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Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 4, 612–640

Persistent URL: http://dml.cz/dmlcz/101130

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

UNIVERSAL APPROXIMATION BY HILL FUNCTIONS*)

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(Received September 30, 1971)

1. INTRODUCTION

Several results closely related to recent work of BABUŠKA [1], [2], and STRANG and FIX [6], [12], [13] are given in this paper. (Its preliminary version appeared as Tech. Note BN-619, Inst. for Fluid Dynamics and Applied Mathematics, University of Maryland in June 1970.)

Basic concepts of the theory of generalized functions are introduced and their principal properties are surveyed in Sec. 2. In addition, the main results of [1] are given in this section. They indicate that, in approximation by hill functions in one dimension, the quality of approximation is determined by the properties of the hill and the smoothness of the function approximated. The properties of the hill are particularly dependent on the length of the support of the hill. In conclusion, the sequence ω_n of hill functions is introduced in this section according to [11].

In Sec. 3, the completeness of this system ω_n of hill functions is proven in $W_k^2(a, b)$ where k is a non-negative integer. Further a function of exponential type with an infinite support is shown to be a limit of the hill functions ω_n as the length of the support increases. Sufficient conditions for a sequence of hill functions given by a convolution formula to have a function of exponential type as its limit are given.

A theorem concerned with the approximation by a function $\omega \in S$ is proven in Sec. 4. This approximation is shown to be universal, i.e., the function ω universally gives the best possible approximation limited only by the smoothness of the function approximated. In this approximation, a certain function $\eta(h)$ making the support of ω "wider" as $h \to 0$ is employed. A proper choice of such a function $\eta(h)$ is shown for a class of functions $\omega \in S$. Finally a possibility of the approximation by a function $\omega \in S$ not having a compact support is studied.

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*) This research was supported in part by the Atomic Energy Commission under Contract No. AEC AT(40-1) 3443/3 with the Institute for Fluid Dynamics and Applied Mathematics of the University of Maryland.

A generalization of most results can be made to n-dimensional case.

In conclusion, a simple numerical example illustrating the statements of Sec. 4 is given in Sec. 5.

2. DEFINITIONS. APPROXIMATION BY HILL FUNCTIONS

Let us confine ourselves to the one-dimensional case (for general definitions and statements, see [1], [7], [14]). The generalization of most results of the following sections to the n-dimensional case can be readily obtained.

Apart from basic definitions and notations, this section contains several results of the theory of generalized functions. The proofs may be found in [7], [14]. In addition, the principal results of [1] are given in this section also without proofs.

Definition 2.1. Let R be a one-dimensional Euclidean space. Let us denote the set of complex-valued continuous functions defined in R with derivatives of all orders continuous in R by $C^\infty(R)$. Let us denote by $S(R)$ the set of all rapidly decreasing (at ∞) functions (i.e., the functions $\varphi \in C^\infty(R)$ satisfying the condition

$$\sup_{x \in R} |x^k \varphi^{(l)}(x)| < \infty$$

for all non-negative integers k, l) with the usual topology (see [14]). Let $S'(R)$ be the space of generalized functions over $S(R)$. We will write simply C^∞, S, S' etc. instead of $C^\infty(R)$, $S(R)$, $S'(R)$ wherever it will not be ambiguous.

Definition 2.2. Let $w(x) = ax + b$ be a non-singular linear mapping of R on R with a, b real, let $f \in S'$. Let us denote by $f(ax + b)$ a function from S' satisfying the relation

$$(f(ax + b), \varphi(x)) = (f(w(x)), \varphi(x)) = |a|^{-1} (f(x), \varphi(w^{-1}(x)))$$

for any $\varphi \in S$. For the sake of brevity we will sometimes use the notation

$$f^{[a]}(x) = f(ax)$$

with $b = 0$.

Definition 2.3. Let $f \in S'$, $\psi \in S$. A function $g \in S'$ is said to be the product of the functions ψ and f, and denoted by $g = \psi f$ if

$$(g, \varphi) = (\psi f, \varphi) = (f, \bar{\psi} \varphi)$$

holds for any $\varphi \in S$.

Definition 2.4. Let us denote the Fourier transform of a function $\varphi \in S$ by

$$F(\varphi)(x) = \tilde{\varphi}(x) = \int_{-\infty}^{\infty} e^{itx} \varphi(t) \, dt .$$

Let us introduce $F(f) = \tilde{f}$ for $f \in S'$ by the equality

$$(F(f), F(\varphi)) = 2\pi(f, \varphi)$$

valid for any $\varphi \in S$.

Remark 2.1. The Fourier transform F is a linear continuous mapping of S on S and of S' on S'. The inverse Fourier transform $F^{-1}(\varphi)$ of the function $\varphi \in S$ is given by the formula

$$F^{-1}(\varphi)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t) \, dt .$$

The inverse Fourier transform $F^{-1}(f)$ of the function $f \in S'$ is defined by the equality

$$(F^{-1}(f), \varphi) = (2\pi)^{-1} (f, F(\varphi))$$

valid for any $\varphi \in S$.

Remark 2.2. Let $f \in S'$ and $a \neq 0$ be real. Then

(2.1) $$F(|a| f^{[a]}) = (F(f))^{[a^{-1}]} .$$

Definition 2.5. Let $\varphi, \psi \in S$. A function $\vartheta \in S$ is said to be the convolution of the functions φ and ψ, and denoted by $\vartheta = \varphi * \psi = \psi * \varphi$ if

$$\vartheta(x) = \int_{-\infty}^{\infty} \varphi(t) \psi(x - t) \, dt = \int_{-\infty}^{\infty} \varphi(x - t) \psi(t) \, dt .$$

Let $f \in S'$, $\varphi \in S$. A function g is said to be the convolution of the functions f and φ, and denoted by $g = f * \varphi$ if

$$\bar{g}(x) = \overline{(f * \varphi)}(x) = (f(t), \bar{\varphi}(x - t)) .$$

Remark 2.3. Let $\varphi \in S$, $f \in S'$. Defining $g = f * \varphi$, we obtain $g \in C^{\infty}$. Moreover,

(2.2) $$F(g) = F(f * \varphi) = F(\varphi) F(f) .$$

Definition 2.6. Let $f \in S'$. A closed set $G = \operatorname{supp} f$ is said to be the support of the function f if $(f, \varphi) = 0$ for all $\varphi \in S$ such that $\varphi(x) = 0$ in some neighborhood of G. (A support in the sense of this definition need not mean the minimal support.)

Definition 2.7. A function $g \in C^\infty$ is said to be a multiplier if $\varphi g \in S$ for any $\varphi \in S$ and $\varphi_n g \to 0$ when $\varphi_n \in S$, $\varphi_n \to 0$ (as $n \to \infty$) with the convergence in the topology of S.

Let g be a multiplier, $f \in S'$. A function $h \in S'$ is said to be the product of the functions g and f, and denoted by $h = gf$ if $(h, \varphi) = (gf, \varphi) = (f, g\varphi)$ holds for any $\varphi \in S$.

A function $f \in S'$ is said to be a convolutor if $f * \varphi = \psi \in S$ for any $\varphi \in S$ and $f * \varphi_n \to 0$ when $\varphi_n \in S$, $\varphi_n \to 0$ (as $n \to \infty$) with the convergence in the topology of S.

Remark 2.4. Let g be a multiplier. Then $f = F^{-1}(g) \in S'$ is a convolutor.

Remark 2.5. Let $f \in S'$ have a compact support. Then $F(f)$ is a multiplier.

Remark 2.6. Let $f \in S'$, $\varphi \in S$. Then

$$\operatorname{supp}(f * \varphi) = \mathsf{E}[w \in R,\ w = x + y,\ x \in \operatorname{supp} f,\ y \in \operatorname{supp} \varphi] .$$

The following theorem is a special case of Theorem 2.1 of [1].

Theorem 2.1. *Let $g \in S'$ have a compact support. Let a complex-valued function $c(k)$ be defined for all integers k, and let there exist constants $0 < C < \infty$ and $0 \leqq \gamma < \infty$ such that $|c(k)| \leqq C|k|^\gamma$. Let us write*

$$f(x) = \sum_{k=-\infty}^{\infty} c(k)\, g(x - ak)$$

for $a \neq 0$ real. Then $f \in S'$ and

$$F(f)(t) = F(g)(t) \sum_{k=-\infty}^{\infty} c(k)\, e^{iakt}$$

where $F(f) \in S'$ and $F(g)$ is a multiplier. The convergence of both the series is considered in the weak topology of S'.

Proof is given in [1].

Remark 2.7. Supposing $g \in S$ (instead of $g \in S'$ with a compact support), we obtain also the statement of Theorem 2.1. The modification of the proof is obvious.

Definition 2.8. Let us denote by $W_2^\alpha(R)$, $\alpha \geqq 0$ the set of all functions $f \in S'$ such that

$$|F(f)|^2 (1 + |x|^{2\alpha}) \in L_1(R) .$$

Let us put

$$\|f\|^2_{W_2{}^\alpha(R)} = (2\pi)^{-1} \| |F(f)|^2 (1 + |x|^{2\alpha})\|_{L_1} . \tag{2.3}$$

The normed linear space $W_2^\alpha(R)$ with the norm (2.3) is said to be a fractional Sobolev space.

Remark 2.8. Apparently $W_2^\alpha(R) \supset W_2^\beta(R)$ for $0 \leqq \alpha \leqq \beta$, and $W_2^0(R) = L_2(R)$.

The following theorem is a special case of Theorem 4.1, the basic approximation theorem of [1].

Theorem 2.2. *Let $0 \leqq \alpha' \leqq \beta$ be real numbers. Let ${}^j\omega \in S'; j = 1, \ldots, r$ be functions with compact supports. Let χ_j; $j = 1, \ldots, r$ be (complex-valued) trigonometric polynomials such that the function*

$$\Lambda = \sum_{j=1}^{r} \lambda_j \chi_j$$

where $\lambda_j = F({}^j\omega)$ has the following properties:

1.

(2.4) $$\Lambda(0) \neq 0 .$$

2. *There exists a function $z(k)$ such that*

(2.5) $$|\Lambda(x - 2\pi k)| \leqq z(k)\, |x|^t$$

for some $t \geqq 0$, all x such that

(2.6) $$|x| < \pi ,$$

and all integers $k \neq 0$, and

3.

(2.7) $$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} z^2(k)\, |k|^{2\alpha'} < \infty .$$

Then there exists an operator A_h,

$$A_h(f)(x) = \sum_{j=1}^{r} \sum_{k=-\infty}^{\infty} c_j(h, f, k)\, {}^j\omega(xh^{-1} - k)$$

mapping W_2^β into W_2^α for any $0 \leqq \alpha \leqq \alpha'$. Moreover,

(2.8) $$\|f - A_h(f)\|_{W_2^\alpha} \leqq Ch^\mu \|f\|_{W_2^\beta}$$

where

(2.9) $$\mu = \min(t - \alpha,\ \beta - \alpha)$$

and $0 < C < \infty$ is a constant independent of h.

If the support T of f is compact then there exists a constant $0 < L < \infty$ independent of h such that $A_h(f)$ has a compact support T' where T' is an Lh-neighborhood of T.

Proof is given in [1].

Remark 2.9. Further analysis in [1] shows that the conditions (2.4) to (2.7) are not only sufficient but also necessary for the estimate (2.8), (2.9). Condition (2.5) is of particular importance. It says that the function Λ has zeros of multiplicity t' at all the points $2\pi k$ where $k \neq 0$ is an integer and t' is the minimal integer not less than t.

Remark 2.10. In the one-dimensional case with $r = 1$, it is shown in [1] that the support of the function $\omega = {}^1\omega$ cannot be less than the interval $\langle -\frac{1}{2}t', \frac{1}{2}t' \rangle$ in order to satisfy (2.5). A sequence of hill functions ω_n given by the relations

$$\omega_1(x) = 1\,, \quad |x| \leqq \tfrac{1}{2}\,, \qquad = 0\,, \quad |x| > \tfrac{1}{2}\,, \tag{2.10}$$

$$\omega_n = \omega_{n-1} * \omega_1 \tag{2.11}$$

is shown to have the desired approximation properties with $t = t' = n$ (see [1], [11]). These functions are piecewise polynomial ones of degree $n - 1$, they are continuous for $n > 1$, and

$$F(\omega_n)(x) = (\sin \tfrac{1}{2}x)^n (\tfrac{1}{2}x)^{-n}\,. \tag{2.12}$$

3. SOME PROPERTIES OF THE HILL FUNCTIONS ω_n

In this section we will closer study the properties of the hill functions ω_n given by (2.10), (2.11).

Fixing the parameter h in Theorem 2.2, we may be interested in the possibility of approximation by hill functions as n increases. Let us put $h = 1$ (the modification for a general h is obvious) and assume a finite closed interval $\langle a, b \rangle \subset R$. Then the system of functions $\omega_n(x - k)$ is complete on this interval in $W_2^l(a, b)$ norm for any non-negative integer l.

Theorem 3.1. *Let $l \geqq 0$ be an integer, $\langle a, b \rangle \subset R$ be a finite interval. For any $\varepsilon > 0$ and any $f \in W_2^l(a, b)$, there exist an integer $n > 0$ and coefficients $c(f, k, n)$ such that*

$$\Big\| f(x) - \sum_{k=-\infty}^{\infty} c(f, k, n)\, \omega_n(x - k) \Big\|_{W_2{}^l(a,b)} \leqq \varepsilon\,.$$

Proof. The existence of a polynomial p_{n-1} of a degree $n - 1$ such that

$$\|f - p_{n-1}\|_{W_2{}^l(a,b)} \leqq \varepsilon$$

follows from the properties of the space $W_2^l(a, b)$. The set $S(a, b)$ of the functions infinitely continuously differentiable in (a, b) is dense in $W_2^l(a, b)$. In turn, polynomials are dense in the set $S(a, b)$ in the $L_2(a, b)$ norm. We approximate the function $f \in W_2^l(a, b)$ by a function $\varphi \in S(a, b)$ and the function $\varphi^{(l)}$ (belonging also to $S(a, b)$) by a polynomial ${}^l r$. Constructing successively the primitive functions ${}^i r$ such that ${}^{i-1}r'(x) = {}^i r(x)$, we find that they approximate the functions $\varphi^{(i)}$ in the $L_2(a, b)$ norm and the quality of approximation is affected only by powers of $b - a$. Finally we may put $p_{n-1} = {}^0 r$.

Let us now show that any polynomial of degree $n - 1$ can be expressed in the basis of the piecewise polynomial functions $\omega_n(x - k)$ on the interval $\langle a, b \rangle$. Let us write

$$\psi_{ns}(x) = \sum_{k=-\infty}^{\infty} k^s \omega_n(x - k) ; \quad s = 0, 1, \ldots, n - 1 .$$

Then $\psi_{ns} \in S'$ is a continuous piecewise polynomial function of degree at most $n - 1$ (cf. Remark 2.10). From Theorem 2.1 we obtain $\tilde{\psi}_{ns} \in S'$ where

$$\tilde{\psi}_{ns}(t) = \tilde{\omega}_n(t) \sum_{k=-\infty}^{\infty} k^s e^{ikt} \tag{3.1}$$

and $\tilde{\omega}_n$ is a multiplier. Using (2.12) we find $\tilde{\omega}_n^{(j)}(2\pi k) = 0$; $k \neq 0$ integer; $j = 0, 1, \ldots \ldots, n - 1$. Differentiating the relation

$$\sum_{k=-\infty}^{\infty} e^{ikx} = 2\pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k)$$

where δ is the Dirac function, and considering the fact that $\widetilde{x^j} = 2\pi(-i)^j \delta^{(j)}$ we finally obtain

$$(\tilde{\psi}_{ns}, \varphi) = (\sum_{j=0}^{s} \binom{s}{j} i^{s-j} \tilde{\omega}_n^{(s-j)}(0) \widetilde{x^j}, \varphi)$$

for any $\varphi \in S$, i.e.,

$$\tilde{\psi}_{ns} = \sum_{j=0}^{s} \binom{s}{j} i^{s-j} \tilde{\omega}_n^{(s-j)}(0) \widetilde{x^j}$$

and

$$\psi_{ns}(x) = \sum_{j=0}^{s} \binom{s}{j} i^{s-j} \tilde{\omega}_n^{(s-j)}(0) x^j \tag{3.2}$$

in the sense of generalized functions. Since both $\psi_{ns}(x)$ and x^j are continuous the equality (3.2) holds identically in R.

The functions x^j may be expressed in terms of $\psi_{ns}(x)$ from the system (3.2) since the matrix of the system is triangular and all the diagonal elements are equal to $\omega_n(0) = 1 \neq 0$. Thus this procedure yields

$$x^j = \sum_{s=0}^{j} d_{js}\, \psi_{ns}(x)\,; \quad j = 0, 1, \ldots, n - 1$$

with the coefficients d_{js} uniquely determined. Eventually we can express any polynomial of degree $n - 1$ as a linear combination of the functions ψ_{ns}; $s = 0, 1, \ldots$ $\ldots, n - 1$. For $x \in \langle a, b \rangle$, any $\psi_{ns}(x)$ is a finite linear combination of the functions $\omega_n(x - k)$. Then we can in turn express any polynomial of degree $n - 1$ as a finite linear combination of the functions $\omega_n(x - k)$, which completes the proof.

Remark 3.1. We have expressed any polynomial of a given degree in the finite interval $\langle a, b \rangle$ as a finite linear combination of the functions $\omega_n(x - k)$ for sufficiently large n in the course of the proof of Theorem 3.1. Therefore the statement of the theorem holds apparently in any normed linear space B of the functions defined on $\langle a, b \rangle$ such that polynomials are dense in B and $\omega_n \in B$ for all n greater than some $N > 0$.

We will study the asymptotic behavior of the hill functions ω_n as $n \to \infty$ in the remaining part of this section. We will use certain results of the probability theory to this end.

The convolution formula (2.11) is analogous to the convolution formula of the probability theory describing the probability density of the sum of n independent random variables with the same density. It leads to the conclusion that the asymptotic behavior of ω_n for n increasing can be characterized by the central limit theorem.

Theorem 3.2. *Let ω_n be given by* (2.10), (2.11). *Then*

$$\lim_{n \to \infty} \omega_n(x \sqrt{n}) \sqrt{n} = \sqrt{(6/\pi)}\, e^{-6x^2}$$

uniformly with respect to all $x \in R$. Moreover,

$$\left|\omega_n(x \sqrt{n}) \sqrt{n} - \sqrt{(6/\pi)}\, e^{-6x^2}\right| \leqq C n^{-1}$$

uniformly with respect to all $x \in R$ where C is a finite positive constant independent of n.

Proof. The statements follow immediately from the corresponding statements of [5] (Ch. XV, Sec. 5, Theorem 2, and Ch. XVI, Sec. 2, Theorem 2).

The central limit theorem holds even under assumptions weaker than in Theorem 3.2. Let us mention a version of this theorem using the so-called Lindeberg condition as a sufficient one.

Theorem 3.3. *Let us have a sequence $\{\varphi_n\}$ of non-negative integrable functions such that the functions ϑ_n are defined by the convolution formula*

$$\vartheta_1 = \varphi_1 \,, \tag{3.3}$$

$$\vartheta_n = \vartheta_{n-1} * \varphi_n \,. \tag{3.4}$$

Let

$$s_n^2 = \sum_{k=1}^{n} |\tilde{\varphi}_k''(0)| < \infty$$

for any positive integer n and

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \int_{ts_n < |x|} x^2 \varphi_k(x) \, dx = 0$$

for each fixed $t > 0$. Then there exist finite positive constants Y, Z such that

$$\lim_{n\to\infty} s_n \vartheta_n(xs_n) = Ye^{-Zx^2}$$

uniformly with respect to all $x \in R$.

Proof. The theorem is a special case of the Lindeberg theorem (see [5], Ch. XV, Sec. 6, Theorem 1).

Using Theorem 3.3 we may examine the asymptotic behavior of more general one-dimensional hills mentioned in [6].

Theorem 3.4. *Let $\varphi_1 \in S'$ be a non-negative integrable function with a compact support and let*

$$|\tilde{\varphi}_1''(0)| = \tfrac{1}{12}\Phi < \infty \,.$$

Let us put

$$\begin{aligned} \varphi_n(x) &= 1 \,, \quad |x| \leqq \tfrac{1}{2} \,, \\ &= 0 \,, \quad |x| > \tfrac{1}{2} \quad \textit{for} \quad n = 2, 3, \ldots . \end{aligned} \tag{3.5}$$

Let the sequence $\{\vartheta_n\}$ of hill functions be given by the formulae (3.3), (3.4) with φ_n from (3.5). Then there exist finite positive constants Y, Z such that

$$\lim_{n\to\infty} \sqrt{(\Phi + n - 1)}\, \vartheta_n(x \sqrt{(\Phi + n - 1)}) = Ye^{-Zx^2}$$

holds uniformly with respect to all $x \in R$.

Proof. It is sufficient to verify the assumptions of the previous theorem. We have

$$\tilde{\varphi}_n(x) = \sin \tfrac{1}{2}x \, (\tfrac{1}{2}x)^{-1} \, ; \quad n = 2, 3, \ldots ,$$

comparing (3.5) with (2.10) and using (2.12), and

$$|\tilde{\varphi}_n''(0)| = \tfrac{1}{12}\,; \quad n = 2, 3, \ldots$$

Then

$$s_n^2 = \tfrac{1}{12}(\Phi + n - 1) \tag{3.6}$$

is an increasing function of n. Writing

$$\operatorname{supp} \varphi_1 = \langle -d, d \rangle\,,$$

we obtain

$$\lim_{n\to\infty} \frac{1}{s_n^2}\left(\int_{ts_n<|x|} x^2\, \varphi_1(x)\, \mathrm{d}x + \sum_{k=2}^{n} \int_{ts_n<|x|} x^2\, \varphi_k(x)\, \mathrm{d}x\right) = 0$$

since all the integrals vanish for

$$s_n > t^{-1} \max\left(\tfrac{1}{2}, d\right). \tag{3.7}$$

According to (3.6), for any $t > 0$ there exists an n_0 such that (3.7) is satisfied for all $n > n_0$. The statement of the theorem follows then from Theorem 3.3.

Remark 3.2. The statement of Theorem 3.2 concerning the convergence follows immediately from Theorem 3.4 if we put $\varphi_1 = \varphi_2$. In addition, Theorem 3.2 yields the explicit values of the constants Y, Z and the rate of the convergence.

4. UNIVERSAL APPROXIMATION

Constructing an approximation according to Theorem 2.2, we are always concerned with a hill function satisfying the condition (2.5) for some fixed t. In practice, the number β may be very large. Then the exponent (2.9) is given by the formula

$$\mu = t - \alpha$$

and, trying to get as good approximation (2.8) as possible, we have to use a function ω with a very high parameter t.

Unfortunately, we usually do not know the exact value of β and this makes the choice of t rather difficult. Therefore there is a question if it is possible to find a function that would give universally the order of approximation (2.8) equal to

$$\mu = \beta - \alpha\,.$$

Let us show now the main idea of our further considerations. Following the asymptotic considerations of Section 3 (in particular, Theorem 3.2), let us define a function ω_n^* by the formula

$$\omega_n^*(x) = \sqrt{(6/(\pi n))}\, e^{-6x^2/n}$$

for any $n > 0$ real. Not paying for a while attention to the fact that supp ω_n^* is not compact, we find easily that

$$\widetilde{\omega_n^*}(x) = e^{-nx^2/24}\,.$$

The function $\widetilde{\omega_n^*}$ does not satisfy (2.5) for any $t \geqq 0$ because it has no zeros at all (cf. Remark 2.9). But this function decreases so rapidly as $|x| \to \infty$ that a statement analogous to that of Theorem 2.2 with $\mu = \beta - \alpha - \varepsilon$ can be proven supposing that $n = n(h)$ increases in a definite way as $h \to 0$.

We will prove a more general statement concerning a "universal hill function" ω in Theorem 4.1 but the main idea is the same. Instead of the parameter n, we will use a function $\eta = \eta(h)$ that corresponds to n^{-1} and tends to 0 as $h \to 0$.

If supp ω is compact another effect of this function $\eta(h)$ is apparent: it makes the support of $\omega^{[\eta(h)]}$ "wider" as $h \to 0$ (cf. Remark 2.10). On the other hand, when supp ω is not compact Theorem 4.3 shows a numerically practicable way for the approximation by this function.

C, D, and L mean general constants (independent, in particular, of the parameter h) taking different finite positive values at different places throughout this section.

Definition 4.1. Let $\Lambda \in S$ be given. A bounded continuous increasing real-valued function $\eta(h)$ defined on the interval $\langle 0, 1\rangle$ is said to be Λ-admissible if it satisfies the following assumptions:

1.

(4.1) $$\eta(0) = 0\,.$$

2. There exists a finite positive constant $C(\eta)$ such that

(4.2) $$h^{\varepsilon}\,\eta^{-1}(h) \leqq C(\eta)$$

for $0 < h \leqq 1$ and any $\varepsilon > 0$.

3. For any $\alpha \geqq 0$ there exists a function $z(k) = z(k, \alpha)$ such that

(4.3) $$|\Lambda((x - 2\pi k)\,\eta^{-1}(h))| \leqq C(\alpha, \gamma)\, h^{\gamma}\, z(k, \alpha)$$

holds for all integers k, $k \neq 0$, any $\gamma \geqq 0$, $0 < h < 1$, and $-\pi < x < \pi$ with some positive constant $C(\alpha, \gamma) < \infty$. Moreover, the series

(4.4) $$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} z^2(k, \alpha)\, |k|^{2\alpha} < \infty$$

converges for any $\alpha \geqq 0$.

Theorem 4.1. *Let $\varepsilon > 0$, $\varepsilon' > 0$ be given. Let $\omega \in S$ and let us denote the Fourier transform of ω by*

$$F(\omega) = \Lambda . \tag{4.5}$$

Further let us suppose that

$$\Lambda(0) \neq 0 . \tag{4.6}$$

Let there exist a Λ-admissible function $\eta(h)$.

Let

$$0 \leqq \alpha \leqq \beta < \infty \tag{4.7}$$

be real numbers and let $f \in W_2^\beta(R)$.

Then there exists an operator $B_{h,\eta}$, $0 < h < 1$,

$$B_{h,\eta}(f) = \eta(h) \sum_{k=-\infty}^{\infty} c(k, h, f)\, \omega((xh^{-1} - k)\, \eta(h)) \tag{4.8}$$

such that

$$\|f - B_{h,\eta}(f)\|_{W_2{}^\alpha(R)} \leqq C(\alpha, \beta, \varepsilon)\, h^{\beta-\alpha-\varepsilon} \|f\|_{W_2{}^\beta(R)} \tag{4.9}$$

for $0 < h < 1$.

If both the support T of f and the support Ω of ω are compact then there exists a constant $L(\alpha, \beta, \varepsilon')$ such that $B_{h,\eta}(f)$ has a compact support T' where T' is an $Lh^{1-\varepsilon'}$-neighborhood of T.

Proof. The course of the proof is based on that of Theorem 2.2 given in [1]. It is divided into four parts. In the first part we approximate the function $f \in W_2^\beta(R)$ by a function $f_h \in C^\infty(R)$ and find the bound (4.14) for the error of this approximation. We construct a function $g \in W_2^\alpha(R)$ approximating f_h in the second part and we find the bound (4.27) for the norm of their difference in the third part. The fourth part is concerned with the statement on compactness of the support of $B_{h,\eta}(f)$. The proof of three auxiliary statements is removed into Lemmas 4.1, 4.2, and 4.3.

1. Let us write

$$\varkappa(x) = \exp\left(1/(x^2 - 1)\right), \quad |x| \leqq 1 ,$$
$$= 0 , \qquad\qquad |x| > 1 .$$

Apparently $\varkappa \in S$ and $\nu \in S$ where

$$\nu = F(\varkappa) ,$$
$$\nu(0) = \int_{-1}^{1} \exp \frac{1}{x^2 - 1}\, dx \neq 0 .$$

According to Lemma 4.1 there exists a trigonometric polynomial P such that

(4.10) $$|1 - \varphi(xh)| \leqq C|x|^{\beta-\alpha} h^{\beta-\alpha}, \quad |xh| < 1$$

where

$$\varphi = vP .$$

Then $\varphi \in S$ since P is a multiplier. Let us put

(4.11) $$\chi = F^{-1}(\varphi) .$$

Then $\chi \in S$. Since $e^{ikx} v(x) \in S$ we may write

$$F^{-1}(e^{ikx} v(x)) (t) = F^{-1}(e^{ikx} F(\varkappa) (x)) (t) = \varkappa(t - k)$$

for any integer k. The support of $\varkappa$ is compact and the degree of P is finite. Thus the support of χ is compact, too.

Defining

(4.12) $$f_h = h^{-1}(f * \chi^{[h^{-1}]})$$

for $f \in W_2^\beta(R)$, we find that $f_h \in C^\infty$ (cf. Remark 2.3). Further $F(f_h) \in S'$ where

(4.13) $$F(f_h) = h^{-1} F(\chi^{[h^{-1}]}) F(f) = \varphi^{[h]} F(f)$$

according to (2.1), (4.11). From (4.7), (4.13) we obtain $F(f) \in L_2(R)$, $F(f_h) \in L_2(R)$.

Let us now show that

(4.14) $$\|f_h - f\|_{W_2^\alpha(R)} \leqq Ch^{\beta-\alpha}\|f\|_{W_2^\beta(R)} .$$

Having proven (4.14) we find that $f_h \in W_2^\alpha(R)$ since then $f_h - f \in W_2^\alpha(R)$ and $f \in W_2^\beta(R) \subset W_2^\alpha(R)$. Writing

$$\|f_h - f\|^2_{W_2^\alpha(R)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(f)(x)|^2 \, |1 - \varphi(xh)|^2 \, (1 + |x|^{2\alpha}) \, dx$$

we split the integral into two parts. Using (4.10) we obtain

$$\int_{|xh|<1} |F(f)(x)|^2 \, |1 - \varphi(xh)|^2 \, (1 + |x|^{2\alpha}) \, dx$$

$$\leqq C\int_{|xh|<1} |F(f)(x)|^2 \, |x|^{2(\beta-\alpha)} \, (1 + |x|^{2\alpha}) \, h^{2(\beta-\alpha)} \, dx \leqq Ch^{2(\beta-\alpha)}\|f\|^2_{W_2^\beta(R)}$$

since according to (4.7) the function

(4.15) $$(1 + |x|^{2\alpha}) \, |x|^{2(\beta-\alpha)} \, (1 + |x|^{2\beta})^{-1}$$

is bounded in R. Because $\varphi(x) - 1$ and (4.15) are bounded functions we further have

$$\int_{1<|xh|} |F(f)(x)|^2 \, |1 - \varphi(xh)|^2 (1 + |x|^{2\alpha}) \, dx$$

$$\leqq C \int_{1<|xh|} |F(f)(x)|^2 (1 + |x|^{2\alpha}) \, |x|^{2(\beta-\alpha)} \, |x|^{-2(\beta-\alpha)} \, dx \leqq Ch^{2(\beta-\alpha)} \, \|f\|^2_{W_2{}^\beta(R)} .$$

Thus (4.14) is valid.

2. According to Lemma 4.2 there exists a trigonometric polynomial P_h such that

$$(4.16) \qquad |\Lambda(xh\,\eta^{-1}(h))\, P_h(xh) - 1| \leqq Ch^{\beta-\alpha}|x|^{\beta-\alpha}, \quad |x| \leqq \pi/h ,$$

and

$$(4.17) \qquad |P_h(x)| \leqq Ch^{-\varepsilon}, \quad x \in R$$

for arbitrary $\varepsilon > 0$. Let us now put

$$(4.18) \qquad \xi_h = F(f_h) = \varphi^{[h]} F(f) ,$$

$$(4.19) \qquad \zeta_h(x) = P_h(xh) \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi kh^{-1}) .$$

Then $\xi_h \in L_2(R)$ (cf. (4.13)) and the series in (4.19) converges in $L_2(-\pi/h, \pi/h)$ since

$$(4.20) \qquad \Big\| \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi kh^{-1}) \Big\|_{L_2(-\pi/h,\pi/h)}$$

$$\leqq \sum_{k=-\infty}^{\infty} \|\xi_h(x - 2\pi kh^{-1})\|_{L_2(-\pi/h,\pi/h)}$$

$$= \sum_{k=-\infty}^{\infty} \left(\int_{-\pi/h}^{\pi/h} |\xi_h(x - 2\pi kh^{-1})|^2 \, dx \right)^{1/2}$$

$$= \sum_{k=-\infty}^{\infty} \left(\int_{(2k-1)\pi/h}^{(2k+1)\pi/h} |\varphi(xh)|^2 \, |F(f)(x)|^2 \, dx \right)^{1/2}$$

$$\leqq C \|f\|_{L_2(R)}$$

where we use the fact that $\varphi \in S$ (cf. the analogous proof of Lemma 4.3 in the following). Therefore $\zeta_h \in L_2(-\pi/h, \pi/h)$ and $\zeta_h \in L_1(-\pi/h, \pi/h)$. Moreover the function ζ_h is apparently periodic with the period $2\pi h^{-1}$. Let us write

$$(4.21) \qquad \zeta_h = P_h^{[h]}(\xi_h + \xi_h^*)$$

where

$$\xi_h^*(x) = \sum_{k \neq 0} \xi_h(x - 2\pi kh^{-1}) .$$

Let us construct a Fourier series for the periodic function $\zeta_h \in L_2(-\pi/h, \pi/h)$,

$$\zeta_h(x) = \sum_{k=-\infty}^{\infty} c_k(h)\, e^{ikhx}, \tag{4.22}$$

where

$$c_k(h) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \zeta_h(x)\, e^{-ikhx}\, dx. \tag{4.23}$$

The series converges in $L_2(-\pi/h, \pi/h)$ and

$$\lim_{|k|\to\infty} c_k(h) = 0. \tag{4.24}$$

Let us define

$$g(x) = \eta(h)\, h^{-1} \sum_{k=-\infty}^{\infty} c_k(h)\, \omega((xh^{-1} - k)\, \eta(h)) \tag{4.25}$$

with $c_k(h)$ given in (4.23). The assumptions of Theorem 2.1 are fulfilled for the function $h\eta^{-1}(h)\, g^{[h/\eta(h)]}$ according to (4.24) and Remark 2.7, and we have

$$\begin{aligned} F(h\eta^{-1}(h)\, g^{[h\eta^{-1}(h)]})\,(x) &= F(\omega)\,(x) \sum_{k=-\infty}^{\infty} c_k(h)\, e^{ik\eta(h)x} \\ &= A(x) \sum_{k=-\infty}^{\infty} c_k(h)\, e^{ik\eta(h)x} \end{aligned}$$

using (4.5). From (2.1) we obtain

$$F(g)\,(x) = F(h\,\eta^{-1}(h)\, g^{[h\eta^{-1}(h)]})\,(h\,\eta^{-1}(h)\, x) = A(h\,\eta^{-1}(h)\, x) \sum_{k=-\infty}^{\infty} c_k(h)\, e^{ikhx}.$$

Using (4.21) and (4.22), we have finally

$$F(g) = \zeta_h A^{[h\eta^{-1}(h)]} = P_h^{[h]} A^{[h\eta^{-1}(h)]} \xi_h + P_h^{[h]} A^{[h\eta^{-1}(h)]} \xi_h^*. \tag{4.26}$$

3. Let us now show

$$\|f_h - g\|_{W_2^\alpha(R)} \leqq Ch^{\beta-\alpha-\varepsilon} \|f\|_{W_2^\beta(R)}. \tag{4.27}$$

Having proven (4.27) we find that $g \in W_2^\alpha(R)$ since then $f_h - g \in W_2^\alpha(R)$, $f_h \in W_2^\alpha(R)$. From (4.18), (4.21), (4.26) we have

$$\begin{aligned} &\|f_h - g\|^2_{W_2^\alpha(R)} = \\ &\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\xi_h(x) - P_h(xh)\, A(h\eta^{-1}(h)\, x)\, \xi_h(x) \\ &\qquad - P_h(xh)\, A(h\,\eta^{-1}(h)\, x)\, \xi_h^*(x)|^2\, (1 + |x|^{2\alpha})\, dx \end{aligned}$$

$$\leq C\Big(\int_{-\pi/h}^{\pi/h} |1 - P_h(xh)\, \Lambda(h\,\eta^{-1}(h)\,x)|^2\, |\xi_h(x)|^2\, (1 + |x|^{2\alpha})\, dx$$

$$+ \int_{-\pi/h}^{\pi/h} |P_h(xh)\, \Lambda(h\,\eta^{-1}(h)\,x)|^2\, |\xi_h^*(x)|^2\, (1 + |x|^{2\alpha})\, dx$$

$$+ \sum_{k\neq 0} \int_{-\pi/h}^{\pi/h} |\zeta_h(x)|^2\, |\Lambda((xh - 2\pi k)\,\eta^{-1}(h))|^2\, (1 + |x + 2\pi k h^{-1}|^{2\alpha})\, dx$$

$$+ \int_{\pi/h < |x|} |\xi_h(x)|^2\, (1 + |x|^{2\alpha})\, dx\Big)$$

$$= C(I_1 + I_2 + I_3 + I_4)$$

since ζ_h is periodic with period $2\pi h^{-1}$.

From (4.7), (4.16), and (4.18) we obtain

(4.28) $$I_1 \leq C h^{2(\beta-\alpha)} \int_{-\pi/h}^{\pi/h} |F(f)\,(x)|^2\, |\varphi(xh)|^2\, |x|^{2(\beta-\alpha)}\, (1 + |x|^{2\alpha})\, dx$$

$$\leq C h^{2(\beta-\alpha)} \|f\|^2_{W_2^{\beta}(R)}$$

since the functions φ and (4.15) are bounded in R.

Putting $\gamma = \alpha$ in Lemma 4.3 and using (4.17), we have

(4.29) $$I_2 \leq C h^{2(\beta-\alpha-\varepsilon)} \|f\|^2_{W_2^{\beta}(R)}$$

for arbitrary $\varepsilon > 0$, because the function Λ is bounded in R.

Further let us write

$$I_3 \leq \sum_{k\neq 0} \int_{-\pi/h}^{\pi/h} |P_h(xh)|^2\, |\xi_h(x)|^2\, |\Lambda((xh - 2\pi k)\,\eta^{-1}(h))|^2\, (1 + |x + 2\pi k h^{-1}|^{2\alpha})\, dx$$

$$+ \sum_{k\neq 0} \int_{-\pi/h}^{\pi/h} |P_h(xh)|^2\, |\xi_h^*(x)|^2\, |\Lambda((xh - 2\pi k)\,\eta^{-1}(h))|^2\, (1 + |x + 2\pi k h^{-1}|^{2\alpha})\, dx$$

$$= I_{31} + I_{32}\,.$$

Putting $\gamma = \beta$ in (4.3) of Definition 4.1 and using (4.17), we may write

(4.30) $$|P_h(xh)|^2\, |\Lambda((xh - 2\pi k)\,\eta^{-1}(h))|^2\, (1 + |x + 2\pi k h^{-1}|^{2\alpha})$$

$$\leq C h^{2(\beta-\alpha-\varepsilon)}\, z^2(k)\, |k|^{2\alpha}$$

for $x \in \langle -\pi/h, \pi/h\rangle$, $0 < h < 1$, and any integer k, $k \neq 0$ since

$$1 + |x + 2\pi k h^{-1}|^{2\alpha} \leq C h^{-2\alpha} |k|^{2\alpha}$$

holds for these values of x, h, and k. Then from (4.18), (4.30), and (4.4) of Definition 4.1 we have

$$I_{31} \leqq Ch^{2(\beta-\alpha-\varepsilon)} \sum_{k \neq 0} z^2(k) |k|^{2\alpha} \int_{-\pi/h}^{\pi/h} |F(f)(x)|^2 |\varphi(xh)|^2 \, dx$$

$$\leqq Ch^{2(\beta-\alpha-\varepsilon)} \int_{-\pi/h}^{\pi/h} |F(f)(x)|^2 (1 + |x|^{2\beta}) \, dx \leqq Ch^{2(\beta-\alpha-\varepsilon)} \|f\|^2_{W_2^\beta(R)}$$

because the function φ is bounded in R. Putting $\gamma = \beta$ in Lemma 4.3 and using (4.30) and (4.4) of Definition 4.1, we obtain

$$I_{32} \leqq Ch^{2(\beta-\alpha-\varepsilon)} \sum_{k \neq 0} z^2(k) |k|^{2\alpha} \int_{-\pi/h}^{\pi/h} |\xi_h^*(x)|^2 (1 + |x|^{2\beta}) \, dx \leqq Ch^{2(\beta-\alpha-\varepsilon)} \|f\|^2_{W_2^\beta(R)} \,.$$

Therefore

$$I_3 \leqq Ch^{2(\beta-\alpha-\varepsilon)} \|f\|^2_{W_2^\beta(R)} \,. \tag{4.31}$$

Finally we use (4.7), (4.18), and the boundedness of the function φ to show

$$\begin{aligned} I_4 &\leqq C \int_{\pi/h < |x|} |F(f)(x)|^2 (1 + |x|^{2\alpha}) \, dx \\ &= C \int_{\pi/h < |x|} |F(f)(x)|^2 (1 + |x|^{2\beta}) |x|^{2(\alpha-\beta)} \, dx \\ &\leqq Ch^{2(\beta-\alpha)} \int_{\pi/h < |x|} |F(f)(x)|^2 (1 + |x|^{2\beta}) \, dx \\ &\leqq Ch^{2(\beta-\alpha)} \|f\|^2_{W_2^\beta(R)} \end{aligned} \tag{4.32}$$

since the function (4.15) is bounded in R.

From (4.28), (4.29), (4.31), and (4.32) we obtain (4.27), which together with (4.14) completes the proof of (4.9). From (4.8), (4.9), (4.25) we have

$$B_{h,\eta}(f) = g \,,$$

i.e.,

$$c(k, h, f) = c_k(h) \, h^{-1} \,. \tag{4.33}$$

Thus $B_{h,\eta}(f) \in W_2^\alpha(R)$.

4. Supposing that the support T of f is compact, we find that $F(f)$ is a multiplier (cf. Remark 2.5) and $\xi_h = F(f_h) \in S$, $f_h \in S$ follows from (4.12), (4.13). There exists a constant L such that

$$\operatorname{supp} \chi = \mathsf{E}[x, |x| \leqq L] \,,$$

i.e.,

$$\text{supp}\, \chi^{[h^{-1}]} = \mathsf{E}[x, |x| \leqq Lh]\,. \tag{4.34}$$

From (4.12), (4.34) we have

$$\text{supp} f_h = \text{supp}\,(f * \chi^{[h^{-1}]}) = \mathsf{E}[w, w = x + y,\ x \in T,\ |y| \leqq Lh]$$

(cf. Remark 2.6). Thus the function f_h has a compact support lying in an Lh-neighborhood of T.

Since P_h is a multiplier we obtain $F^{-1}(\xi_h P_h^{[h]}) \in S$. In particular, $e^{ikhx}\, \xi_h(x) \in S$ and we may write

$$F^{-1}(e^{ikhx}\, \xi_h(x))\,(t) = F^{-1}(e^{ikhx}\, F(f_h)\,(x))\,(t) = f_h(t - kh)$$

for any integer k. Since the compact support of f_h lies in an Lh-neighborhood of T and the degree of P_h is finite there exists a constant L (the value of which may be different from that of L above) such that

$$T_{Lh} = \mathsf{E}[w, w = x + y,\ x \in T,\ |y| \leqq Lh] \tag{4.35}$$

is a support of the function $F^{-1}(\xi_h P_h^{[h]})$. Therefore

$$F^{-1}(\xi_h P_h^{[h]})\,(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixw}\, \xi_h(x)\, P_h(xh)\, dx = 0 \tag{4.36}$$

for all $w \notin T_{Lh}$. From (4.19), (4.23) we have

$$\begin{aligned} c_k(h) &= \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} P_h(xh)\, e^{-ikhx} \sum_{j=-\infty}^{\infty} \xi_h(x - 2\pi j h^{-1})\, dx \\ &= \frac{h}{2\pi} \int_{-\infty}^{\infty} P_h(xh)\, \xi_h(x)\, e^{-ikhx}\, dx \end{aligned} \tag{4.37}$$

because $\xi_h \in S$ and $P_h^{[h]}$, e^{-ikhx} are periodic functions with the period $2\pi h^{-1}$. Comparing (4.36) with (4.37) we see that $c_k(h) = 0$ in (4.22) for all k such that $kh \notin T_{Lh}$. Therefore the sum in (4.8) is finite, i.e., the summation goes over all integers $k \in K$ where

$$K = \mathsf{E}[k, kh \in T_{Lh}] = \mathsf{E}[k, kh = v + y,\ v \in T,\ |y| \leqq Lh] \tag{4.38}$$

according to (4.33), (4.35).

Let the support Ω of ω be also compact, i.e.,

$$\Omega = \text{supp}\, \omega = \mathsf{E}[x, |x| \leqq D]$$

with some constant D. Then

$$\Omega_k = \operatorname{supp} \omega(\eta(h)(xh^{-1} - k)) = \mathsf{E}[x, |\eta(h) h^{-1}(x - kh)| \leqq D]$$
$$= \mathsf{E}[x, kh - Dh\,\eta^{-1}(h) \leqq x \leqq kh + Dh\,\eta^{-1}(h)] .$$

Finally with respect to (4.38) we find

(4.39) $$T' = \operatorname{supp} B_{h,\eta}(f) = \bigcup_{k \in K} \Omega_k$$
$$= \mathsf{E}[w, w = v + x + y, v \in T, |y| \leqq Lh, |x| \leqq Dh\,\eta^{-1}(h)]$$
$$= \mathsf{E}[w, w = v + z, v \in T, |z| \leqq Lh^{1-\varepsilon'}]$$

since the function $h^{\varepsilon'}(L + D\,\eta^{-1}(h))$ is bounded for $0 < h < 1$ and arbitrary $\varepsilon' > 0$ according to (4.2) of Definition 4.1. Apparently (4.39) is an $Lh^{1-\varepsilon'}$-neighborhood of T. Thus the statement of the theorem in the case of compact supports of both f and ω holds, which completes the proof of the theorem.

The following three lemmas are used in the proof of Theorem 4.1.

Lemma 4.1. *Let $v \in S$, $v(0) \neq 0$, and $0 \leqq \alpha \leqq \beta$. Then there exists a trigonometric polynomial $P(x)$ such that*

(4.40) $$|v(x) P(x) - 1| \leqq C|x|^{\beta - \alpha}$$

for $|x| < 1$.

Proof. If $\beta = \alpha$ then we may put $P(x) \equiv 1$. Thus let $\alpha < \beta$ and let us denote the minimal integer not less than $\beta - \alpha$ by B. Let us choose two integers $N \geqq M$ in such a way that

(4.41) $$N - M = B - 1 \geqq 0 .$$

Assuming that $P(x)$ is of the form

(4.42) $$P(x) = \sum_{k=M}^{N} b_k e^{ikx}$$

we will find its coefficients b_k; $k = M, \ldots, N$. We may write a Taylor series for the function vP,

(4.43) $$v(x) P(x) = \sum_{j=0}^{B-1} (vP)^{(j)}(0)(j!)^{-1} x^j$$
$$+ (vP)^{(B)}(\vartheta x)(B!)^{-1} x^B , \quad |x| < 1 , \quad 0 < \vartheta < 1 ,$$

where

$$(vP)^{(j)}(y) = \frac{d^j}{dx^j}(v(x) P(x))\bigg|_{x=y} .$$

Putting

$$(4.44) \qquad (vP)(0) = 1\,, \qquad (vP)^{(j)}(0) = 0\,; \quad j = 1, \ldots, B-1\,,$$

we obtain from (4.43)

$$|v(x)\,P(x) - 1| \leqq C|x|^B \leqq C|x|^{\beta-\alpha}\,, \quad |x| < 1$$

and thus (4.40) is satisfied. Let us further examine the conditions (4.44). Differentiating (4.42) and substituting the derivatives into (4.44), we have

$$(vP)^{(j)}(0) = \sum_{l=0}^{j} \sum_{k=M}^{N} \binom{j}{l} i^l k^l v^{(j-l)}(0)\, b_k\,; \quad j = 0, 1, \ldots, B-1\,,$$

i.e.,

$$(4.45) \qquad \sum_{k=M}^{N} a_{jk} b_k = \delta_{0j}\,; \quad j = 0, 1, \ldots, B-1$$

with

$$(4.46) \qquad a_{jk} = \sum_{l=0}^{j} \binom{j}{l} i^l k^l v^{(j-l)}(0)$$

where δ_{kj} is the Kronecker symbol. (4.45) is a system of B linear algebraic equations for B unknown coefficients b_k; $k = M, \ldots, N$. Making use of the form (4.46) of the elements of the matrix of this system and expressing the determinant $\det(a_{jk})$ of this matrix in terms of the sums in (4.46), we finally obtain

$$\det(a_{jk}) = i^{B(B-1)/2}\, v^B(0)\, V_B(M, M+1, \ldots, N) \neq 0$$

where $V_B(M, M+1, \ldots, N)$ is the Vandermonde determinant formed of the B integers $M, M+1, \ldots, N$. It is non-zero since these integers differ from each other. Thus the system (4.45) has a unique solution b_k; $k = M, \ldots, N$ for any right-hand part and the trigonometric polynomial (4.42) satisfying (4.40) has been constructed.

Lemma 4.2. *Let the assumptions of Theorem* 4.1 *be fulfilled. Then there exists a trigonometric polynomial* $P_h(x)$ *such that*

$$(4.47) \qquad |A(x\eta^{-1}(h))\, P_h(x) - 1| \leqq C|x|^{\beta-\alpha}$$

for $|x| \leqq \pi$ *and*

$$|P_h(x)| \leqq Ch^{-\varepsilon}$$

for $x \in R$ *and arbitrary* $\varepsilon > 0$.

Proof. The existence of P_h follows from a modification of the proof of Lemma 4.1. The case $\alpha = \beta$ being again trivial, let us suppose that $\alpha < \beta$. Using (4.6) and the fact that $\Lambda \in S$, let us find $B > 0$ and choose $N \geqq M$ according to (4.41), and assume

$$P_h(x) = \sum_{k=M}^{N} b_k(h)\, e^{ikx} \tag{4.48}$$

instead of (4.42). Writing $\Lambda(x\eta^{-1}(h))$ instead of $v(x)$ and $P_h(x)$ instead of $P(x)$ in (4.43), we obtain finally the system

$$\sum_{k=M}^{N} a_{jk}(h)\, b_k(h) = \delta_{0j}\,; \quad j = 0, 1, \ldots, B - 1 \tag{4.49}$$

for the coefficients $b_k(h)$; $k = M, \ldots, N$ of (4.48) with

$$a_{jk}(h) = \sum_{l=0}^{j} \binom{j}{l} i^l k^l\, \eta^{l-j}(h)\, \Lambda^{(j-l)}(0)\,.$$

The determinant of the system (4.49) is

$$\det\left(a_{jk}(h)\right) = i^{B(B-1)/2}\, \Lambda^B(0)\, V_B(M, M + 1, \ldots, N) \neq 0$$

and is independent of h. Thus the system (4.49) has a unique solution $b_k(h)$; $k = M, \ldots, N$ for any right-hand part and $0 < h < 1$. The trigonometric polynomial (4.48) satisfying (4.47) has been constructed.

Let us solve the system (4.49) using Cramer's rule. Denoting the matrix, obtained by replacing the mth column in the matrix $(a_{jk}(h))$ by the column of the right-hand parts, by $({}^m a_{jk}(h))$ and treating this matrix in the same way as above, we find that its determinant is of the form

$$\det\left({}^m a_{jk}(h)\right) = C\, \eta^{1-B}(h) + o(\eta^{1-B}(h))\,, \quad h \to 0\,.$$

Since the determinant $\det(a_{jk}(h))$ of the system (4.49) is independent of h we have

$$|b_k(h)| \leqq C\, \eta^{1-B}(h)$$

and according to (4.2) of Definition 4.1 we obtain finally

$$|b_k(h)| \leqq Ch^{-\varepsilon}\,, \quad 0 < h < 1$$

for any $\varepsilon > 0$. This completes the proof, because then

$$|P_h(x)| \leqq \sum_{k=M}^{N} |b_k(h)| \leqq Ch^{-\varepsilon}\,, \quad x \in R\,.$$

Lemma 4.3. *Let the assumptions of Theorem* 4.1 *be fulfilled, let* $\beta \geqq \gamma \geqq 0$. *Then*

$$I(\gamma) = \int_{-\pi/h}^{\pi/h} |\xi_h^*(x)|^2\, (1 + |x|^{2\gamma})\, dx \leqq Ch^{2(\beta-\gamma)} \|f\|^2_{W_2^\beta(R)}\,. \tag{4.50}$$

Proof. Since $1 + |x|^{2\gamma} \leqq Ch^{-2\gamma}$ for $|x| \leqq \pi/h$ we obtain

$$\sqrt{I(\gamma)} \leqq Ch^{-\gamma} \Big\| \sum_{k \neq 0} \xi_h(x - 2\pi k h^{-1}) \Big\|_{L_2(-\pi/h,\pi/h)}$$
$$\leqq Ch^{-\gamma} \sum_{k \neq 0} \left(\int_{-\pi/h}^{\pi/h} |\xi_h(x - 2\pi k h^{-1})|^2 \, dx \right)^{1/2}$$

in the same way as in (4.20). Using (4.18) we may further write

$$\sqrt{I(\gamma)} \leqq Ch^{-\gamma} \sum_{k \neq 0} \left(\int_{-\pi/h}^{\pi/h} |\varphi(xh - 2\pi k)|^2 \, |F(f)\,(x - 2\pi k h^{-1})|^2 \, dx \right)^{1/2}$$
$$\leqq Ch^{-\gamma} \sum_{k \neq 0} |k|^{-p} \left(\int_{-\pi/h}^{\pi/h} |F(f)\,(x - 2\pi k h^{-1})|^2 \, dx \right)^{1/2}$$
$$\leqq Ch^{-\gamma} \left(\int_{\pi/h \leqq |x|} |F(f)\,(x)|^2 \, dx \right)^{1/2} \sum_{k \neq 0} |k|^{-p}$$
$$\leqq Ch^{-\gamma} \left(\int_{\pi/h \leqq |x|} |F(f)\,(x)|^2 \, dx \right)^{1/2}$$

because $\varphi \in S$ and

$$|\varphi(xh - 2\pi k)| \leqq C(p)\,|k|^{-p}, \quad k \neq 0$$

holds for every $p > 0$ and $|x| < \pi/h$. We may find such p that the series $\sum_{k \neq 0} |k|^{-p}$ converges. Finally we obtain

$$\sqrt{I(\gamma)} \leqq Ch^{-\gamma} \left(\int_{\pi/h \leqq |x|} |F(f)\,(x)|^2 \, (1 + |x|^{2\beta})\,(1 + |x|^{2\beta})^{-1} \, dx \right)^{1/2}$$
$$\leqq Ch^{\beta-\gamma} \left(\int_{\pi/h \leqq |x|} |F(f)\,(x)|^2 \, (1 + |x|^{2\beta}) \, dx \right)^{1/2}$$
$$\leqq Ch^{\beta-\gamma} \|f\|_{W_2{}^{\beta}(R)},$$

from which (4.50) follows since $(1 + |x|^{2\beta})^{-1} \leqq Ch^{2\beta}$ for $\pi/h \leqq |x|$.

Remark 4.1. The statement of Theorem 4.1 shows that the spaces $W_2^{\alpha}(R)$ are not convenient for a closer analysis of the universal approximation considered. A Hilbert space H_γ of functions $f \in S'$ with the norm

$$\|f\|_\gamma^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(x)|^2 \, \gamma(x^2) \, dx$$

may be introduced for this purpose where γ is an entire function,

$$\gamma(x) = \sum_{k=0}^{\infty} \gamma_k x^k,$$

$\gamma_k \geqq 0$, $\gamma_0 > 0$, $\gamma_l > 0$ for an $l > 0$, and

$$\lim_{k \to \infty} \gamma_k^{1/k} = 0 .$$

A Λ-admissible function η fulfilling the conditions of Definition 4.1 may be readily found for a class of functions from S the Fourier transform of which decreases (at ∞) as rapidly as $e^{-|x|}$.

Theorem 4.2. *Let $\Lambda \in S$ satisfy the condition*

$$|\Lambda(x)| \leqq Le^{-D|x|}, \quad x \in R \tag{4.51}$$

with some constants L, D which are positive and finite. Then the function

$$\eta(h) = \eta_0(\eta_1 + \log^{1+\varepsilon_0} h^{-1})^{-1} \tag{4.52}$$

where $\eta_0, \eta_1, \varepsilon_0$ are arbitrary positive numbers is Λ-admissible independently of L, D.

Proof. It is necessary to verify the conditions (4.1) to (4.4) of Definition 4.1. The function $\eta(h)$ is apparently bounded, continuous, and increasing in $\langle 0, 1 \rangle$ and

$$\eta(0) = 0 , \quad \eta(1) = \eta_0/\eta_1 .$$

Thus (4.1) is fulfilled. The function

$$h^\varepsilon \eta^{-1}(h) = h^\varepsilon \eta_0^{-1} \eta_1 + h^\varepsilon \eta_0^{-1} \log^{1+\varepsilon_0} h^{-1}$$

is bounded in $\langle 0, 1 \rangle$ for any $\varepsilon > 0$, which proves (4.2).

Let us consider the conditions (4.3), (4.4). From (4.51) we obtain

$$\begin{aligned} |\Lambda((x - 2\pi k)\, \eta^{-1}(h))| &\leqq L\exp(-D|2\pi k - x|\, \eta^{-1}(h)) \\ &\leqq L(\exp(-|k|\, \eta^{-1}(h)))^D \end{aligned}$$

because $|2\pi k - x| \geqq C|k|$ for all integers k, $k \neq 0$ and $x \in \langle -\pi, \pi \rangle$. Since

$$(1 - |k|)\, \eta^{-1}(h) + s \log |k| \leqq C(s)$$

for all $k \neq 0$, $0 < h \leqq 1$ and arbitrary non-negative s we further have

$$|\Lambda((x - 2\pi k)\, \eta^{-1}(h))| \leqq C(s) \exp(-D\eta^{-1}(h))\, |k|^{-s} .$$

Showing

$$\exp(-D\eta^{-1}(h)) = Ch^\sigma \leqq C(\gamma)\, h^\gamma , \quad \sigma = D\eta_0^{-1} \log^{\varepsilon_0} h^{-1} \tag{4.53}$$

for any $\gamma \geqq 0$ and $0 < h \leqq 1$, we obtain finally

$$|\Lambda((x - 2\pi k)\, \eta^{-1}(h))| \leqq C(s, \gamma)\, h^\gamma |k|^{-s} \tag{4.54}$$

for $0 < h \leqq 1$, $-\pi \leqq x \leqq \pi$, all $k \neq 0$ and arbitrary non-negative numbers γ, s. Putting $s = \alpha + 1$ in (4.54) we may define

$$z(k) = z(k, \alpha) = |k|^{-\alpha-1}, \quad k \neq 0 .$$

Then

$$\sum_{k \neq 0} z^2(k, \alpha) |k|^{2\alpha} = \sum_{k \neq 0} |k|^{-2} < \infty ,$$

which completes the proof of the conditions (4.3), (4.4) of Definition 4.1.

Approximating a function f the support of which is compact we expect the support of $A_h(f)$ and $B_{h,\eta}(f)$ to be also compact. This is possible only in the case when the approximating function ω has also a compact support (cf. Theorems 2.2 and 4.1).

In practice, this $\omega(x)$ need not have a compact support but may decrease (at ∞) so rapidly that (from a numerical point of view) its values are negligible for $|x|$ greater than some $Y > 0$ (this is e.g. the case of the "limit" function $\exp(-x^2)$). The approximation by a class of such functions is considered in Theorem 4.3.

Theorem 4.3. *Let the assumptions of Theorem* 4.1 *be fulfilled. Further let J be a non-negative integer such that*

$$|\omega^{(j)}(x)| \leqq L_j e^{-D_j|x|} ; \quad x \in R ; \quad j = 0, 1, \ldots, J \tag{4.55}$$

with some finite positive constants D_j, L_j; $j = 0, 1, \ldots, J$.

Let us introduce a function ω_Y and its derivatives up to the order J by the formula

$$\omega_Y^{(j)}(x) = \omega^{(j)}(x) , \quad |x| < Y , \qquad = 0 , \quad |x| \geqq Y \tag{4.56}$$

where

$$Y = Y(h) = Y_0 + Y_1 \log^{1+\varepsilon_1} h^{-1} \tag{4.57}$$

for $0 < h < 1$ with arbitrary positive constants Y_0, Y_1, and ε_1. Writing

$$B_{h,\eta}^{(j)}(f)(x) = \eta^{j+1}(h) h^{-j} \sum_{k=-\infty}^{\infty} c(k, h, f) \omega^{(j)}((xh^{-1} - k) \eta(h)) , \tag{4.58}$$

$$B_{h,\eta,Y}^{(j)}(f)(x) = \eta^{j+1}(h) h^{-j} \sum_{k=-\infty}^{\infty} c(k, h, f) \omega_Y^{(j)}((xh^{-1} - k) \eta(h)) \tag{4.59}$$

for $0 < h < 1$ with $c(k, h, f)$ given in (4.8), *and*

$$\varrho_{h,\eta,Y}^{(j)} = B_{h,\eta}^{(j)} - B_{h,\eta,Y}^{(j)} ; \quad j = 0, 1, \ldots, J , \tag{4.60}$$

we have

$$\sup_{x \in R} |\varrho_{h,\eta,Y}^{(j)}(f)(x)| \leqq C(j, s) h^s \|f\|_{L_2(R)} \tag{4.61}$$

for any $s \geqq 0$.

Proof. Fixing an $x \in R$ and an integer j, $0 \leqq j \leqq J$, and substituting (4.58) and (4.59) into (4.60), we obtain

$$(4.62) \qquad \varrho_{h,\eta,Y}^{(j)}(f)(x) = \eta^{j+1}(h)\, h^{-j} \sum_{k=-\infty}^{\infty} c(k,h,f)\,(\omega^{(j)}((xh^{-1}-k)\,\eta(h)) - \omega_Y^{(j)}((xh^{-1}-k)\,\eta(h)))\,.$$

According to (4.56) we have

$$\omega_Y^{(j)}((xh^{-1}-k)\,\eta(h)) = \omega^{(j)}((xh^{-1}-k)\,\eta(h))\,, \quad kh - Yh\,\eta^{-1}(h) < x < kh + Yh\,\eta^{-1}(h)\,,$$
$$= 0\,, \quad x \leqq kh - Yh\,\eta^{-1}(h)\,, \quad kh + Yh\,\eta^{-1}(h) \leqq x$$

for any integer k.

Let $0 \leqq \sigma_1 < 1$ be such a real number that

$$K_1 = xh^{-1} - Y\eta^{-1}(h) - \sigma_1$$

is the maximal integer not greater than $xh^{-1} - Y\eta^{-1}(h)$. Similarly let

$$K_2 = xh^{-1} + Y\eta^{-1}(h) + \sigma_2$$

be the minimal integer not less than $xh^{-1} + Y\eta^{-1}(h)$ where $0 \leqq \sigma_2 < 1$. Using the notation

$$\sum_k{}' = \sum_{k=-\infty}^{K_1} + \sum_{k=K_2}^{\infty}$$

we may rewrite (4.62) and use the Schwarz inequality:

$$(4.63) \qquad |\varrho_{h,\eta,Y}^{(j)}(f)(x)| = \eta^{j+1}(h)\, h^{-j} \Big|\sum_k{}' c(k,h,f)\,\omega^{(j)}((xh^{-1}-k)\,\eta(h))\Big|$$
$$\leqq \eta^{j+1}(h)\, h^{-j} \Big(\sum_{k=-\infty}^{\infty} c^2(k,h,f)\Big)^{1/2} \Big(\sum_k{}' (\omega^{(j)}((xh^{-1}-k)\,\eta(h)))^2\Big)^{1/2}.$$

We will study the two sums separately. Using the notation of the proof of Theorem 4.1 we obtain

$$\sum_{k=-\infty}^{\infty} c_k^2(h) = \|\zeta_h\|^2_{L_2(-\pi/h,\pi/h)}$$
$$= Ch^{-2\varepsilon} \Big\| \sum_{k=-\infty}^{\infty} \xi_h(x - 2\pi k h^{-1}) \Big\|^2_{L_2(-\pi/h,\pi/h)} \leqq Ch^{-2\varepsilon} \|f\|^2_{L_2(R)}$$

according to (4.17) to (4.20), and (4.22). Finally from (4.33) we have

$$(4.64) \qquad \Big(\sum_{k=-\infty}^{\infty} c^2(k,h,f)\Big)^{1/2} = h^{-1} \Big(\sum_{k=-\infty}^{\infty} c_k^2(h)\Big)^{1/2} \leqq Ch^{-1-\varepsilon} \|f\|_{L_2(R)}\,.$$

Now let us show that

$$\sum_k{}' (\omega^{(j)}((xh^{-1} - k)\,\eta(h)))^2 \leqq C(j, \gamma)\, h^{2\gamma}\, \eta^{-2j-2}(h) \tag{4.65}$$

is valid for any $\gamma \geqq 0$. Substituting (4.55) for $\omega^{(j)}$, we obtain

$$\sum_k{}' (\omega^{(j)}((xh^{-1} - k)\,\eta(h)))^2 \leqq$$

$$\leqq L_j^2\Big(\sum_{k=-\infty}^{K_1} \exp(-2D_j\,\eta(h)\,|xh^{-1} - k|) + \sum_{k=K_2}^{\infty} \exp(-2D_j\,\eta(h)\,|xh^{-1} - k|)\Big)$$

$$= L_j^2\Big(\sum_{k=-K_1}^{\infty} \exp(-2D_j\,\eta(h)\,(xh^{-1} + k)) + \sum_{k=K_2}^{\infty} \exp(-2D_j\,\eta(h)\,(k - xh^{-1}))\Big)$$

$$= L_j^2 \exp(-2D_j\,Y(h))\,(1 - \exp(-2D_j\,\eta(h)))^{-1}\,(\exp(-2D_j\,\eta(h)\,\sigma_1)$$

$$+ \exp(-2D_j\,\eta(h)\,\sigma_2)) \leqq C(j) \exp(-2D_j\,Y(h))\,(1 - \exp(-2D_j\,\eta(h)))^{-1}\,.$$

Showing

$$(1 - \exp(-2D_j\,\eta(h)))^{-1} \leqq C(j)\,\eta^{-2j-2}(h)\,,$$

we finally have

$$\sum_k{}' (\omega^{(j)}((xh^{-1} - k)\,\eta(h)))^2 \leqq C(j)\,\eta^{-2j-2}(h) \exp(-2D_j\,Y(h)) \tag{4.66}$$

independently of x. Analogously to (4.53) we may prove that

$$\exp(-2D_j\,Y(h)) \leqq C(j, \gamma)\, h^{\gamma}$$

for any $\gamma \geqq 0$ and $0 < h < 1$. From this and (4.66) we obtain (4.65). Eventually (4.63), (4.64), and (4.65) imply (4.61), which completes the proof of the theorem.

Remark 4.2. Let us put

$$\begin{aligned} \hat{\omega}_Y(x) &= \omega(x)\,, & |x| &\leqq Y\,, \\ &= 0\,, & |x| &> Y\,. \end{aligned}$$

If we substitute $\hat{\omega}_Y$ for ω_Y in Theorem 4.3 the theorem remains true. In the proof, only the definition of integers K_1, K_2 has to be changed in an apparent way.

Remark 4.3. Returning to the hill functions $\omega_n(x)$ given by (2.10), (2.11), we may readily find that for large n their values are negligible for such $x \in \operatorname{supp} \omega_n$ that lie out of some vicinity of the origin, i.e., that their "substantial support" (in the numerical sense mentioned above) is significantly narrower than their support. Then a question of employing this "substantial support" for practical computation arises here, too.

5. A NUMERICAL EXAMPLE

The following simple numerical example illustrates the statements of Sec. 4.

In [4], the problem

$$(5.1) \qquad -u''(x) + cu(x) = f(x)\,, \quad x \in (0, \pi)\,, \quad c > 0$$

with the boundary conditions

$$(5.2) \qquad u'(0) = u'(\pi) = 0$$

and the right-hand part

$$(5.3) \qquad f(x) = -\sin\left(d(x - \tfrac{1}{2}\pi)\right), \quad d > 0$$

is solved by the finite element method using the hill functions ω_n given by (2.10), (2.11). The exact solution of this problem is

$$u(x) = -\frac{1}{d^2 + c}\sin\left(d(x - \tfrac{1}{2}\pi)\right) + \frac{d}{(d^2 + c)\sqrt{c}}\,\frac{\cos\left(\tfrac{1}{2}\pi d\right)}{\operatorname{ch}\left(\tfrac{1}{2}\pi\sqrt{c}\right)}\operatorname{sh}\left((x - \tfrac{1}{2}\pi)\sqrt{c}\right).$$

Let us solve the problem (5.1), (5.2), (5.3) by the finite element method using a universal hill function ω. Let us denote the approximate solution of the problem sought in the form (4.8) by $u_{h,\eta}$. Since $f \in W_2^\beta(0, \pi)$ for any $\beta \geqq 0$ we obtain from Theorem 4.1

$$\|u_{h,\eta} - u\|_{L_2(0,\pi)} \leqq C(\beta, \varepsilon)\, h^{\beta+2-\varepsilon}\|f\|_{W_2^\beta(0,\pi)}$$

for arbitrary $\varepsilon > 0$ in the way analogous to [2], [3]. Employing Theorem 4.3 and denoting the approximate solution of the problem in the form (4.59) by $u_{h,\eta,Y}$, we have finally

$$\sup_{x \in \langle 0,\pi \rangle} |u_{h,\eta,Y}(x) - u(x)| \leqq C(\beta, \varepsilon)\, h^{\beta+2-\varepsilon}\|f\|_{W_2^\beta(0,\pi)}\,.$$

The universal hill function

$$\omega(x) = e^{-x^2}$$

was used for approximation. Because

$$\Lambda(x) = \tilde{\omega}(x) = e^{-t^2/4}\sqrt{\pi}\,,$$

the Λ-admissible function $\eta(h)$ of the form (4.52) may be chosen according to Theorem 4.2. The function $\omega_Y(x)$ of the form (4.56) with $Y(h)$ given in (4.57) was used for actual computation according to Theorem 4.3.

The computation has been carried out in single precision on a Minsk 22 computer for various values of the parameters $\eta_0, \eta_1, \varepsilon_0, Y_0, Y_1, \varepsilon_1$ of the functions $\eta(h)$, $Y(h)$, and the parameters c, d of the problem. The system of linear algebraic equations was solved by Gauss elimination.

A typical result is shown in Fig. 5.1 where the scale of the variable $M = \pi/h$ is linear while the scale of the error is logarithmic. The actual values of the parameters used in computation of the solution in Fig. 5.1 are: $\eta_0 = 3{\cdot}8$, $\eta_1 = 0{\cdot}5$, $\varepsilon_0 = 0{\cdot}0001$, $Y_0 = \frac{10}{3}$, $Y_1 = \frac{2}{3}$, $\varepsilon_1 = 0{\cdot}0001$, $c = 0{\cdot}25$, and $d = 1$ and 3. The error of the solution is measured by the two quantities,

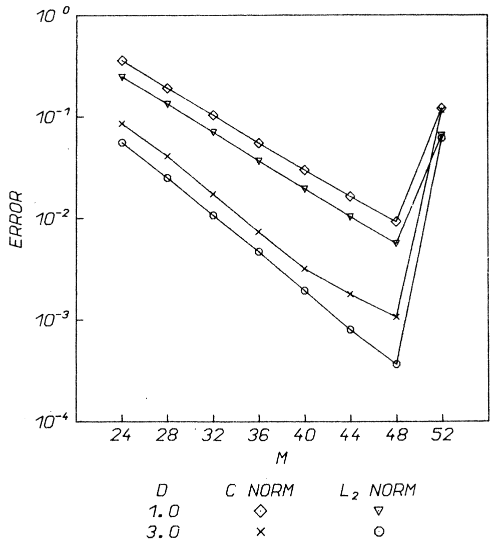

Fig. 5.1.

$$Q_1 = U \max_{0 \leqq j \leqq M} |u_{h,\eta,Y}(jh) - u(jh)|$$

(denoted by "C norm" in the figure) and

$$Q_2 = U((M+1)^{-1} \sum_{j=0}^{M} |u_{h,\eta,Y}(jh) - u(jh)|^2)^{1/2}$$

(denoted by "L_2 norm") where $U^{-1} = \max_{x \in \langle 0, \pi \rangle} |u(x)|$.

The graph shows that the error decreases (as $M \to \infty$) rapider than any polynomial of a finite degree. For $M > 48$, the error increases due to the round-off.

In general, the dependence of the error on the choice of $Y(h)$ has shown rather weak. On the other hand, an appropriate choice of $\eta(h)$ can influence the results very strongly.

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