\, \|U_\varphi\|_{L_p}^{p/2} \leqq \tilde{\tilde{C}}|t|^{(p-2)/4}\, \|\varphi\|_{L_p}$$

which yields a contradiction for $t \to +\infty$.

Lemma 10. *Let* $\sum_{i=1}^{n} a_{ij} y_i y_j$, $a_{ij} = a_{ji}$ *be a quadratic form with real coefficients. If for some* $p \in \langle 1, +\infty \rangle$, $p \neq 2$ *the estimate*

(8) $$\|F^{-1} \exp\{i \sum a_{ij} y_i y_j\}\, F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p}$$

holds for all $\varphi \in S$, *where* C *is a constant, then* $a_{ij} = 0$, $i, j = 1, 2, \ldots, n$.

Proof. Let α be the orthogonal matrix such that $\sum_{i,j=1}^{n} a_{ij} y_i y_j = \sum_{k=1}^{n} b_k \eta_k^2$ for $\eta = \alpha y$. Using the substitution $\eta = \alpha y$ we get from (8)

$$\|F^{-1} \exp\{i \sum b_k \eta_k^2\}\, F\varphi\|_{L_p} \lesseqqgtr C\|\varphi\|_{L_p}\,.$$

Putting $\varphi(x) = \prod_1^n \varphi_k(x_k)$, $\varphi_k(x_k) \in S(R_1)$, we obtain

$$\prod_{k=1}^n \|F^{-1} \exp\{ib_k\eta_k^2\} F\varphi_k\|_{L_p} \leqq C \prod_{k=1}^n \|\varphi_k\|_{L_p}$$

for all $\varphi_k \in S(R_1)$, which implies

$$\|F^{-1} \exp\{ib_k\eta_k^2\} F\varphi\|_{L_p} \leqq C_k \|\varphi\|_{L_p}$$

for $k = 1, 2, \ldots, n$, $\varphi \in S(E_1)$. By Lemma 9 $b_k = 0$ and hence $a_{ij} = 0$, $i, j = 1, 2, \ldots$ $\ldots, n$.

Lemma 11. ([7]) *Let $\lambda(y)$ be a real function, $\lambda(y) \in S$, such that for some $p \in$ $\in \langle 1, +\infty \rangle$, $p \neq 2$ the estimation*

$$\|F^{-1} \exp\{i\,\lambda(y)\,t\} F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p} \tag{9}$$

holds for $t \geqq 0$, $F\varphi \in D(B)$, $B = B(y^0, \varrho)$, $y^0 \in R_n$, $\varrho > 0$, $C = const$. Then $\lambda(y) =$ $= \lambda_0 + \sum_{k=1}^n \lambda_k y_k$ on B, where $\lambda_0, \lambda_1, \ldots, \lambda_n$ are real numbers.

Proof. It is sufficient to prove that $\partial^2\lambda(y)/\partial y_i\, \partial y_j = 0$ on B, $i, j = 1, 2, \ldots, n$. Let $\tilde{y} \in B$, $\tilde{\varrho} > 0$ such that $\tilde{B} = B(\tilde{y}, \tilde{\varrho}) \subset B$. Then (9) holds for all φ, $F\varphi \in D(\tilde{B})$ and hence

$$\|F^{-1} \exp\{i(\lambda(y - \tilde{y}))\,t\} F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p}$$

for $F\varphi \in D(B')$, $B' = B(0, \tilde{\varrho})$, (by the substitution $y - \tilde{y} = z$). Thus we may suppose $\tilde{y} = 0$. Let

$$\lambda(y) = a + \sum_{j=1}^n b_j y_j + \sum_{i,j=1}^n a_{ij} y_i y_j + o(|y|^2)\,,$$

where $a_{ij} = \frac{1}{2} \cdot \partial^2\lambda(0)/\partial y_i\, \partial y_j$, $i, j = 1, 2, \ldots, n$. Then

$$\begin{aligned} C\|\varphi\|_{L_p} &\geqq \|F^{-1} \exp\{i\,\lambda(y)\,t\} F\varphi\|_{L_p} = \\ &= \|F^{-1} \exp\{i(a + \textstyle\sum b_j y_j + \sum a_{ij} y_i y_j + o(|y|^2))\,t\} F\varphi\|_{L_p} = \\ &= \|F^{-1} \exp\{i(\textstyle\sum a_{ij} y_i y_j + o(|y|^2))\,t\} F\varphi\|_{L_p} \end{aligned}$$

for $F\varphi \in D(B')$. By the substitution $y\sqrt{t} = z$ we get

$$\|F^{-1} \exp\{i(\textstyle\sum a_{ij} y_i y_j + t o(|y/\sqrt{t}|^2))\} F\varphi\|_{L_p} \leqq C\|\varphi\|_p \tag{10}$$

for $F\varphi \in \mathscr{D}(B_t)$, $B_t = B(0, \varrho\sqrt{t})$. Let B^* be a fixed ball in R_n. Then for t sufficiently large $B_t \supset B^*$ and for $t \to +\infty$ and $F\varphi \in \mathscr{D}(B^*)$ we have

1) $\exp\{i(\sum a_{ij}y_iy_j + to(|y/\sqrt{t}|^2))\} \to \exp\{i\sum a_{ij}y_iy_j\}$

uniformly on B^*

2) $\exp\{i(\sum a_{ij}y_iy_j + to(|y/\sqrt{t}|^2))\}\, F\varphi \to \exp\{i\sum a_{ij}y_iy_j\}\, F\varphi$ in $L_1(R_n)$,

3) $F^{-1}\exp\{i(\sum a_{ij}y_iy_j + to(|y/\sqrt{t}|^2))\}\, F\varphi \to F^{-1}\exp\{i\sum a_{ij}y_iy_j\}\, F\varphi$

in R_n and by the Fatou lemma we get from (10)

$$\|F^{-1}\exp\{i(\sum a_{ij}y_iy_j)\}\, F\varphi\|_{L_p} \leqq C\|\varphi\|_{L_p} \tag{11}$$

for $F\varphi \in \mathscr{D}(B^*)$, and hence (since B^* was arbitrary) for all $\varphi \in \mathscr{D}$, and by Lemma 8 for all $\varphi \in \mathscr{S}$. By Lemma 10 $a_{ij} = 0$, $i, j = 1, 2, \ldots, n$.

Proof of Theorem 1. By Lemma 4, $A(y)$ has only real eigenvalues and is diagonalizable for all $y \in R_n$. Then by Lemma 5 there exists $B = B(y^0, \varrho_0)$ such that in B all eigenvalues $\lambda_i(y)$ and the corresponding eigenvectors $v_i(y)$ may be taken infinitely differentiable in B. If $\tilde{B} = B(y^0, \frac{1}{2}\varrho_0)$, one can change these eigenvalues and eigenvectors outside of $\tilde{B}$ in such a way that they belong to $S(\mathscr{S})$. By Lemma 7 there exists $B' \subset \tilde{B}$ and a constant C_1 such that

$$C_1\|F^{-1}v_1\varphi\|_{L_p} \geqq \|F^{-1}\varphi\|_{L_p}$$

for $F\varphi \in \mathscr{D}(B')$. Then by Lemma 4

$$C\|F^{-1}v_1\, F\varphi\|_{L_p} \geqq \|F^{-1}(\exp\{i\, A(y)\, t\}\, v_1\, F\varphi)\|_{L_p} =$$

$$= \|F^{-1}\exp\{i\,\lambda_1(y)\, t\}\, v\, F\varphi\|_{L_p} \geqq \frac{1}{C_1}\|F^{-1}\exp\{i\,\lambda_1(y)\, t\}\, F\varphi\|_{L_p}$$

and then

$$\|F^{-1}\exp\{i\,\lambda_1(y)\, t\}\, F\varphi\|_{L_p} \leqq \tilde{C}\|F^{-1}v_1\|_{L_1}\,\|\varphi\|_{L_p}$$

for $F\varphi \in \mathscr{D}(B')$. By Lemma 11, $\lambda_1(y)$ is a polynomial of degree $\leqq 1$ in B'. Similarly we can proceed with the other eigenvalues and, finally, obtain a ball $B_1 = B(y^1, \varrho_1)$ such that in B_1 $\lambda_i(y) = a_i + \sum_{j=1}^{n} b_{ij}y_j$, $i = 1, 2, \ldots, N$. Thus

$$\det(\lambda I - A(y)) = \prod_{i=1}^{N}(\lambda - a_i - \sum_{j=1}^{n} b_{ij}y_j)$$

in B_1. Since there are polynomials on both sides, we get $\lambda_i(y) = \sum_{j=1}^{n} b_{ij}y_j + a_i$ for all $y \in R_n$, $i = 1, 2, \ldots, N$ and $a_i = 0$. By Lemma 6 $A_iA_j = A_jA_i$, $i, j = 1, 2, \ldots, n$. However, an arbitrary set of commuting diagonalizable matrices is formed by simultaneously diagonalizable matrices (see [12], p. 10). The theorem is proved.

Corollary. *Let the system* (1) *satisfy* (H) *and let for some* $p \in \langle 1, +\infty \rangle$, $p \neq 2$, $T > 0$, $C > 0$, $\varrho > 0$, $x^0 \in R_n$ *and* $K = K(x^0, \varrho)$ *the estimate*

$$\|u_\varphi(t, x)\|_{L_p} \leqq C \|\varphi\|_{L_p} \tag{2'}$$

hold for all $t \in \langle 0, T \rangle$ *and all* $\varphi \in \mathscr{D}(K)$. *Then there exist* $\tau > 0$, $C_1 = \text{const}$ *such that*

$$\|u_\varphi(t, x)\|_{L_p} \leqq C \|\varphi\|_{L_p}$$

holds for all $t \in \langle 0, \tau \rangle$ *and all* $\varphi \in \mathscr{S}$ *and thus the assertion of Theorem* 1 *is valid.*

Proof. The system (1) has a finite domain of dependence which means that there exist R such that for arbitrary $t_0 > 0$, $x^0 \in R_n$ the solution of (1) at the point (t_0, x^0) does not depend on its values at the points $(0, x)$, $x \,\bar{\in}\, K(x^0, Rt_0)$. (See e.g. [6], pp. 58–63, 89–90.) If (2) holds for all $\varphi \in \mathscr{D}(K(\varrho x^0, \varrho))$, then it holds for $\varphi \in \mathscr{D}(K(\tilde{x}, \varrho))$, $\tilde{x}$ arbitrary $\in R_n$. Let $\varphi \in \mathscr{D}(K(0, m\varrho))$, m natural. Let $K_j = K(j\varrho, \varrho)$, $j = j_1, j_2, \ldots, j_n$, $j\varrho = j_1\varrho, j_2\varrho, \ldots, j_n\varrho$, j_s integer. If ψ_j (j_s integer) is the decomposition of unity on R_n corresponding to the system of domains K_j (j_s integer), i.e. $\psi_j \in D(K_j)$, $1 \geqq \psi \geqq 0$, $\sum_j \psi_j(x) = 1$ for all $x \in R_n$, denote $\varphi_j = \psi_j \varphi$. Since $K(0, m\varrho) \subset \bigcup_{|j_s| \leqq m} K_j$, it is $\sum_{|j_s| \leqq m} \varphi_j = \varphi$, $u_\varphi = \sum_{|j_s| \leqq m} u_{\varphi_j}$. Then there exist natural numbers $\nu(n)$, $\mu(n)$ such that for $t \in \langle 0, \tau \rangle$, $\tau = \min(\varrho/2R, T/2)$, $u_\varphi(t, .) \equiv 0$ outside of $\bigcup_{|j_s| \leqq m} K_j$, at most $\nu(n)$ functions $u_{\varphi_j}(t, .)$ are not identically zero on K_k, $|j_s| \leqq m$, $|k_s| \leqq m$ and each $u_{\varphi_j}(t, .)$, $|j_s| \leqq m$ is not identically zero at most on $\mu(n)$ cubes K_k. Thus we have

$$\begin{aligned}
\|u_\varphi(t, .)\|_{L_p}^p &\leqq \sum_{|k_s| \leqq m} \int_{K_k} |u_\varphi(t, x)|^p \, dx = \sum_{|k_s| \leqq m} \int_{K_k} \Big| \sum_{j \in J_k} u_{\varphi_j}(t, x) \Big|^p \, dx \leqq \\
&\leqq \sum_{|k_s| \leqq m} \nu^{p-1} \int_{K_k} \sum_{j \in J_k} |u_{\varphi_j}|^p \, dx \leqq \nu^{p-1} \mu \sum_{|j_s| \leqq m} \int_{R_n} |u_{\varphi_j}|^p \, dx \leqq \\
&\leqq \nu^{p-1} \mu C^p \int_{R_n} \sum_{|j_s| \leqq m} |\varphi_j|^p \, dx = \nu^{p-1} \mu C^p \int_{R_n} \sum_{|j_s| \leqq m} |\varphi|^p \, |\psi_j|^p \, dx \leqq \\
&\leqq \nu^{p-1} \mu C^p \int_{R_n} |\varphi|^p \sum_{|j_s| \leqq m} \psi_j = \nu^{p-1} \mu C^p \|\varphi\|_{L_p}^p
\end{aligned}$$

where J_k is the set of such j that u_{φ_j} is not identically zero on K_k. For $\tilde{C} = C\nu^{1/q}\mu^{1/p}$ we have then

$$\|u_\varphi(t; x)\|_{L_p} \leqq \tilde{C} \|\varphi\|_{L_p}$$

for $t \in \langle 0, \tau \rangle$, $\varphi \in \mathscr{D}(K(0, m\varrho))$. Since $\tilde{C}$ does not depend on m, this estimate is valid for all $\varphi \in \mathscr{D}$, and by the density of $\mathscr{D}$ in $\mathscr{S}$ it is valid for all $\varphi \in \mathscr{S}$. The corollary is proved.

5. Proof of Theorem 2. We begin with two lemmas.

Lemma 12. *For each* $q \in (1, +\infty)$, $p = q/(1-q)$ *there exists a constant* $C_q > 0$ *such that*

$$\sup_{\varphi \in \mathscr{W}_p^k} \frac{\left| \sum_{|\alpha| \leq k} \int_{R_n} (D^\alpha \psi, D^\alpha \varphi)\, dx \right|}{\|\varphi\|_{\mathscr{W}_p^k}} \geqq C_q \|\psi\|_{\mathscr{W}_q^k} \tag{12}$$

for all $\psi \in \mathscr{W}_q^k$.

Proof. For fixed $\psi \in \mathscr{W}_q^k$ denote

$$f_l^\alpha = \mu_{l\alpha}^{-1} |D^\alpha \psi_l|^{q-1}$$

for $l = 1, 2, \ldots, N$, $|\alpha| \leqq k$, where $\bar{\mu}_{l\alpha} . D^\alpha \psi_l = |D^\alpha \psi_l|$, $|\mu_{l\alpha}| = 1$. Then

$$\|f_l^\alpha\|_{L_p} = \left(\int |D^\alpha \psi_l|^{(q-1)/p} \right)^{1/p} = \|D^\alpha \psi_l\|_{L_q}^{q-1} .$$

There exists a sequence $f_m^\alpha = (f_{m,1}^\alpha, \ldots, f_{N,m}^\alpha)$ converging to $f^\alpha = f_1^\alpha, f_2^\alpha, \ldots, f_N^\alpha$ in L_p. Putting

$$\varphi_m = F^{-1} g(\xi) \sum_{|\alpha| \leqq k} (-1)^{|\alpha|} (i\xi)^\alpha F^{-1} f_m^\alpha$$

where $g(\xi) = (\sum_{|\alpha| \leqq k} \xi^{2\alpha})^{-1}$, $\xi^{2\alpha} = \xi_1^{2\alpha_1} \xi_2^{2\alpha_2} \ldots \xi_n^{2\alpha_n}$. By Michlin's theorem on multipliers [12] we have

$$\|\varphi_m\|_{W_p^k} \leqq C_p \sum_{|\alpha| \leqq k} \|f_m^\alpha\|_{L_p}$$

$\varphi_m \to \tilde{\varphi}$ in $\mathscr{W}_p^k$ as $m \to \infty$, satisfying

$$\sum_{|\alpha| \leqq k} \int_{R_n} (D^\alpha \psi, D^\alpha \tilde{\varphi})\, dx = \sum_{|\alpha| \leqq k} \int_{R_n} (D^\alpha \psi, f^\alpha)\, dx .$$

From the definition of f^α we obtain

$$\sum_{|\alpha| \leqq k} \int_{R_n} (D^\alpha \psi, D^\alpha \tilde{\varphi})\, dx = \sum_{|\alpha| \leqq k} \int_{R_n} (D^\alpha \psi, f^\alpha)\, dx = \sum_{|\alpha| \leqq k} \sum_{l=1}^N \int_{R_n} |D^\alpha \psi_l|^q\, dx$$

and hence

$$\sup_{\varphi \in \mathscr{W}_p^k} \frac{\left| \int \sum (D^\alpha \psi, D^\alpha \varphi)\, dx \right|}{\|\varphi\|_{\mathscr{W}_p^k}} \geqq C \frac{\sum_{|\alpha| \leqq k} \sum_{l=1}^N |D^\alpha \varphi_l|^q\, dx}{\sum_{|\alpha| \leqq k} \sum_{l=1}^N \left(\int |D^\alpha \psi_l|^q\, dx \right)^{(q-1)/q}} \geqq C_q \|\psi\|_{\mathscr{W}_q^k} .$$

Lemma 13. *Let A be a linear operator from $\mathscr{D}(E_n)$ to $\mathscr{S}(R_n)$, $p \in (1, +\infty)$, k natural, satisfying*

$$\|A\varphi\|_{\mathscr{W}_p{}^k} \leqq C_1 \|\varphi\|_{\mathscr{W}_p{}^k},$$
$$\|A\varphi\|_{\mathscr{W}_p{}^{-k}} \leqq C_2 \|\varphi\|_{\mathscr{W}_p{}^{-k}}$$

for all $\varphi \in \mathscr{D}(R_n)$, where C_1, C_2 are constants. Then there exists a constant C depending only on C_1, C_2, N, n and n such that

$$\|A\varphi\|_{L_p} \leqq C \|\varphi\|_{L_p}$$

holds for all $\varphi \in \mathscr{D}(R_n)$.

For the proof see e.g. [13] (theorems 7, 9, 10) and [14].

Now we are able to prove Theorem 2. Putting for $\varphi \in \mathscr{S}$, $t \in \langle 0, T\rangle$

$$S_t\,\varphi(x) \equiv F^{-1} \exp\{(i\,A(y) + B)\,t\}\,F\varphi\,, \quad S_t^*\,\varphi(x) = F^{-1} \exp\{(-i\,A^*(y) + B^*)\,t\}\,F\varphi$$

where $A^*(y)$, B^* are the adjoint matrices to $A(y)$ and B, respectively, we have by $D^\alpha S_t \varphi = S_t D^\alpha \varphi$, $D^\alpha S_t^* \varphi = S_t^* D^\alpha \varphi$ and the assumption of Theorem 2:

$$\left|\sum_{|\alpha| \leqq k} \int_{R_n} (D^\alpha \varphi, D^\alpha S_t^* \psi)\,\mathrm{d}x\right| = \left|\sum_{|\alpha| \leqq k} \int_{R_n} (D^\alpha S_t \varphi, D^\alpha \psi)\,\mathrm{d}x\right| \leqq$$
$$\leqq \tilde{C} \|S_t \varphi\|_{\mathscr{W}_p{}^k} \|\psi\|_{\mathscr{W}_q{}^k} \leqq C_1 \tilde{C} \|\varphi\|_{\mathscr{W}_p{}^k} \|\psi\|_{\mathscr{W}_q{}^k},$$

$\tilde{C}$ being a constant, $q = (p - 1)/p$. Since $\mathscr{S}$ is dense in $\mathscr{W}_p^k$, this estimate holds for all $\varphi \in \mathscr{W}_p^k$. Now by Lemma 12

$$\sup_{\varphi \in \mathscr{W}_p{}^k} \frac{\left|\sum_{|\alpha| \leqq k} \int_{R_n} (D^\alpha \varphi, D^\alpha S_t^* \psi)\,\mathrm{d}x\right|}{\|\varphi\|_{\mathscr{W}_p{}^k}} \geqq C_q |S_t^* \psi\|_{\mathscr{W}_q{}^k}$$

and thus

$$\|S_t^* \psi\|_{\mathscr{W}_q{}^k} \leqq C_3 \|\psi\|_{\mathscr{W}_q{}^k}$$

for all $\psi \in \mathscr{S}$, $t \in \langle 0, T\rangle$. This implies for all $\varphi, \psi \in \mathscr{S}$, $t \in \langle 0, T\rangle$

$$\left|\int_{R_n} (S_t^* \psi, \varphi)\,\mathrm{d}x\right| \leqq C_4 \|\varphi\|_{\mathscr{W}_p{}^{(-k)}} \,.\, \|\psi\|_{\mathscr{W}_q{}^k}$$

and hence

$$\left|\int_{R_n} (S_t \varphi, \psi)\,\mathrm{d}x\right| = \left|\int_{R_n} (S_t^* \psi, \varphi)\,\mathrm{d}x\right| \leqq C_4 \|\varphi\|_{\mathscr{W}_p{}^{(-k)}} \,.\, \|\psi\|_{\mathscr{W}_q{}^k}$$

and

$$\|S_t \varphi\|_{\mathscr{W}_p{}^{(-k)}} \leqq C_4 \|\varphi\|_{\mathscr{W}_p{}^{(-k)}}$$

for all $\varphi \in \mathscr{S}$, $t \in \langle 0, T\rangle$. Applying Lemma 13 to S_t, $t \in \langle 0, T\rangle$, we get Theorem 2.

Remark. The cases $p = 1$, $+\infty$ are not covered by this theorem. This seems to be a defect of the method. The following example of a strongly hyperbolic system

(1) ($n = 2, N = 2$) which does not satisfy $A_1A_2 = A_2A_1$ and the estimate $\|u_\varphi\|_{\mathscr{W}_\infty^1} \leqq \leqq C\|\varphi\|_{\mathscr{W}_\infty^1}$ may be of some interest.

Example. Let us consider the system

$$(13) \qquad \frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} = 0\,, \quad \frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0$$

with initial conditions $u(0, x) = \varphi(x)$, $\varphi \in \mathscr{D}(R_2)$, $x = x_1, x_2$. If we take $\varphi_2 \equiv 0$, $\varphi_1 = J_\varepsilon \tilde{\tilde{\varphi}}_h$, where J_ε is the mollifier with the radius $\varepsilon > 0$,

$$(14) \quad \varphi_h = \begin{cases} \int_0^{x_2} y \arcsin\left(\frac{x_1}{\sqrt{[(1+h)^2 - y^2]}}\right) \mathrm{d}y & x_2 \geqq 0 \quad \text{for} \quad |x| \leqq (1 + h/2) \\ 0 & x_2 < 0 \quad \text{for} \quad 0 < h < 1 \end{cases}$$

(it is $\|\varphi_h\|_{\mathscr{W}_\infty^1(|x| \leqq 1 + h/2)} \leqq \text{const}$), $\tilde{\varphi}_h$ the extension of φ_h on R_2 satisfying $\|\tilde{\varphi}_h\|_{\mathscr{W}^1_\infty(R_2)} \leqq \leqq \text{const}$, $\tilde{\tilde{\varphi}}_h = \tilde{\varphi}_h \zeta$ where $\zeta = 1$ for $|x| \leqq \frac{3}{2}$, $\zeta = 0$ for $|x| \geqq 2$, $\zeta \in \mathscr{D}(R_2)$, $0 \leqq \zeta \leqq 1$, then we have for the corresponding solution u_φ: $(\partial/\partial x_1)(u_\varphi)_1(1, 0, 0)$ is unbounded as $h \to 0$, $\varepsilon(h) \to 0$. Thus the system (13) does not satisfy the estimate $\|u_\varphi\|_{\mathscr{W}_\infty^1} \leqq \leqq C\|\varphi\|_{\mathscr{W}_\infty^1}$. This may be seen as follows: if u_φ is the solution of (13) with the initial condition φ, then applying to (13) the operator

$$B(D) = \begin{pmatrix} \frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\,, & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2}\,, & \frac{\partial}{\partial t} - \frac{\partial}{\partial x_1} \end{pmatrix}$$

we obtain that $(u_\varphi)_i$ is the solution of

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)(u_\varphi)_i = 0\,,$$

$i = 1, 2$, with initial conditions $u_\varphi(0, x) = \varphi(x)$,

$$\frac{\partial (u_\varphi)_1}{\partial t}(0, x) = \frac{\partial \varphi_1}{\partial x_1} + \frac{\partial \varphi_2}{\partial x_2}\,, \quad \frac{\partial (u_\varphi)_2}{\partial t}(0, x) = \frac{\partial \varphi_1}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_1}\,.$$

Then $(u_\varphi)_2$ may be written by the well known formula

$$(u_\varphi)_2(t, x) = \frac{\partial}{\partial t}\frac{1}{2\pi}\int_{\sqrt{(\xi^2+\eta^2)} \leqq t} \frac{\varphi_2(x_1 + \xi, x_2 + \eta)}{\sqrt{(t^2 - \xi^2 - \eta^2)}}\, \mathrm{d}\xi\, \mathrm{d}\eta + $$
$$+ \frac{1}{2\pi}\int_{\sqrt{(\xi^2+\eta^2)} \leqq t} \frac{\left(\frac{\partial \varphi_1}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_1}\right)(x_1 + \xi, x_2 + \eta)}{\sqrt{(t^2 - \xi^2 - \eta^2)}}\, \mathrm{d}\xi\, \mathrm{d}\eta\,.$$

From this representation the desired result follows.

6. Proof of Theorem 3. First of all we remark that if $|\exp\{i(A(y)+B)t\}| \leqq C$ for $t \in \langle 0, T\rangle$, $T > 0$, $y \in R_n$ then $|\exp\{i A(y)\}| \leqq C$ for $y \in R_n$, $|\exp\{i A(y) t\}| \leqq C$ for $t \in R_1$, $y \in R_n$, $|\exp\{(i A(y) + \tilde{B}) t\}| \leqq C \exp\{C|\tilde{B}| |t|\}$ for arbitrary B and hence

$$\|u_\varphi^{(\tilde{B})}\|_{L_2} \leqq C \exp\{C|\tilde{B}| |t|\} \|\varphi\|_{L_2}$$

for $t \in R_1$, $\varphi \in \mathscr{S}$, $u_\varphi^{(\tilde{B})} = F^{-1} \exp\{(i A(y) + \tilde{B}) t\} F\varphi$ (C is the same as in the first inequality). The system (1) has a finite domain of dependence and the constant R (see the proof of Corollary in § 4) depends only on $A_1, A_2, \ldots, A_n$.

Now we prove that there exists $C(p, n, N)$ such that

$$\|u_\varphi^{(B)}(t, x)\|_{\mathscr{L}_p(R_n)} \leqq C(p, n, N, R)\, e^{C|B|} \|\varphi\|_{\mathscr{W}_p^k(R_n)} \tag{14}$$

for all $\varphi \in \mathscr{D}(K(x^0, 1))$, $|t| \leqq 1$, $x^0 \in R_n$. In virtue of the invariance of (14) with respect to a translation one may suppose $x^0 = 0$.

a) $p \in \langle 1, 2)$. If $\varphi \in \mathscr{D}(K(0, 1))$, then $u_\varphi^{(B)} \in \mathscr{D}(K(0, 1 + R))$ for $|t| \in \langle 0, 1\rangle$. Thus we have

$$\begin{aligned}\|u_\varphi^{(B)}\|_{\mathscr{L}_p(R_n)} &= \|u_\varphi^{(B)}\|_{\mathscr{L}_p(K(0,1+R))} \leqq \\ &\leqq \tilde{C}(p, N, n, R) \|u_\varphi^{(B)}\|_{\mathscr{L}_2(K(0,1+R))} \leqq \\ &\leqq \tilde{C}(p, N, n, R)\, C\, e^{C|B|} \|\varphi\|_{L_2(K(0,1))} \leqq \\ &\leqq \tilde{\tilde{C}}(p, N, n, R)\, C\, e^{C|B|} \|\varphi\|_{\mathscr{W}_p^k(K(0,1))}\end{aligned}$$

(with k defined in the formulation of Theorem 3) by imbedding theorems [16] and Hölder inequality.

b) $p \in (2, +\infty\rangle$.

$$\begin{aligned}\|u_\varphi^{(B)}\|_{\mathscr{L}_p(R_n)} &= \|u_\varphi^{(B)}\|_{\mathscr{L}_p(K(0,1+R))} \leqq \\ &\leqq \tilde{C}(p, N, n) \|u_\varphi^{(B)}\|_{\mathscr{W}_2^k(K(0,1+R))} \leqq \\ &\leqq \tilde{C}(p, N, n)\, C\, e^{C|B|} \|\varphi\|_{\mathscr{W}_2^k(K(0,1))} \leqq \\ &\leqq \tilde{\tilde{C}}(p, N, n, R)\, C\, e^{C|B|} \|\varphi\|_{\mathscr{W}_p^k(K(0,1))}\end{aligned}$$

Thus (13) is proved.

Let $\varphi \in \mathscr{D}(0, m)$, m natural. We shall use the notation as in the proof of Corollary of Theorem 1 with $T = \varrho = 1$. We have $\varphi = \sum_{|j_s| \leqq m} \varphi_j$, $u_\varphi^{(B)} = \sum_{|j_s| \leqq m} u_{\varphi_j}^{(B)}$. Now it is

$$\|u_\varphi^{(B)}(t, x)\|_{\mathscr{L}_p(R_n)}^p \leqq \sum_{|l_s| \leqq m} \int_{K_l} |u_\varphi|^p \, dz = \sum_{|l_s| \leqq m} \int_{K_l} |\sum_{|j_s| \leqq m} u_{\varphi_j}^{(B)}|^p \, dx \leqq$$

$$\leqq \sum_{|l_s| \leqq m} \nu^{p-1} \int_{K_l} \sum_{j \in J_l} |u_{\varphi_j}^{(B)}|^p \, dx \leqq \nu^{p-1} \mu \sum_{|j_s| \leqq m} \int_{R_n} |u_{\varphi_j}^{(B)}|^p \, dx \leqq$$

$$\leqq \nu^{p-1}\mu\, C^p(p, n, N, R)\, e^{pC|B|} \sum_{|j_s| \leqq m} (\|\varphi_j\|_{\mathscr{W}_p{}^k})^p \leqq$$

$$\leqq \nu^{p-1}\mu\, C^p(p, n, N, R)\, e^{pC|B|} \sum_{|j_s| \leqq m} \sum_{|\alpha| \leqq k} \int_{R_n} |D^\alpha \varphi_j|^p \, dx \leqq \tilde{C}^p\, e^{pC|B|} \|\varphi\|^p_{\mathscr{W}_p{}^k} .$$

We have used inequalities

$$(|\sum_{j=1}^{s} a_j|)^p \leqq s^{p-1} \sum |a_j|^p , \quad \sum |\psi_j|^p \leqq \sum \psi_j = 1 ,$$

$$\sum_{|j_s| \leqq m} \int_{K_j} |\varphi|^p \, dx \leqq \text{const} \int_{R_n} |\varphi|^p \, dx .$$

Thus we have proved that there exist $\tau > 0$, $C^*(p, n, N, R)$ such that

$$\|u_\varphi^{(B)}\|_{\mathscr{L}_p(R_n)} \leqq C^*(p, n, N, R)\, e^{C|B|} \|\varphi\|_{\mathscr{W}_p{}^k(R_n)} \tag{15}$$

for $|t| \leqq \tau$, $\varphi \in \mathscr{D}(R_n)$, B arbitrary. Let $\varphi \in \mathscr{D}(R_n)$, $u_\varphi^{(B)}$ as above, $\gamma > 0$. Putting $u_\gamma(t, x) = u_\varphi^{(B)}(\gamma t, \gamma x)$, then u_γ is the solution of

$$\frac{\partial u_\gamma}{\partial t} = \sum_{i=1}^{n} A_i \frac{\partial u_\gamma}{\partial x_i} + \gamma\, B u_\gamma \tag{16}$$

with the initial condition $\varphi(\gamma x)$. Thus by (15)

$$\|u_\gamma(t, x)\|_{\mathscr{L}_p} \leqq C^*\, e^{C\gamma|B|}\, \|\varphi(\gamma x)\|_{\mathscr{W}_p{}^k(R_n)} , \quad |t| < \tau .$$

If t_0 is arbitrary but fixed $\in R_1$, then

$$\|u_\varphi^{(B)}(t_0, x)\|_{\mathscr{L}_p} = \|u_\gamma(t_0/\gamma, x/\gamma)\|_{\mathscr{L}_p} =$$

$$= \gamma^{n/p} \|u_\gamma(t_0/\gamma, x)\|_{\mathscr{L}_p} \leqq \gamma^{n/p} C^*\, e^{C\gamma|B|} \sum_{s=0}^{k} \|\varphi(\gamma x)\|_{\mathscr{L}_p{}^s} =$$

$$= C^*\, e^{C\gamma|B|} \sum \gamma^s \|\varphi\|_{\mathscr{L}_p{}^s}$$

for $|t_0|/\gamma \leqq \tau$.

Then for $\gamma = |t_0|/\tau$

$$\|u_\varphi(t_0, x)\|_{L_p} \leqq \tilde{C}\, e^{C/\tau|B||t_0|} \sum_{s=0}^{k} (|t_0|/\tau)^s\, \|\varphi\|_{L_p{}^s} = \tilde{\tilde{C}}\, e^{C_1|B||t_0|} \sum |t_0|^s\, \|\varphi\|_{L_p{}^s} .$$

The theorem is proved.

Remark. It is seen that Theorem 3 gives a better result then in [5] for all $p \in \langle 1, +\infty)$ when n is even and for p close to 2 if n is odd.

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Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83 (Matematicko-fyzikální fakulta UK).