Czechoslovak Mathematical Journal

Charles L. Belna

On the 3-segment property for complex-valued functions

Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 2, 238–241

Persistent URL: http://dml.cz/dmlcz/101095

Czechoslovak Mathematical Jornal, 22 (1972), Praha

ON THE 3-SEGMENT PROPERTY FOR COMPLEX-VALUED FUNCTIONS*)

CHARLES L. BELNA, Dayton

(Received January 6, 1971)

1. A function f from the open upper half plane H into the Riemann sphere W is said to have the 3-segment property at the point p on the real line R if there exist three segments in H emanating from p such that the intersection of the cluster sets of f at p relative to these segments is empty. The following question was posed in [1]: does there exist a continuous function from H into W having the 3-segment property at each point of a set of positive measure or of second category on R? In partial answer to this question, we introduce the 3-segment property relative to two direction functions and show that the set of points at which a continuous function has this property is of first category and measure zero on R.

2. For $s \in (0, \pi)$ and $p \in R$, the set $\{z : \arg(z - p) = s\}$ is called a *segment at* p and s is said to be its *direction*. A monotone function from R into the open interval $(0, \pi)$ which is absolutely continuous on finite intervals is called a *direction function*. Throughout this paper λ_1 and λ_2 will denote two direction functions, while $S_1(p)$ and $S_2(p)$ will denote the segments at $p \in R$ whose corresponding directions dir $S_1(p)$ and dir $S_2(p)$ are equal to $\lambda_1(p)$ and $\lambda_2(p)$, respectively.

If f is a function from H into W, the *cluster set* $CS_j(p)$ *of* f *at* $p \in R$ *relative to the segment* $S_j(p)$ is defined to be the set of all points $w \in W$ for which there exists a sequence $\{z_k\} \subset S_j(p)$ with $z_k \to p$ and $f(z_k) \to w$.

Definition. The function $f : H \to W$ has the 3-segment property at $p \in R$ relative to the direction functions λ_1 and λ_2 if there exists a segment $S_3(p)$ at p distinct from the segments $S_1(p)$ and $S_2(p)$ such that

$$CS_1(p) \cap CS_2(p) \cap CS_3(p) = \emptyset .$$

*) This paper is a generalization of a portion of the author's doctoral thesis written at Michigan State University under the direction of Professor PETER LAPPAN. The research was performed in part at the USAF Aerospace Research Laboratories while in the capacity of an Ohio State University Research Foundation Research Analyst under Contract F33615-67-C-1758.

Theorem. *If $f : H \to W$ is continuous, the set Q of points at which f has the 3-segment property relative to the direction functions λ_1 and λ_2 is of first category and measure zero on R.*

To prove this theorem we make use of the following lemma (for a related result see [2, p. 158]). Before stating the lemma we introduce some terminology: The point $p \in R$ is a *point of density* of a measurable set $E \subset R$ if

$$\lim_{a \to 0^+} \frac{m[E \cap (p - a, p + a)]}{2a} = 1 ,$$

where m denotes linear Lebesgue measure; and p is a *point of right density* of E if

$$\lim_{a \to 0^+} \frac{m[E \cap (p, p + a)]}{a} = 1 .$$

A measurable function $g : R \to R$ is (*right*) *approximately continuous at* $p^* \in R$ if there exists a set $E \subset R$ such that (1) p^* is a point of (right) density of E and (2) $f(p) \to f(p^*)$ as $p \to p^*$ provided $p \in E$.

Lemma. *Let $g : [0, +\infty) \to [0, +\infty]$ be monotone and absolutely continuous on $[0, a]$ for some $a > 0$ with $g(0) = 0$ and $g'(0) > 0$, and let g' be right approximately continuous at 0. If 0 is a point of right density of a measurable set E, then 0 is a point of right density of $g(E)$.*

Proof. Choose a set $F \subset [0, +\infty)$ such that 0 is a point of right density of F and $g'(p) \to g'(0)$ as $p \to 0$ provided $p \in F$. Set $E^* = E \cap F$, and let ϕ denote the characteristic function of E^*. Then for each $p \in (0, a)$ we have

$$\frac{m[g(E) \cap (0, g(p))]}{g(p)} \geqq \frac{m[g(E^*) \cap (0, g(p))]}{g(p)} = \frac{1}{g(p)} \int_0^{g(p)} \phi(g^{-1}(\sigma))\, d\sigma =$$

$$= \frac{1}{g(p)} \int_0^p \phi(\tau)\, g'(\tau)\, d\tau = \frac{g'(0)}{g(p)} \int_0^p \phi(\tau)\, d\tau + \frac{1}{g(p)} \int_0^p \phi(\tau)\, [g'(\tau) - g'(0)]\, d\tau .$$

Let $\varepsilon > 0$ be given, and choose a point $p_\varepsilon < a$ so that (1) $|g'(\tau) - g'(0)| < \varepsilon g'(0)/2$ for all $\tau \in [0, p_\varepsilon] \cap E^*$ and (2) $g(p)/p \geqq g'(0)/2$ for all $p \in (0, p_\varepsilon] \cap E^*$. Then for all $p \in (0, p_\varepsilon]$,

$$\left| \frac{1}{g(p)} \int_0^p \phi(\tau)\, [g'(\tau) - g'(0)]\, d\tau \right| < \varepsilon ;$$

and so

$$\lim_{p \to 0^+} \frac{1}{g(p)} \int_0^p \phi(\tau)\, [g'(\tau) - g'(0)]\, d\tau = 0 .$$

Consequently

$$\lim_{p\to 0^+} \frac{m[E \cap (0, g(p))]}{g(p)} \geqq \lim_{p\to 0^+} \frac{g'(0)}{g(p)} \int_0^p \phi(\tau)\, d\tau =$$

$$= g'(0) \lim_{p\to 0^+} \left(\frac{p}{g(p)}\right)\left(\frac{m[E^* \cap (0, p)]}{p}\right) = 1\,;$$

hence the lemma is proved.

Proof of theorem. Let $\mathscr{B}$ be a countable basis for the usual topology on W, and let $\mathscr{G}$ be the countable collection of all finite unions of the sets $B \in \mathscr{B}$. Then let $\mathscr{G}^*$ be the set of all 3-tuples (G_1, G_2, G_3) of sets in $\mathscr{G}$ for which $\bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3 = \emptyset$, where $\bar{G}_j$ denotes the closure of G_j. For each 3-tuple (G_1, G_2, G_3) in $\mathscr{G}^*$, each pair of rational numbers α, β satisfying $0 < \alpha < \beta < \pi$, and each positive rational $r \leqq 1$, let $Q(G_1, G_2, G_3; \alpha, \beta; r)$ be the set of points $p \in R$ at which there exists a segment $S_3(p)$ such that

(1) $$\alpha \leqq \operatorname{dir} S_3(p) \leqq \beta$$

(2) $$\lambda_j(p) \notin (\alpha - r, \beta + r) \quad (j = 1, 2)$$

and

(3) $$f(S_j(p, r)) \subset \bar{G}_j \quad (j = 1, 2, 3)\,,$$

where $S_j(p, r) = S_j(p) \cap \{z : \operatorname{Im}(z) \leqq r\}$. One can easily see that each of the sets $Q(G_1, G_2, G_3; \alpha, \beta; r)$ is closed and that Q is the countable union of all of them. Thus, if we let Q_0 denote the set $Q(G_1, G_2, G_3; \alpha, \beta; r)$, it suffices to show that Q_0 is of measure zero on R.

Let Q_0^* be the set of points in Q_0 which are points of density of Q_0, and assume that $mQ_0^* > 0$. Since λ_j' $(j = 1, 2)$ is approximately continuous a.e. on R, there exists a point $p_0 \in Q_0^*$ at which λ_1' and λ_2' are approximately continuous. For convenience we take $p_0 = 0$ and assume that $\operatorname{dir} S_3(0) > \operatorname{dir} S_j(0)$ $(j = 1, 2)$, where $S_3(0)$ is the segment at 0 guaranteed by $0 \in Q_0$. For $j = 1, 2$ define the function $\mu_j : (0, +\infty) \to \to S_3(0)$ by

$$\{\mu_j(p)\} = S_3(0) \cap S_j(p)\,.$$

Then define the function $g_j : [0, +\infty) \to [0, +\infty)$ by $g_j(p) = |\mu_j(p)|$ for $p > 0$ and $g_j(0) = 0$. It is easily verified that near 0

$$g_j(p) = p\{\sin \lambda_j(p)/\sin [\lambda_j(p) - \operatorname{dir} S_3(0)]\}$$

and that g_j satisfies the hypotheses of the lemma for $j = 1, 2$. It follows that 0 is a point of right density for the set $g_1(Q_0) \cap g_2(Q_0)$. Consequently there exist points $p_1, p_2 \in Q_0 \cap (0, +\infty)$ such that

$$\mu_1(p_1) = \mu_2(p_2) \in S_3(0, r)\,.$$

Setting $z = \mu_1(p_1)$ we have

$$f(z) \in \bar{G}_1 \cap \bar{G}_2 \cap \bar{G}_3$$

which contradicts $(G_1, G_2, G_3) \in \mathscr{G}^*$. Thus we must have $mQ_0^* = 0$, and it follows from the Lebesgue Density Theorem that $mQ_0 = 0$. This completes the proof of the theorem.

Remark. In effect BAGEMIHL, PIRANIAN and YOUNG [1, p. 30] have constructed a continuous function from H into W having the 3-segment property relative to the constant direction functions $\lambda_1(p) \equiv \pi/4$ and $\lambda_2(p) \equiv \pi/2$ at each point of the Cantor "middle halves" set in $[0, 1]$; hence the set Q of the above theorem need not be countable.

References

[1] *F. Bagemihl, G. Piranian,* and *G. S. Young,* Intersections of cluster sets, Bul. Inst. Politehn. Iaşi (N.S.) *5* (*9*), (1959), no. 3—4, 29—34.

[2] *J. L. Doob,* The boundary values of analytic functions, Trans. Amer. Math. Soc. *34* (1932), 153—170.

Author's address: Wright State University, Dayton, Ohio 45431, U.S.A.