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ON MINIMAL SEQUENCES OF TYPE l_+ AND BOUNDED BIORTHOGONAL SYSTEMS IN BANACH SPACES

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A sequence $\{x_n\}$ in a Banach space E is said to be *minimal* if there exists a (not necessarily unique) sequence of functionals $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$ $(i,j = 1, 2, \ldots)$; any such pair of sequences (x_n, f_n) is called a *biorthogonal system*. A minimal sequence is called [5] *(E, E^*)-bounded* if there exists a sequence $\{f_n\} \subset E^*$ such that (x_n, f_n) is a biorthogonal system which is *bounded* in the sense of V. PTÁK [3], i.e., $\sup_n \|x_n\| < \infty$ and $\sup_n \|f_n\| < \infty$. A minimal sequence $\{x_n\}$ is said to be *of type l_+*, if it is (E, E^*)-bounded and if there exists a constant $\eta > 0$ such that we have, for every finite sequence $\alpha_1, \ldots, \alpha_n \geqq 0$,

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geqq \eta \sum_{i=1}^{n} \alpha_i . \tag{1}$$

This notion was introduced by V. Pták [3] (for the term "of type l_+" see [4], [5]), who has shown in [3] that minimal sequences of type l_+ exist both in every non-reflexive Banach space and in the Hilbert space $L^2([0, 1])$. In [5] the question was raised ([5], page 166, problem 3.9) whether a minimal sequence $\{x_n\}$ of type l_+ exists in *every* Banach space. In the present Note we shall prove that the answer to this problem is affirmative and that for a wide class of *separable* Banach spaces (including all spaces having a finite dimensional decomposition) $\{x_n\}$ can be chosen to be also *complete* in E, i.e. such that the closed linear span $[x_n]$ of $\{x_n\}$ coincides with E. The problem whether every separable Banach space E has a complete minimal sequence $\{x_n\}$ of type l_+ remains still open and it turns out to be equivalent to the problem whether every separable Banach space E has an (E, E^*)-bounded complete minimal sequence ([5], p. 169, problem 3.10).

Theorem 1. *For a Banach space E the following two statements are equivalent:*

1°. *E has a complete minimal sequence $\{x_n\}$ of type l_+.*

2°. *E has an (E, E^*)-bounded complete minimal sequence $\{y_n\}$.*

Proof. The implication $1^\circ \Rightarrow 2^\circ$ is obvious since by definition every minimal sequence of type l_+ is (E, E^*)-bounded.

Conversely, assume that (y_n, g_n) $(\{y_n\} \subset E, \{g_n\} \subset E^*)$ is a bounded biorthogonal system such that $[y_n] = E$. Put

(2) $$x_n = y_1 + y_{n+1} \quad (n = 1, 2, \ldots),$$

(3) $$f_n = g_{n+1} \quad (n = 1, 2, \ldots).$$

Then (x_n, f_n) is a bounded biorthogonal system and for every finite sequence $\alpha_1, \ldots, \alpha_n \geqq 0$ we have

(4) $$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geqq \left| \frac{g_1}{\|g_1\|} \left(\sum_{i=1}^{n} \alpha_i x_i \right) \right| = \frac{1}{\|g_1\|} \left| \sum_{i=1}^{n} \alpha_i g_1(x_i) \right| = \frac{1}{\|g_1\|} \sum_{i=1}^{n} \alpha_i ,$$

i.e. (1) with $\eta = 1/\|g_1\|$. Therefore, if $[x_n] = E$, then $\{x_n\}$ is a complete minimal sequence of type l_+. If $[x_n] \neq E$, then there exists a $g \in E^*$ such that $g \neq 0$, $g(x_n) = 0$ $(n = 1, 2, \ldots)$, whence $g(y_1) = -g(y_2) = -g(y_3) = \ldots$ and thus, since $[y_n] = E$ and $g \neq 0$, it follows that $g(y_1) \neq 0$. Put

(5) $$x_0 = y_1 , \quad f_0 = \frac{1}{g(y_1)} g .$$

Then $[x_n]_0^\infty = E$ and $(x_n, f_n)_0^\infty$ is a bounded biorthogonal system such that for every finite sequence $\alpha_0, \alpha_1, \ldots, \alpha_n \geqq 0$ we have (4) with $\sum_{i=1}^{n}$ replaced by $\sum_{i=0}^{n}$. Therefore $\{x_n\}_0^\infty$ is a complete minimal sequence of type l_+, which completes the proof of theorem 1.

Remark 1. The problem, whether every separable Banach space has property 2° ([5], p. 169, problem 3.10), is apparently slightly "easier" then the unsolved problem of S. Banach [1], whether in every separable Banach space E there exists an (E, E^*)-bounded *M-basis* $\{y_n\}$, that is, a bounded biorthogonal system (y_n, g_n) such that $[y_n] = E$ and that $\{g_n\}$ is total on E (i.e., $\{x \in E \mid g_n(x) = 0 \ (n = 1, 2, \ldots)\} = \{0\}$).

We shall give now a class of separable Banach spaces having property 2° (and hence 1°) of theorem 1. We recall that a sequence of finite-dimensional subspaces $\{E_n\}$ of a Banach space E is called a *finite dimensional decomposition* (f.d.d.) of E if for every $x \in E$ there exists a unique sequence $\{z_n\} \subset E$ with $z_n \in E_n$ $(n = 1, 2, \ldots)$ such that $x = \sum_{i=1}^{\infty} z_i$. It is well known that in this case for each n the operator $P_n(x) = z_n$ $(x = \sum_{i=1}^{\infty} z_i \in E)$ is a bounded linear projection, called "the natural projection" of E onto E_n, and that $\sup_n \|P_n\| < \infty$. Let us also recall that a sequence $\{z_n\} \subset E$

is called a *basis* of E if the one-dimensional subspaces $E = \{\alpha z_n \mid \alpha \text{ scalar}\}$ constitute an f.d.d. of E, i.e., if for every $x \in E$ there exists a unique sequence of scalars $\{\alpha_n\}$ such that $x = \sum_{i=1}^{\infty} \alpha_i z_i$. In this case $\{z_n\}$ is a complete minimal sequence in E, namely, for $h_n(x) = \alpha_n$ $(x = \sum_{i=1}^{\infty} \alpha_i z_i \in E)$ we have $h_n \in E^*$ $(n = 1, 2, \ldots)$ and $h_i(x_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$. Moreover, it is also known that if $\inf_n \|z_n\| > 0$ then $\sup_n \|h_n\| < \infty$.

Lemma 1. *Every Banach space E with an f.d.d. $\{E_n\}$ has an (E, E^*)-bounded M-basis $\{y_n\}$.*

Proof. Since $\dim E_n < \infty$ $(n = 1, 2, \ldots)$, for each n there exists (see e.g. [1]) a biorthogonal system $(y_i, \varphi_i)_{i=m_{n-1}+1}^{m_n}$ $(\{y_i\}_{i=m_{n-1}+1}^{m_n} \subset E_n,\ \{\varphi_i\}_{i=m_{n-1}+1}^{m_n} \subset E_n^*)$ such that

$$(6) \qquad \|y_i\| = \|\varphi_i\| = 1 \quad (i = m_{n-1} + 1, \ldots, m_n;\ n = 1, 2, \ldots;\ m_0 = 0).$$

For each $i = m_{n-1} + 1, \ldots, m_n$ $(n = 1, 2, \ldots)$ put

$$(7) \qquad g_i(x) = \begin{cases} \varphi_i(x) & \text{for } x \in E_n \\ 0 & \text{for } x \in \bigcup_{j \neq n} E_j \end{cases}$$

and extend g_i by linearity to the (dense) linear subspace of E spanned by $\bigcup_{j=1}^{\infty} E_j$; this is possible, since $E_n \cap \bigcup_{j \neq n} E_j = \{0\}$. Then for every finite sum $x = \sum_{k=1}^{p} z_k \in E$ with $z_k \in E_k$ $(k = 1, \ldots, p)$ we have

$$|g_i(x)| = |g_i(\sum_{k=1}^{p} z_k)| = \begin{cases} |\varphi_i(z_n)| & \text{for } i = m_{n-1} + 1, \ldots, m_n;\ n = 1, \ldots, p \\ 0 & \text{for } i = m_{n-1} + 1, \ldots, m_n;\ n = p + 1, p + 2, \ldots \end{cases}$$

whence, by (6), we obtain for $i = m_{n-1} + 1, \ldots, m_n$ and $n = 1, \ldots, p$

$$|g_i(x)| \leqq \|\varphi_i\| \|z_n\| = \|z_n\| = \|P_n(x)\| \leqq \sup_j \|P_j\| \|x\|$$

where P_n is the natural projection of E onto E_n $(n = 1, \ldots, p)$. Since the set of all finite sums $\sum_{k=1}^{p} z_k$ with $z_k \in E_k$ $(k = 1, \ldots, p)$ is dense in E, it follows that $\{g_n\} \subset E^*$ and that $\sup_n \|g_n\| < \infty$. Furthermore, obviously $[y_n] = E$ and $g_i(y_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$. Finally, $\{g_n\}$ is total on E, because $x = \sum_{k=1}^{\infty} z_k \in E$ and $g_n(x) = 0$ $(n = 1, 2, \ldots)$ imply, by (7), $\varphi_i(z_n) = 0$ $(i = m_{n-1} + 1, \ldots, m_n;\ n = 1, 2, \ldots)$, whence $z_n = 0$ $(n = 1, 2, \ldots)$ and $x = 0$.

From theorem 1 and lemma 1 it follows

Corollary 1. *Every Banach space E with an f.d.d. (in particular, every Banach space E with a basis) has a complete minimal sequence* $\{x_n\}$ *of type* l_+.

Dropping the assumption that E is separable (and hence the requirement that $\{x_n\}$ be complete in E), we have

Theorem 2. *Every Banach space E contains a minimal sequence* $\{x_n\}$ *of type* l_+.

Proof. It is well known (see e.g. [1]) that every Banach space has a "basic sequence" $\{z_n\}$ (i.e., a sequence $\{z_n\}$ which is a basis of $[z_n]$). Then $\{y_n\} = \{z_n/\|z_n\|\}$ is a basis of $[y_n] = [z_n]$ with $\|y_n\| = 1$ $(n = 1, 2, \ldots)$ and hence for $\{g_n\} \subset [y_n]^*$ with $g_i(y_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$ we have $\sup_n \|g_n\| < \infty$. Hence, by theorem 1, the subspace $[y_n]$ of E has a complete minimal sequence $\{x_n\}$ of type l_+. Since $\{x_n\}$ is obviously a minimal sequence of type l_+ in E, the proof of theorem 2 is complete.

Remark 2. Note that in a direct proof of theorem 2 the case $[x_n] \neq E$ of the proof of theorem 1 can be omitted.

Remark 3. Such a result is no longer true if we require, in addition, that $\{f_n \mid_{[x_j]}\}$ be total on $[x_j]$. Indeed, V. Pták has observed [3] that if a Banach space E has a minimal sequence $\{x_n\}$ of type l_+ with this additional property, then E is non-reflexive. The converse of this latter statement is also true, since every non-reflexive Banach space E has ([4], [2]) even a basic sequence $\{x_n\}$ of type l_+.

Added in proof. Our problem mentioned before theorem 1 (and in remark 1) has been solved in the affirmative by W. J. Davis and W. B. Johnson (to appear in Studia Math.).

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