Czechoslovak Mathematical Journal

Ján Jakubík

Center of a complete lattice

Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 1, 125–138

Persistent URL: http://dml.cz/dmlcz/101151

Czechoslovak Mathematical Journal, 23 (98) 1973, Praha

CENTER OF A COMPLETE LATTICE

JÁN JAKUBÍK, Košice

(Received February 28, 1972)

1. INTRODUCTION

Let L be a complete lattice. We denote by $C = C(L)$ the center of L. It is well-known that $C(L)$ is a sublattice of L. If L is infinitely distributive, then $C(L)$ is a closed sublattice of L [2]. In this Note we show (Thm. 2) that $C(L)$ is a closed sublattice of L if and only if the following weakened infinite distributive law is valid in L:

For any $x, y \in L$, $x \geqq y$, and any subset $\{a_i\} \subset C(L)$,

$$y \vee (x \wedge (\bigwedge a_i)) = \bigwedge(y \vee (x \wedge a_i)), \tag{1}$$

$$x \wedge (y \vee (\bigvee a_i)) = \bigvee(x \wedge (y \vee a_i)). \tag{2}$$

In [1] there is proposed the problem wheather the center of any complete lattice is a closed sublattice (p. 131, Problem 34). In §4 below there is described a complete distributive lattice L and a subset $\{a_i\} \subset C(L)$ such that the element $\bigwedge a_i$ has no complement in L; thus $C(L)$ is not a closed sublattice of L.

In §5 there are investigated relative centers and direct factors of a conditionally complete lattice L. There are found necessary and sufficient conditions under which each nonempty intersection of direct factors of L is a direct factor of L (Thm. 3). As a corollary, we obtain the assertion: If for each interval $[u, v] \subset L$ the center $C([u, v])$ is a closed sublattice of $[u, v]$, then each nonempty intersection of direct factors of L is a direct factor of L. Let us remark that the first condition has a local character (concerning intervals of L) while the second one has a global character.

Let us recall some basic notions and denotations (cf. [1]). The lattice operations will be denoted by $\wedge$, $\vee$ (unless otherwise stated). The direct product $A \times B$ of two lattices A, B is the set of all pairs (a, b) with $a \in A$, $b \in B$, the lattice operations in $A \times B$ being defined componentwise. If a lattice L has the least element (the greatest element), then we denote this element by $0(L)$ or $1(L)$, respectively, and analogously for other lattices. Let c, $0(L)$, $1(L) \in L$ and assume that there are lattices A, B and an isomorphism φ of L onto $A \times B$ such that $\varphi(c) = (c_1, c_2)$ with $c_1 = 1(A)$, $c_2 = 0(B)$.

Then c is said to be a central element of L and the set $C(L)$ of all central elements of L is the center of L. The set $C(L)$ is a sublattice of L and $C(L)$ is a Boolean algebra. Each element $c \in C(L)$ has a unique complement in L. This complement will be always denoted by c'; this element also belongs to $C(L)$. Each element $a \in C(L)$ is neutral, i.e., if $x, y \in L$, then the sublattice of L generated by the elements a, x, y is distributive.

2. WEAKENED INFINITE DISTRIBUTIVITY

In §2–4 we assume that L is a complete lattice. Let $\{a_i\} \subset C(L)$. If there exists the least upper bound of $\{a_i\}$ in $C(L)$, then we denote it by $\bigvee^* a_i$ and analogously for the greatest lower bound in $C(L)$. Since $C(L)$ is a sublattice of L, for a finite set $\{a_i\}$ we have $\bigvee^* a_i = \bigvee a_i$, and dually. If $\bigvee^* a_i$ exists, then clearly $\bigvee^* a_i \geqq \bigvee a_i$, and dually.

Lemma 1. *Let* $\emptyset \neq \{a_i\}_{i \in I} \subset C(L)$, $a = \bigwedge a_i \in C(L)$, $b = \bigvee a'_i \in C(L)$. *Then* $a \wedge b = 0(L)$, $a \vee b = 1(L)$.

Proof. Since $\bigwedge a_i \in C$, $\bigvee a'_i \in C$, we have

$$\bigwedge a_i = \bigwedge^* a_i\,, \quad \bigvee a'_i = \bigvee^* a'_i\,.$$

Any Boolean algebra is infinitely distributive, therefore

$$a \wedge (\bigvee^* a'_i) = \bigvee^*(a \wedge a'_i)\,.$$

Hence

$$a \wedge b = a \wedge (\bigvee a'_i) = a \wedge (\bigvee^* a'_i) = \bigvee^*(a \wedge a'_i)\,.$$

Further we have $a \wedge a'_i \leqq a_i \wedge a'_i = 0(L)$ for each $i \in I$, thus $a \wedge b = 0(L)$. In a dual way we prove that $a \vee b = 1(L)$.

Lemma 2. *Let* $x \in L$, $c \in C(L)$. *Then* $x \wedge c \in C([0(L), x])$, $x \vee c \in C([x, 1(L)])$.

Proof. There exist lattices A, B and an isomorphism φ of L onto $A \times B$ such that $\varphi(c) = (1(A), 0(B))$. Denote $L_1 = [0(L), x]$ and let φ^* be the corresponding partial mapping of the set L_1 into $A \times B$, $\varphi^*(x) = (x_1, x_2)$. Then φ^* is an isomorphism of L_1 onto $[0(A), x_1] \times [0(B), x_2]$. In fact, if $y \in L_1$, then $\varphi^*(y) = (y_1, y_2) \in [0(A), x_1] \times [0(B), x_2]$. Let $(z_1, z_2) \in [0(A), x_1] \times [0(B), x_2]$, $z \in L$, $\varphi(z) = (z_1, z_2)$; then $z \in [0(L), x]$. We have $x \wedge c \in L_1$ and $\varphi^*(x \wedge c) = (x_1, 0(B))$. Therefore $x \wedge c \in C([0(L), x])$. The second assertion can be proved dually.

Lemma 3. *Let* $x, z \in L$, $\emptyset \neq \{a_i\} \subset C(L)$, $a = \bigwedge a_i \in C(L)$, $b = \bigvee a'_i \in C(L)$. *Then* $x \wedge b = \bigvee(x \wedge a'_i)$, $z \vee a = \bigwedge(z \vee a_i)$.

Proof. According to lemma 2, the elements $x \wedge a_i$, $x \wedge a'_i$ belong to the center of the lattice $[0(L), x]$. Further from the fact that $a_i \in C(L)$ we infer that a_i is neutral, hence the sublattice of L generated by the elements a_i, a'_i, x is distributive. Therefore $x \wedge a'_i$ is a relative complement of $x \wedge a_i$ in the interval $[0(L), x]$. Denote $a_0 = \bigwedge(x \wedge a_i)$, $b_0 = \bigvee(x \wedge a'_i)$. From Lemma 1 applied to the lattice $[0(L), x]$ we obtain $a_0 \wedge b_0 = 0(L)$, $a_0 \vee b_0 = x$. Put $v = x \wedge b = x \wedge (\bigvee a'_i)$. Clearly $a_0 = \bigwedge(x \wedge \wedge a_i) = x \wedge (\bigwedge a_i) = x \wedge a$, hence according to Lemma 1, $a_0 \wedge v = 0(L)$. Further, since $b \in C(L)$ is neutral, we have $a_0 \vee v = (x \wedge a) \vee (x \wedge b) = x \wedge (a \vee b) = x$. Therefore both elements b_0 and v are relative complements of a_0 in the interval $[0(L), x]$. Since (by Lemma 2) a_0 belongs to the center of $[0(L), x]$ we get $b_0 = v$. Thus we have $x \wedge b = \bigvee(x \wedge a'_i)$. By a dual method we verify the second assertion.

Lemma 4. *Let $x, y \in L$, $y \leqq x$, $\emptyset \neq \{a_i\} \subset C(L)$, $a = \bigwedge a_i \in C(L)$, $b = \bigvee a'_i \in C(L)$. Then*

$$y \vee (x \wedge (\textstyle\bigwedge a_i)) = \bigwedge(y \vee (x \wedge a_i))\,, \tag{3}$$

$$x \wedge (y \vee (\textstyle\bigvee a'_i)) = \bigvee(x \wedge (y \vee a'_i))\,. \tag{4}$$

Proof. We have $\bigwedge(x \wedge a_i) = x \wedge (\bigwedge a_i) = x \wedge a \in C([0(L), x])$ by Lemma 2. Now according to Lemma 3 (applied to the lattice $[0(L), x]$ instead of L) we obtain

$$y \vee (x \wedge a) = \bigwedge(y \vee (x \wedge a_i))\,.$$

Therefore (3) is valid. By a dual method we verify (4).

From Lemma 4 we obtain as a corollary:

Lemma 5. *Assume that $C(L)$ is a closed sublattice of L. Then (1) and (2) are valid for each subset $\emptyset \neq \{a_i\} \subset C(L)$ and each $x, y \in L$, $x \geqq y$.*

3. SUFFICIENT CONDITION FOR THE CENTER TO BE CLOSED

In this section we show that the validity of (3), (4) for each $x, y \in L$, $y \leqq x$ is sufficient in order that the elements $\bigwedge a_i$, $\bigvee a'_i$ belong to the center $C(L)$ of a complete lattice L.

Let us remark that by putting $x = 1(L)$ we get from (3)

$$y \wedge (\textstyle\bigvee a_i) = \bigvee(y \wedge a_i)\,, \tag{3'}$$

and by putting $y = 0(L)$ we obtain from (4)

$$x \wedge (\textstyle\bigvee a'_i) = \bigvee(x \wedge a'_i)\,. \tag{4'}$$

Let $\emptyset \neq \{a_i\}$ be a fixed subset of $C(L)$, $\bigwedge a_i = a$, $\bigvee a'_i = b$.

Lemma 6. *Assume that (4') holds for each $x \in L$. Then $a \wedge b = 0(L)$.*

Proof. We have

$$a \wedge b = a \wedge (\bigvee a_i') = \bigvee(a \wedge a_i').$$

Since $a \leqq a_i$ and $a_i \wedge a_i' = 0(L)$, we obtain $a \wedge b = 0(L)$.

In a dual way we get:

Lemma 6'. *Assume that* (3') *is valid for each* $y \in L$. *Then* $a \vee b = 1(L)$.

Lemma 7. *Assume that* (3) *and* (4) *are valid for each pair of elements* $x, y \in L$ *with* $y \leqq x$. *Then* $x \wedge a$ *is a complement of* $x \wedge b$ *in the lattice* $[0(L), x]$ *for each* $x \in L$.

Proof. By Lemma 6, $a \wedge b = 0(L)$, whence $(x \wedge a) \wedge (x \wedge b) = 0(L)$. Denote $z = x \wedge (\bigvee a_j')$ $(j \in I)$. According to (3) we have $z \vee (x \wedge (\bigwedge a_i)) = \bigwedge(z \vee (x \wedge a_i))$. Further from (4') we obtain (by using the neutrality of a_i)

$$(x \wedge a_i) \vee z = (x \wedge a_i) \vee (x \wedge (\bigvee_j a_j')) = \bigvee_j((x \wedge a_i) \vee (x \wedge a_j')) =$$
$$= \bigvee_j(x \wedge (a_i \vee a_j')).$$

Since $a_i \vee a_i' = 1(L)$, we get

$$(x \wedge a_i) \vee z = x \quad \text{for each} \quad i \in I.$$

Thus $z \vee (x \wedge (\bigwedge a_i) = x$. The proof is complete.

Analogously we verify (by using Lemma 6'):

Lemma 7'. *Assume that* (3) *and* (4) *are valid for each pair of elements* $x, y \in L$ *with* $y \leqq x$. *Then* $x \vee a$ *is a complement of* $x \vee b$ *in the interval* $[x, 1(L)]$ *for each* $x \in L$.

Lemma 8. *Let the same assumptions as in Lemma* 7 *be valid. Let* $x \in L$ *and denote* $x \wedge a = u_1$, $x \wedge b = u_2$. *Let* $v_1, v_2 \in L$, $v_1 \leqq u_1$, $v_2 \leqq u_2$, $v_1 \vee v_2 = x$. *Then* $v_i = u_i$ $(i = 1, 2)$.

Proof. According to Lemma 7' we have

$$v_1 = (v_1 \vee a) \wedge (v_1 \vee b).$$

Since $v_1 \vee a = a$, $v_1 \vee b = v_1 \vee v_2 \vee b = x \vee b$, we obtain $v_1 = a \wedge (x \vee b) \geqq$ $\geqq a \wedge x = u_1$. This shows that $u_1 = v_1$. Analogously we verify that $u_2 = v_2$.

Now consider the mapping

$$\psi : x \to (x \wedge a, x \wedge b)$$

of the lattice L into the direct product $[0(L), a] \times [0(L), b]$.

Lemma 9. *Let the assumptions as in Lemma 7 be fulfilled. Then the mapping* ψ *is an isomorphism of the lattice* L *onto* $[0(L), a] \times [0(L), b]$ *and* $\psi(a) = (a, 0(L))$, $\psi(b) = (0(L), b)$.

Proof. Let $x, y \in L$. The mapping ψ is monotone and by Lemma 7, $\psi(x) \leqq \psi(y)$ implies that $x \leqq y$. Let $a \geqq u \in L$, $b \geqq v \in L$, $x = u \vee v$. Under the denotations as above we have $u \leqq u_1$, $v \leqq u_2$, hence according to Lemma 8, $u = u_1$, $v = u_2$. Therefore the mapping ψ is onto and thus ψ is an isomorphism. By Lemma 6 we have $\psi(a) = (a, 0(L))$, $\psi(b) = (0(L), b)$.

Theorem 1. *Let* L *be a complete lattice and let* $\{a_i\} \neq \emptyset$ *be a subset of the center* $C(L)$ *of the lattice* L. *The following conditions are equivalent*:

(i) *The elements* $\bigwedge a_i$ *and* $\bigvee a'_i$ *belong to* $C(L)$.

(ii) *If* $x, y \in L$, $x \geqq y$, *then* (3) *and* (4) *are valid.*

Proof. By Lemma 4, (i) ⇒ (ii). From Lemma 9 it follows that (ii) ⇒ (i).

As an immediate consequence we obtain:

Theorem 2. *Let* L *be a complete lattice. Then the following conditions are equivalent*:

(i) *The center* $C(L)$ *is a closed sublattice of* L.

(ii) *If* $\emptyset \neq \{a_i\} \subset C(L)$, $x \in L$, $y \in L$ *and* $x \geqq y$, *then* (1) *and* (2) *are valid.*

Corollary. (Cf. [2].) *If* L *is an infinitely distributive complete lattice, then* $C(L)$ *is a closed sublattice of* L.

4. AN EXAMPLE

Now we describe an example showing that the center of a complete distributive lattice L need not be a closed sublattice of L.

Let L_0 be the lattice of all real functions defined on the interval $[0, 1] = X$ with the natural partial order. The lattice operations in L_0 are denoted $\bigwedge^*$, $\bigvee^*$. Let L be the subset of L_0 consisting of all functions f that satisfy the following conditions:

(i) If $0 \leqq x < 1$, then $f(x) \in \{0, 2\}$.

(ii) $f(1) \in \{0, 1, 2\}$.

(iii) $f(1) = 2$ if and only if the set $s(f) = \{x : 0 \leqq x < 1, f(x) = 2\}$ is infinite.

The set L is partially ordered by the induced order. The least and the greatest element of L will be denoted by f_0 and f_1, respectively. Let $f \in L_0$, $f(x) \in \{0, 1, 2\}$ for each $x \in X$. We define $f^-, f^+ \in L$ as follows. If $s(f)$ is finite and $f(1) = 2$, then we put $f^-(x) = f(x)$ for each $x \in X$, $x \neq 1$, and $f^-(1) = 1$; otherwise we put $f^- = f$. If $s(f)$ is infinite and $f(1) \neq 2$, we set $f^+(x) = f(x)$ for each $x \in X$, $x \neq 1$, and $f^+(1) = 2$; otherwise we put $f^+ = f$. If $f \in L$, then $f^- = f = f^+$.

Lemma 10. *The partially ordered set L is a complete lattice.*

Proof. Let $\emptyset \neq \{f_i\}$ $(i \in I) \subset L$. Denote $\bigwedge^* f_i = f$, $\bigvee^* f_i = g$. The functions f, g satisfy the conditions (i) and (ii). If $f \in L (g \in L)$, then clearly $f = \inf \{f_i\}$ $(g = \sup \{f_i\})$ in L.

Assume that $f \notin L$. Suppose that $s(f)$ is finite. Then $f(1) = 2$, $f_i > f^-$ for each $i \in I$ and $g_1 \leqq f^-$ whenever $g_1 \in L$, $g_1 \leqq f_i$ for each $i \in I$. Thus $f^- = \inf \{f_i\}$ in L. Assume that $s(f)$ is infinite. Then each $s(f_i)$ is infinite, whence $f_i(1) = 2$ for each $i \in I$ and therefore $f(1) = 2$. Thus $f \in L$, a contradiction.

Assume that $g \notin L$. If $s(g)$ is finite, then each $s(f_i)$ is finite, hence $f_i(1) < 2$ for each $i \in I$, thus $g(1) \leqq 1$ and so $g \in L$, a contradiction. Therefore $s(g)$ is infinite and from $g \notin L$ we obtain $g(1) < 2$. Then $g^+ \geqq f_i$ for each $i \in I$. Moreover $g_1 \in L$, $g_1 \geqq f_i$ for each $i \in I$ implies $g_1 \geqq g^+$. Thus $g^+ = \sup \{f_i\}$ in L. The proof is complete.

The lattice operations in L will be denoted by $\bigwedge, \bigvee$. We have shown that $\bigwedge f_i = (\bigwedge^* f_i)^-$, $\bigvee f_i = (\bigvee^* f_i)^+$ for any subset $\emptyset \neq \{f_i\} \subset L$. From this it follows that for each $f, g \in L$ and each $x \in X$, $x \neq 1$ we have

$$(5) \qquad (f \wedge g)(x) = f(x) \wedge g(x), \quad (f \vee g)(x) = f(x) \vee g(x).$$

From (5) we obtain that

$$(6) \qquad s(f \wedge g) = s(f) \cap s(g), \quad s(f \vee g) = s(f) \cup s(g).$$

Lemma 11. *The lattice L is distributive.*

Proof. Let $f, g, h \in L$ and denote

$$(f \wedge g) \vee h = F, \quad (f \vee h) \wedge (g \vee h) = G.$$

Obviously $G \geqq F$ and according to (5), $F(x) = G(x)$ for each $x \in X$, $x \neq 1$. Hence we have to verify that $F(1) = G(1)$.

According to (6) we have

$$s(G) = s(f \vee h) \cap s(g \vee h) = (s(f) \cup s(h)) \cap (s(g) \cup s(h)) =$$
$$= s(f) \cap s(g)) \cup s(h) = s(F),$$

hence either both $G(1)$, $F(1)$ are less than 2, or $G(1) = F(1) = 2$. Thus if $F(1) \geqq 1$, then $F \geqq G$. Assume that $F(1) = 0$. Then $h(1) = 0$, and either $f(1) = 0$ or $g(1) = 0$.

Therefore either $(f \vee h)(1) = 0$ or $(g \vee h)(1) = 0$. From this we get $G(1) = 0$. Hence $F(1) = G(1)$. The proof is complete.

For each $y \in X$, $y \neq 1$ we define the functions $f_y, \bar{f}_y \in L$ by the rule

$$f_y(y) = 2\,, \quad \bar{f}_y(y) = 0\,,$$
$$f_y(x) = 0\,, \quad \bar{f}_y(x) = 2 \quad \text{for each} \quad x \in X\,, \quad x \neq y\,.$$

Further let $g_0 \in L$ be such that $g_0(x) = 0$ for each $x \in X$, $x \neq 1$ and $g_0(1) = 1$. Then we have

$$f_y \wedge \bar{f}_y = f_0\,, \quad f_y \vee \bar{f}_y = f_1$$

for each $y \in X$, $y \neq 1$. Since L is distributive, each element of L is neutral. Therefore an element $f \in L$ belongs to the center of L if and only if f has a complement. Thus all elements f_y belong to the center of L. We have

$$\bigwedge \bar{f}_y = g_0\,. \tag{7}$$

Let $h \in L$, $h \wedge g_0 = f_0$. Then $h(1) = 0$, thus $s(h)$ is finite. Hence $f_1 \neq h \vee^* g_0 \in L$ and so $h \vee^* g_0 = h \vee g_0 \neq f_1$. Therefore the element g_0 has no complement in L. This implies that g_0 does not belong to $C(L)$. In view of (7), the center of L is not a closed sublattice of L.

On the other hand we have $\bigvee f_y = f_1 \in C(L)$. Thus if $\{a_i\}$ is a subset of the center of a complete lattice L and if $\bigvee a_i'$ belongs to $C(L)$, then $\bigwedge a_i$ need not belong to $C(L)$.

5. DIRECT FACTORS IN A CONDITIONALLY COMPLETE LATTICE

In this paragraph we assume that L is a conditionally complete lattice.

Let φ be an isomorphism of L onto a direct product $A \times B$, $x_0 \in L$, $\varphi(x_0) = (a_0, b_0)$. Put

$$A(x_0) = \{y \in L : \varphi(y) = (a, b_0),\ a \in A\}\,,$$
$$B(x_0) = \{y \in L : \varphi(y) = (a_0, b),\ b \in B\}\,.$$

For each $z \in L$ with $\varphi(z) = (a_1, b_1)$ let

$$z_1 = \varphi^{-1}((a_1, b_0))\,, \quad z_2 = \varphi^{-1}((a_0, b_1))\,.$$

We denote by $\varphi'[x_0]$ the mapping of L onto $A(x_0) \times B(x_0)$ defined by the rule

$$\varphi'[x_0](z) = (z_1, z_2)$$

for each $z \in L$. It is easy to verify that $\varphi'[x_0]$ is an isomorphism of L onto $A(x_0) \times B(x_0)$. If the element x_0 is fixed we write φ' instead of $\varphi'[x_0]$.

All lattices $A(x_0)$ constructed in this way will be called direct factors of L with respect to x_0 and the system of all direct factors of L with respect to x_0 will be denoted by $F(x_0)$. (Cf. [3], [4].) Each lattice $A \in \bigcup F(x_0)$ $(x_0 \in L)$ will be called a direct factor of L.

Let φ be as above, $u, v \in L$, $u \leqq v$, $\varphi(u) = (u_1, u_2)$, $\varphi(v) = (v_1, v_2)$, $c = \varphi^{-1}((v_1, u_2))$. Then c is said to be a relative central element of L with respect to the interval $[u, v]$. The set $C'([u, v])$ of all relative central elements with respect to $[u, v]$ will be called the relative center of L with respect to $[u, v]$. Let us consider the following condition on L:

(*) *For each $x_0 \in L$ and each set $\emptyset \neq \{A_i(x_0)\}$ of direct factors of L with respect to x_0 the intersection $\bigcap A_i(x_0)$ is a direct factor of L with respect to x_0.*

If $A(x_0)$ is a direct factor of L and $x_1 \in A(x_0)$, then $A(x_1) = A(x_0)$; therefore the condition (*) is equivalent with the condition:

(**) *Each nonempty intersection of direct factors of L is a direct factor of L.*

The following lemma shows the relation between the condition (*) and the properties of the center of L in the case when L has the greatest and the least element:

Lemma 12. *Let $0(L), 1(L) \in L$. Then L satisfies (*) if and only if the center $C(L)$ is a closed sublattice of L.*

At first we prove the following lemma:

Lemma 12.1. *Let φ be an isomorphism of L onto $A \times B$, $x, x_0 \in L$, $0(L), 1(L) \in L$, $a = \varphi^{-1}((1(A), 0(B)))$. Then $x \in A(x_0)$ if and only if $a \vee x = a \vee x_0$.*

Proof. Let $\varphi(x_0) = (a_0, b_0)$, $\varphi(x) = (a_1, b_1)$. We have

$$(a_1, b_1) \vee (1(A), \quad 0(B)) = (1(A), b_1),$$
$$(a_0, b_0) \vee (1(A), \quad 0(B)) = (1(A), b_0).$$

The element x belongs to $A(x_0)$ if and only if $b_1 = b_0$; since φ is an isomorphism, this is true if and only if $a \vee x = a \vee x_0$.

Proof of Lemma 12:

(a) Assume that $C(L)$ is a closed sublattice of L and let $x_0 \in L$, $\emptyset \neq \{A_i(x_0)\}$ $(i \in I) \subset F(x_0)$. For each $i \in I$ there exist lattices A_i, B_i and an isomorphism φ_i of L onto $A_i \times B_i$. Under the analogous denotations as above let φ'_i be the corresponding isomorphism of L onto $A_i(x_0) \times B_i(x_0)$. Since $1(L) \in L$, there exists a greatest element c_i in $A_i(x_0)$ and a least element d_i in $B_i(x_0)$. The element $a_i = (\varphi'_i)^{-1}((c_i, d_i))$ belongs to the center of L, hence

$$(\varphi'_i)^{-1}((c_i, d_i) \vee (x_0, x_0)) = (\varphi'_i)^{-1}((c_i, x_0)) = c_i$$

belongs to the center of the lattice $[x_0, 1(L)]$ (cf. Lemma 2). Put $c = \bigwedge c_i$, $a = \bigwedge a_i$.

Then, since $C(L)$ is a closed sublattice of L, $a \in C(L)$ and according to Thm. 2 we have

$$c = \bigwedge c_i = \bigwedge(x_0 \vee a_i) = x_0 \vee (\bigwedge a_i) = x_0 \vee a$$

and $c \in C([x_0, 1(L)])$ by Lemma 2. There exist lattices A, B and an isomorphism φ of L onto $A \times B$ such that $a = \varphi^{-1}((1(A), 0(B))$. Consider the direct factor $A(x_0) \in \in F(x_0)$. Let $z \in A(x_0)$. By Lemma 12.1 we have $z \vee a = x_0 \vee a$, thus for each a_i,

$$z \vee a_i = z \vee (a \vee a_i) = (z \vee a) \vee a_i = (x_0 \vee a) \vee a_i = x_0 \vee a_i ,$$

hence $z \in A_i(x_0)$. Conversely, let $z \in A_i(x_0)$. Then

$$z \vee a_i = x_0 \vee a_i \quad \text{for each} \quad i \in I .$$

Since the center of L is a closed sublattice, we have

$$z \vee a = z \vee (\bigwedge a_i) = \bigwedge(z \vee a_i) = \bigwedge(x_0 \vee a_i) = x_0 \vee (\bigwedge a_i) = x_0 \vee a ,$$

therefore $z \in A(x_0)$. Thus $\bigcap A_i(x_0) = A(x_0) \in F(x_0)$.

(b) Let $(*)$ be valid and let $\{c_i\}$ $(i \in I) \subset C(L)$. For each $i \in I$ there are lattices A_i, B_i and an isomorphism φ_i of L onto $A_i \times B_i$ such that $c_i = \varphi_i^{-1}((1(A_i), 0(B_i))$. Put $x_0 = 0(L)$. Then c_i is the greatest element of $A_i(x_0)$. According to the assumption, there exist lattices A, B and an isomorphism φ of L onto $A \times B$ such that $A(x_0) = \bigcap A_i(x_0)$. Thus $\bigcap A_i(x_0)$ has a greatest element c and $c \in C(L)$. Obviously $c = \bigwedge c_i$ $(i \in I)$, hence $\bigwedge c_i \in C(L)$. Further consider the lattices $B_i(y_0)$ for $y_0 = 1(L)$. The element c_i is the least element of $B_i(y_0)$. According to $(*)$, $\bigcap B_i(y_0)$ belongs to $F(y_0)$, therefore $\bigcap B_i(y_0)$ has a least element d and $d \in C(L)$. Clearly $d = \bigvee a_i$. The proof is complete.

Our purpose is to prove the following assertion:

Theorem 3. *Let L be a conditionally complete lattices. Then the following conditions are equivalent*:

(a) = $(*)$.

(b) *For each interval $[u, v] \subset L$, the relative center $C'([u, v])$ is a closed sublattice of L.*

(c) *If $x, y, u, v \in L$, $u \leqq y \leqq x \leqq v$ and $\{a_i\} \subset C'([u, v])$, then the relations* (1) *and* (2) *are valid.*

At first we introduce some auxiliary notions and prove some lemmas. Let us remark that for any $u, v \in L$, $C'([u, v])$ is a closed sublattice of L if and only if $C'([u, v])$ is a closed sublattice of $[u, v]$. Let L be a lattice, $x_0 \in L$. For each subset $\emptyset \neq X \subset L$ we denote by $X^\delta(x_0)$ the set of all $y \in L$ satisfying

$$(x \vee x_0) \wedge (y \vee x_0) = x_0 = (x \wedge x_0) \vee (y \wedge x_0) \tag{8}$$

for each $x \in X$. Let A, B be lattices and let φ be an isomorphism of L onto $A \times B$. Let $p, q \in L$. If $p \in A(q)$ we write $p \equiv q(R(A))$. Analogously we define the relation $p \equiv q(R(B))$. Then $R(A), R(B)$ are permutable congruence relations on L, $R(A) \wedge \wedge R(B)$ is the least congruence relation on L and $R(A) \vee R(B)$ is the greatest congruence relation on L. (Cf. [1].)

Lemma 13. *Let* $z \in L$, $\varphi'(z) = (z_1, z_2)$. *Then*

$$z_1 \in A(x_0) \cap B(z), \quad z_2 \in B(x_0) \cap A(z).$$

This is an immediate consequence of the definition of the sets $A(x)$ and $B(x)$ for $x \in L$.

Lemma 14. *Let* $x_0, z \in [u, v] \subset L$, $\varphi'(z) = (z_1, z_2)$. *Then* $z_1, z_2 \in [u, v]$.

Proof. According to Lemma 13 we have

$$x_0 \equiv z_1(R(A)), \quad z_1 \equiv z(R(B)),$$

and hence (because $R(A), R(B)$ are congruence relations on L)

$$x_0 \equiv (z_1 \vee u) \wedge v(R(A)), \quad (z_1 \vee u) \wedge v \equiv z(R(B)).$$

From this we infer that

$$z_1 \equiv (z_1 \vee u) \wedge v(R(A) \wedge R(B)).$$

Since $R(A) \wedge R(B)$ is the least congruence on L, we obtain $z_1 = (z_1 \vee u) \wedge v$. Thus $z_1 \in [u, v]$.

Lemma 15. $B(x_0) = (A(x_0))^\delta (x_0)$.

Proof. Let $y \in L$, $\varphi(y) = (a, b)$, $\varphi(x_0) = (a_0, b_0)$, $x \in A(x_0)$, $\varphi(x) = (a_1, b_0)$. If $y \in B(x_0)$, then $a = a_0$, thus

$$(\varphi(x) \vee \varphi(x_0)) \wedge (\varphi(y) \vee \varphi(x_0)) = (a_1 \vee a_0, b_0) \wedge (a_0, b \vee b_0) = (a_0, b_0),$$

therefore $(x \vee x_0) \wedge (y \vee y_0) = x_0$. Dually, $(x \wedge x_0) \vee (y \wedge y_0) = x_0$, hence $y \in (A(x_0))^\delta (x_0)$.

Let $x_0 \leqq y \in (A(x_0))^\delta (x_0)$. Then $(a, b_0) \in A(x_0)$, $(a, b_0) \wedge (a, b) = (a, b_0)$ and hence by the definition of the set $(A(x_0))^\delta (x_0)$ we obtain $(a, b_0) = (a_0, b_0)$, therefore $y \in B(x_0)$. Similarly, if $x_0 \geqq y \in (A(x_0))^\delta (x_0)$, then $y \in B(x_0)$. Now let y be any element of the set $(A(x_0))^\delta (x_0)$ and denote $y_1 = y \vee x_0$, $y_2 = y \wedge x_0$. Then y_1 and y_2 fulfil (8), hence $y_1, y_2 \in (A(x_0))^\delta (x_0)$, thus $y_1, y_2 \in B(x_0)$. Since $B(x_0)$ is a convex subset of L, we obtain $y \in B(x_0)$.

Lemma 16. *Let $x_0 \in [u, v] \subset [u_1, v_1] \subset L$. Assume that φ is an isomorphism of $[u, v]$ onto $A \times B$ and that φ_1 is an isomorphism of $[u_1, v_1]$ onto $A_1 \times B_1$ such that $A(x_0) = A_1(x_0) \cap [u, v]$. Then $B(x_0) = B_1(x_0) \cap [u, v]$.*

Proof. Let $y \in B_1(x_0) \cap [u, v]$, $x \in A(x_0)$. Then $x \in A_1(x_0)$ and hence according to Lemma 15 the relation (8) is valid. Thus $y \in B(x_0)$. Conversely, let $y \in B(x_0)$ and let $x \in A_1(x_0)$. Then since $A_1(x_0)$ is a convex sublattice of L we have $x_0 \leqq (x \vee x_0) \wedge \wedge v \in A(x_0)$ and therefore by Lemma 15

$$(x \vee x_0) \wedge (y \vee x_0) = (x \vee x_0) \wedge [v \wedge (y \vee x_0)] = \\ = [(x \vee x_0) \wedge v] \vee (y \vee x_0) = x_0 .$$

Dually we obtain $(x \wedge x_0) \vee (y \wedge x_0) = x_0$, thus by Lemma 15, $y \in B_1(x_0) \cap \cap [u, v]$.

Under the same assumptions as in Lemma 16 the following two lemmas are valid:

Lemma 17. *Let $z \in [u, v]$. Then*

$$A(z) = A_1(z) \cap [u, v] , \quad B(z) = B_1(z) \cap [u, v] .$$

Proof. According to Lemma 13 there exist $z_1 \in A(x_0)$, $z_2 \in B(x_0)$ such that

$$z_1 \in B(z) , \quad z_2 \in A(z) .$$

Thus $A(z_1) = A(z_0)$ and so according to the assumption we have $A(z_1) = A_1(z_1) \cap \cap [u, v]$. Hence by Lemma 16, $B(z_1) = B_1(z_1) \cap [u, v]$. Therefore $z \in B_1(z_1)$ and thus $B(z) = B_1(z) \cap [u, v]$. From this and from Lemma 16 we infer that $A(z) = = A_1(z) \cap [u, v]$.

Lemma 18. *Let $z \in [u, v]$. Then $\varphi'(z) = \varphi_1'(z)$.*

Proof. Let z_1, z_2 be as in the proof of Lemma 17. We have

$$z_1 \in A(x_0) \cap B(z) , \quad z_2 \in B(x_0) \cap A(z)$$

and hence according to Lemma 17,

$$z_1 \in A_1(x_0) \cap B_1(z) , \quad z_2 \in B_1(x_0) \cap A_1(z) .$$

Therefore from Lemma 13 and from the fact that $R(A_1) \wedge R(B_1)$ is the least congruence relation on $[u_1, v_1]$ we get $\varphi'(z) = (z_1, z_2) = \varphi_1'(z)$.

Lemma 19. *Let $L = [u, v]$ and let*

$$\varphi : L \rightarrow A \times B , \quad \varphi_i : L \rightarrow A_i \times B_i$$

be isomorphisms of L *onto* $A \times B$ *and* $A_i \times B_i$, *respectively* $(i \in I)$. *Denote* $a_i = \varphi_i^{-1}((1(A_i), 0(B_i))$ *and assume that*

(c_1) $t \vee (\bigwedge a_i) = \bigwedge(t \wedge a_i)$ *for each* $t \in L$

is valid. Let $x_0 \in L$, $\bigcap A_i(u) = A(u)$. *Then* $\bigcap A_i(x_0) = A(x_0)$.

Proof. There exists $a \in C([u, v])$ such that $a = 1(A(u))$. Clearly $a_i = 1(A_i(u))$. From $\bigcap A_i(u) = A(u)$ it follows $\bigwedge a_i = a$. Now by using (c_1) and by the same method as in the part (a) of the proof of Lemma 12 we obtain that $\bigcap A_i(x_0) = A(x_0)$.

Lemma 20. (c) $\Rightarrow$ (*) *for each conditionally complete lattice* L.

Proof. Assume that L satisfies (c) and let $\{A_i(x_0)\}$ $(i \in I)$ be a nonempty subset of $F(x_0)$ for some $x_0 \in L$. Let $z \in L$. Choose $u, v \in L$ such that $u \leqq v$, $[x_0 \wedge z, x_0 \vee z] \subset [u, v]$.

For each $i \in I$ there is a lattice B_i and an isomorphism φ_i of L onto $A_i \times B_i$. Let $\varphi_i(u) = (u_1^i, u_2^i)$, $\varphi_i(v) = (v_1^i, v_2^i)$ and let $\bar{\varphi}_i$ be the corresponding partial mapping of the interval $[u, v]$ into $A_i \times B_i$. Then $\bar{\varphi}_i$ is an isomorphism of $[u, v]$ onto

$$[u_1^i, v_1^i] \times [u_2^i, v_2^i] = \bar{A}_i \times \bar{B}_i .$$

Let $a_i = \bar{\varphi}_i^{-1}(v_1^i, u_2^i)$, $a_i' = \bar{\varphi}_i^{-1}(u_1^i, v_2^i)$. The elements a_i, a_i' belong to the relative center $C'([u, v]) \subset C([u, v])$ and a_i is the complement of a_i' in the interval $[u, v]$. According to the assumption the condition (c) is valid and thus by Thm. 1 the elements $a = \bigwedge a_i$, $b = \bigvee a_i'$ belong to the center of the lattice $[u, v]$. Hence there are lattices X and Y and an isomorphism $\bar{\varphi}$ of $[u, v]$ onto $X \leqq Y$ such that $\bar{\varphi}(a) = (1(X), 0(Y))$, $\bar{\varphi}(b) = (0(X), 1(Y))$. Clearly

$$X(u) = [u, a] , \quad \bar{A}_i(u) = [u, a_i]$$

and therefore

$$X(u) = \bigcap \bar{A}_i(u) \, (i \in I) .$$

Hence by Lemma 19 (the condition (c_1) of this lemma is valid because of (c)), we have

$$X(x_0) = \bigcap \bar{A}_i(x_0) \subset \bigcap A_i(x_0) .$$

Denote

$$A = \bigcap A_i(x_0) , \quad B = A^\delta(x_0) .$$

Let $x \in A$, $y \in Y(x_0)$ and denote $(x \vee x_0) \wedge v = z$. Then $z \in X(x_0)$ and hence according to Lemma 15,

$$z \wedge (y \vee x_0) = (z \vee x_0) \wedge (y \vee x_0) = x_0 .$$

Therefore

$$(x \vee x_0) \wedge (y \vee x_0) = (x \vee x_0) \wedge [v \wedge (y \vee x_0)] =$$
$$= [(x \vee x_0) \wedge v] \wedge (y \vee x_0) = z \wedge (y \vee x_0) = x_0$$

and dually we obtain

$$(x \wedge x_0) \vee (y \wedge x_0) = x_0 .$$

Thus $y \in A^\delta(x_0)$ and hence $Y(x_0) \subset A^\delta(x_0)$. Let $\varphi'(z) = (z_1, z_2)$. Then $z_1 \in A$, $z_2 \in B$. From Lemma 18 it follows that the elements z_1, z_2 do not depend from the particular choice of elements u, v. We write

$$z_1 = z[A], \quad z_2 = z[B].$$

If $t \in L$, we may choose $u, v \in L$ such that $\{x_0, z, t\} \subset [u, v]$ and then we obtain that

$$(z \wedge t)[A] = z[A] \wedge t[A], \quad (z \vee t)[A] = z[A] \vee t[A]$$

and analogously for B. Further $z \neq t$ implies $(z_1, z_2) \neq (t_1, t_2)$. Hence the mapping $\varphi : z \to (z_1, z_2)$ is an isomorphism of L into $A \times B$.

Let $p \in A$, $q \in B$ and choose $u, v \in L$ such that $\{x_0, p, q\} \subset [u, v]$. Then we have (by the same notations as above) $p \in X(x_0)$. From Lemma 15 it follows $q \in Y(x_0)$. Thus there is $z \in [u, v]$ such that $\bar{\varphi}'(z) = (p, q)$. Hence we obtain $p = z_1$, $q = z_2$. Therefore the mapping φ is onto. We have $\varphi(x_0) = (x_0, x_0)$ and if $z \in A (z \in B)$, then $\varphi(z) = (z, x_0)$ $(\varphi(z) = (x_0, z))$. Thus $A(x_0) = A$, $B(x_0) = B$. We have proved that $\bigcap A_i(x_0) = A$ belongs to $F(x_0)$.

Proof of Thm. 3.

(a) ⇒ (b). Let (a) be valid. Let $[u, v] \subset L$, $\emptyset \neq \{c_i\}$ $(i \in I) \subset C'([u, v])$. For each $i \in I$ there is an isomorphism φ_i of L onto $A_i \times B_i$ such that the condition from the definition of $C'([u, v])$ is fulfilled. Put $x_0 = u$. According to (a), there are lattices A, B and an isomorphism φ of L onto $A \times B$ such that $A(x_0) = \bigcap A_i(x_0)$. The lattice $X = [u, v]$ is isomorphic with the direct product $(X \cap A(x_0)) \times (X \cap B(x_0))$, and $X \cap A(x_0) = \bigcap(X \cap A_i(x_0))$. Then the lattice $X \cap A(x_0)$ has a greatest element c and $c \in C'([u, v])$. The element c_i is the greatest element of $X \cap A_i(x_0)$, hence $\bigwedge c_i = c$ and so $\bigwedge c_i \in C'([u, v])$. By a dual method we can prove that $\bigvee c_i \in C'([u, v])$.

(b) ⇒ (c). Assume that (b) holds. Let $x, y, u, v \in L$, $u \leqq y \leqq x \leqq v$, $\{a_i\} \subset$ $\subset C'([u, v])$. Let a'_i be the relative complement of a_i with respect to the interval $[u, v]$. Then $a'_i \in C'([u, v])$ and hence according to (b) we have $a = \bigwedge a_i \in C'([u, v])$, $b =$ $= \bigvee a'_i \in C'([u, v])$. Thus the elements a_i, a, b belong to $C([u, v])$ and therefore from Lemma 4 we infer that the relations (3) and (4) are valid. Thus (1) and (2) hold whenever the assumptions of (c) are fulfilled.

The implication (c) ⇒ (a) was proved in Lemma 20.

Corollary 1. *Let L be a complete lattice. Then the following conditions are equivalent:*

(a) *The center of L is a closed sublattice of L.*

(b) *Each relative center of L is a closed sublattice of L.*

Proof. Since the center of L is a relative center of L, (b) $\Rightarrow$ (a). From Lemma 12 and Thm. 3 it follows that (a) implies (b).

Corollary 2. *Let L be a conditionally complete lattice, $x_0 \in L$. If for each interval $[u, v]$ of L the center $C([u, v])$ is a closed sublattice of $[u, v]$, then for each set $\emptyset \neq \{A_i(x_0)\}$ of direct factors of L with respect to x_0 the intersection $\bigcap A_i(x_0)$ is a direct factor of L with respect to x_0.*

References

[1] *G. Birkhoff:* Lattice theory, Third edition, Amer. Math. Soc. Colloquium Publications Vol. *XXV*, Providence 1967.

[2] *J. Jakubík:* Center of infinitely distributive lattices (slovak), Matem. fyz. časopis *8* (1957), 116–120.

[3] *J. Jakubík:* Weak product decompositions of discrete lattices, Czechoslov. Math. J. *21* (*96*) (1971), 399–412.

[4] *J. Jakubík:* Weak product decompositions of partially ordered sets, Colloquium mathem. *25* (1972), 13–26.

Author's address: 040 00 Košice, Zbrojnícká 7, ČSSR (Vysoká škola technická).