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KINEMATIC GEOMETRY IN n-DIMENSIONAL EUCLIDEAN AND SPHERICAL SPACE

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In the present paper the autor gives a full system of invariants of the one parametric motion in S^{m-1} and E^m in the "general" case. The Frenet formulas for motion are obtained; in the classical case they reduce to the well known facts. The paper is a continuation of papers [7] and [8] by a slightly different method, but in terms of the theory of compact Lie algebras. By author's method it is possible to study the motion in other compact Lie groups, not only in $O(m)$ (and then in $E(m)$). At the end some geometric applications are given.

Part I

INVARIANTS OF CURVES IN G/H, WHERE G IS $O(m)$ OR $E(m)$ AND $\mathfrak{H}$ IS A CARTAN SUBALGEBRA OF $\mathfrak{G}$

1. Notions

Let $O(m)$ be the group of all orthogonal transformations of the m-dimensional Euclidean vector space V^m. Let $E(m)$ be the group of all congruencies of the m-dimensional Euclidean space E^m. Let $\mathfrak{O}(m)$, $\mathfrak{E}(m)$ be the Lie algebras of the group $O(m)$ and $E(m)$ respectively. Throughout the paper we shall use the consequent symbolic notation for the groups $O(m)$ and $E(m)$ in the fixed frame $\mathscr{R}_0$ in V^m or E^m:

$\mathbf{A}^{10}$ and $\mathbf{A}^{12}$ will be columns of the type $n \times 1$

$\mathbf{A}^{21}$ will be a row of the type $1 \times n$

$\mathbf{A}^{20}$ and $\mathbf{A}^{22}$ will be real numbers (matrices of the type 1×1)

$\mathbf{A}^{11}$ will be a matrix of the type $n \times n$.

If the elements of the above matrices are matrices of the type

2×1 for the indices 10 and 12
1×2 for the index 21
2×2 for the index 11,

and not real numbers, we shall write small letters instead of capitals. In this case we shall write lower-case letters for real numbers with indices 20 or 21, too. For instance: $\mathbf{T}^{10}$ is a column of n real numbers, $\mathbf{b}^{21}$ is a row of the type $\mathbf{b}^{21} = \{b_1, b_1'; \dots, b_n, b_n'\}$ and so on. If this agreement is used, we can write the elements of $O(m)$ or $E(m)$ as matrices $\mathbf{a} \equiv (\mathbf{a}^{ij})$, where

$$\text{(1)} \qquad \begin{aligned} &\text{a) } i, j = 1 && \text{for } O(m) = O(2n) \\ &\text{b) } i, j = 1, 2 && \text{for } O(m) = O(2n + 1) \\ &\text{c) } i, j = 0, 1 && \text{for } E(m) = E(2n) \\ &\text{d) } i, j = 0, 1, 2 && \text{for } E(m) = E(2n + 1) . \end{aligned}$$

In c) and d) it is $\mathbf{a}^{00} = 1$, $\mathbf{a}^{01}$ is a row of the type $1 \times n$ with every element a zero row of the type 1×2. $\mathbf{a}^{02} = 0$ in case d). We can write the Lie algebras $\mathfrak{O}(m)$ and $\mathfrak{E}(m)$ of the groups $O(m)$ and $E(m)$ in a quite analogous way. Here we certainly have for $a \in \mathfrak{O}(m)$ or $a \in \mathfrak{E}(m)$ (a^* is the transpose matrix of a; $a_{\alpha\beta}$ are elements of the matrix a; $\alpha, \beta = 1, \dots, n$):

$$\mathbf{a}^{0j} = 0 , \quad j = 0, 1, 2 ; \quad (\mathbf{a}^{12*}) + (\mathbf{a}^{21}) = 0 ; \quad \mathbf{a}^{22} = 0 ; \quad \mathbf{a}^{11}_{\alpha\beta} + (\mathbf{a}^{11}_{\beta\alpha})^* = 0$$

for indices as in (1).

Let us denote for latter use:

$\mathbf{d}^{11}\{\varrho_1, \dots, \varrho_n\}$ is a diagonal matrix and its elements on the diagonal are the matrices $\varrho_1, \dots, \varrho_n$ of type 2×2,

$\mathbf{D}^{11}\{\lambda_1, \dots, \lambda_n\}$ is a diagonal matrix with the real numbers $\lambda_1, \dots, \lambda_n$ on the diagonal.

$$\text{(2)} \qquad \sigma_\alpha \equiv (\varphi\varepsilon)(\alpha) = \begin{pmatrix} \cos \varphi_\alpha, & \varepsilon(\alpha) \sin \varphi_\alpha \\ -\sin \varphi_\alpha, & \varepsilon(\alpha) \cos \varphi_\alpha \end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\text{(3)} \qquad \mathbf{e}_\alpha \equiv \mathbf{e}_\alpha^{11} = \mathbf{d}^{11}\{\varrho_\beta\} , \quad \text{where} \quad \varrho_\beta = 0 \quad \text{for} \quad \alpha \neq \beta \quad \text{and} \quad \varrho_\alpha = \mathbf{e} .$$

1.1. Definition. (See [2], Chap. 3, §1.) a) Let $\mathfrak{G}$ be a Lie algebra and let $\mathfrak{H}$ be its subalgebra. The normalizer $N(\mathfrak{H})$ of $\mathfrak{H}$ in $\mathfrak{G}$ is the subalgebra

$$N(\mathfrak{H}) = \{X \in \mathfrak{G} \mid [X, \mathfrak{H}] \in \mathfrak{H}\} .$$

b) A subalgebra $\mathfrak{H}$ of a Lie algebra $\mathfrak{G}$ is called a Cartan subalgebra if it is nilpotent and $N(\mathfrak{H}) = \mathfrak{H}$.

c) Let G be a Lie group with the Lie algebra $\mathfrak{G}$ and let a subalgebra $\mathfrak{H}$ in $\mathfrak{G}$ be given. Then the normalizer $\mathcal{N}(\mathfrak{H})$ of $\mathfrak{H}$ in G is $\mathcal{N}(\mathfrak{H}) = \{g \in G \mid \text{ad } g\mathfrak{H} = \mathfrak{H}\}$.

1.2. Proposition. (See [3], Chap. 9, § 65.) *Let us denote*

$$\mathbf{e}'_\alpha = \begin{pmatrix} \mathbf{e}_\alpha, & 0 \\ 0, & 0 \end{pmatrix}, \tag{4}$$

where $\mathbf{e}_\alpha$ *is defined as in* (2). *Then* $\mathbf{e}_\alpha$, $\mathbf{e}'_\alpha$, $\alpha = 1, \ldots, n$ *are bases of some Cartan subalgebras of* $\mathfrak{O}(2n)$, $\mathfrak{O}(2n+1)$ *respectively.*

We shall denote the Cartan subalgebras of Proposition 1.2 respectively by $\mathfrak{o}(2n)$, $\mathfrak{o}(2n+1)$. For latter use we rewrite from [3], theorem 111: $\mathfrak{O}(2n+1)$ is a compact simple Lie algebra of the type B_n for $n \geqq 1$, $\mathfrak{O}(2n)$, $n \geqq 3$ is a compact simple Lie algebra of the type D_n. They both have the rank n and $\mathbf{e}_\alpha$, $\mathbf{e}'_\alpha$ are orthogonal bases in $\mathfrak{o}(2n)$, $\mathfrak{o}(2n+1)$ respectively, consisting of vectors of the same length. The roots are $\{\pm e'_\alpha; \pm e'_\alpha \pm e'_\beta, \alpha < \beta; \alpha, \beta = 1, \ldots, n\}$ for $\mathfrak{O}(2n+1)$ and $\{\pm e_\alpha \pm e_\beta, \alpha < \beta; \alpha, \beta = 1, \ldots, n\}$ for $\mathfrak{O}(2n)$.

1.3. Lemma. *The algebras of matrices*

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{x}^{11} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{x}^{11} & 0 \\ y^{02} & 0 & 0 \end{pmatrix} \tag{5}$$

where $y^{02} \in \mathbf{R}$, $\mathbf{x}^{11} \in \mathfrak{o}(2n)$, *are Cartan subalgebras of* $\mathfrak{E}(2n)$ *or* $\mathfrak{E}(2n+1)$, *respectively.*

Proof. The lemma is an easy consequence of Definition 1.1 and Proposition 1.2.

2. Groups $\mathcal{N}(\mathfrak{o}(m))$ and $\mathcal{N}(\mathfrak{e}(m))$

In this section we shall give an explicit construction of the groups $\mathcal{N}(\mathfrak{o}(m))$ and $\mathcal{N}(\mathfrak{e}(m))$. We shall start by repeating some known facts from the theory of compact Lie groups.

Let G be a compact semisimple Lie group and let $\mathfrak{G}$ be its Lie algebra. Let us denote by Z the centre of G, let G_0 be the connected component of the neutral element of G. The image of G in the adjoint representation will be denoted by ad$G$. Then we have ad$G \sim G/Z$ and ad$G_0 \sim G_0/G_0 \cap Z$. Let us suppose that the group ad$G$ of G is identical with the group of all automorphisms of $\mathfrak{G}$. This means that every automorphism of $\mathfrak{G}$ is realizable as ad$g$ for some $g \in G$. (If this automorphism is outer, g certainly cannot be in G_0.)

Let us consider a maximal torus T of the group G with the Lie algebra $\mathfrak{t}$. If $h \in \mathcal{N}(\mathfrak{t})$, then the restriction of ad$h$ on $\mathfrak{t}$ will be denoted by $\vartheta(h)$; $\vartheta(h) = \text{ad}h|_{\mathfrak{t}}$. ϑ is a homomorphism of $\mathcal{N}(\mathfrak{t})$ into the group of linear transformations of the vector space $\mathfrak{t}$. Let us denote by K the kernel of this homomorphism.

2.1. Definition. (See [4], Chap. 13.) An orthogonal transformation $\mu : \mathfrak{t} \to \mathfrak{t}$ is called a rotation, if it leaves invariant the system of all roots.

Let us denote by W' the group of all rotations. Let W be the Weyl group of $\mathfrak{G}$. W is the group generated by all symmetries $s_\alpha : x \to x - 2(x, \alpha)(\alpha, \alpha)^{-1} . \alpha$ of $\mathfrak{t}$, where α is a root, $x \in \mathfrak{t}$ and (x, α) is the scalar multiplication in $\mathfrak{t}$.

2.2. Theorem. (*See* [1], *Chap.* 9, §2.) *Let A be an automorphism of $\mathfrak{G}$ leaving $\mathfrak{t}$ invariant. Then the restriction of A on $\mathfrak{t}$ is a rotation. Conversely, every rotation in $\mathfrak{t}$ can be extended to an automorphism of $\mathfrak{G}$.*

2.3. Theorem. (*See* [1], *Chap.* 9, § 2.) *W is an invariant subgroup of W' and* ad $G/$ad $G_0 \sim W'/W$.

2.4. Theorem. (*See* [6], *Chap.* 8, § 11.) *Let $\mathfrak{G}$ be a compact simple Lie algebra Let us denote* $\mathrm{E}(\mathfrak{G}) = \mathrm{Aut}\mathfrak{G}/\mathrm{Int}\mathfrak{G}$. *Then we have*

$\mathrm{E}(A_l) = Z_2, \ l > 1$

$\mathrm{E}(D_l) = Z_2, \ l \neq 4$

$\mathrm{E}(D_4) = D_3$

$\mathrm{E}(E_6) = Z_2$, *and the other groups are trivial. Here Z_2 is the cyclic group of order* 2, *D_3 is the trihedral group.*

From theorem 2.2 we have (as $\mathrm{Aut}\mathfrak{G} \equiv \mathrm{ad}G$):

$$\mathcal{N}(\mathfrak{t})/K \sim W' . \tag{6}$$

We shall now find the group W' of the algebra $\mathfrak{o}(m)$.

a) Let us suppose first that $m = 2n$. Then the group W has the generators

$$S_{e_\alpha - e_\beta} : e_\alpha \to e_\beta , \quad e_\beta \to \quad e_\alpha , \quad e_\gamma \to e_\gamma$$

$$S_{e_\alpha + e_\beta} : e_\alpha \to e_\beta , \quad e_\beta \to -e_\alpha , \quad e_\gamma \to e_\gamma$$

where

$$\gamma \neq \alpha , \quad \gamma \neq \beta , \quad \alpha \neq \beta , \quad \alpha, \beta, \gamma = 1, \ldots, n .$$

Let $n \neq 4$, $n \neq 1$. Then $W'/W = Z_2$ from Theorem 2.4. Let us note that the mapping $Sg_\alpha : e_\alpha \to -e_\alpha$, $e_\beta \to e_\beta$, where $\alpha \neq \beta$; $\alpha, \beta = 1, \ldots, n$, is not in W. From these facts we can see that $s_{e_\alpha \pm e_\beta}$, Sg_α are the generators of W' for $n \neq 4$.

b) Let now $m = 2n + 1$. We have $W' = W$ from Theorem 2.4 and W' has the generators $s_{e_\alpha' - e_\beta'}$, $s_{e_\alpha' + e_\beta'}$, $s_{e_\alpha'} \equiv Sg_\alpha$. The group W' is then the same as in a), but the rotations Sg_α are now induced by inner automorphisms.

2.5. Lemma. *The group W' regarded as a group of linear transformations of the vector space $\mathfrak{o}(2n)$, $n \neq 4$, $n \geqq 1$ or $\mathfrak{o}(2n + 1)$, $n \geqq 1$ with the base e_α from* (3) *or e'_α from* (4) *is given by the matrices*

$$\mathbf{A}^{11} = (X_{\alpha\beta}), \quad \text{where} \quad X_{\alpha\beta} = \varepsilon(\alpha)\, \delta_{\alpha\pi^{-1}(\beta)} \tag{7}$$

$\alpha, \beta = 1, \ldots, n$, δ is Kronecker's delta and π is an arbitrary permutation of numbers $1, \ldots, n$. ε is an arbitrary function on the set $\{1, \ldots, n\}$ assuming values in the set $\{1, -1\}$.

Proof. The lemma is an easy corollary of the above discussion.

2.6. Lemma. *The group $sg = \{X_{\alpha\beta} \in W' \mid X_{\alpha\beta} = \varepsilon(\alpha)\, \delta_{\alpha\beta}\}$ is an invariant subgroup of W.*

Finally we shall find the group K. If we note that a matrix from K must be diagonal, we find that the group K is given by matrices

$$\mathbf{d}^{11}(\varphi_\alpha) \quad \text{for} \quad O(2n) \quad \text{and} \quad \begin{pmatrix} \mathbf{d}^{11}(\varphi_\alpha) & 0 \\ 0 & \varepsilon \end{pmatrix} \quad \text{for} \quad O(2n+1), \tag{8}$$

where $\varphi_\alpha = (\varphi\varepsilon)(\alpha)$ from (2) with $\varepsilon(\alpha) = 1$, $\varepsilon = \pm 1$, $\alpha = 1, \ldots, n$.

In the case of the group $O(2n)$ we can easily find an outer automorphism of the algebra $\mathfrak{O}(2n)$, which is of the type ad$g$ for some $g \in O(2n)$. We can now use formula (5) and the groups $\mathcal{N}(\mathfrak{o}(m))$ are described for $m \neq 8$.

For the group $O(8)$ the assumption Aut$\mathfrak{G} \equiv$ ad$G$ is not satisfied as $G/G_0 \sim$ $\sim$ ad$G$/ad$G_0 \sim Z_2$ because the centre Z of G is in G_0 and $W'/W \sim D_3$. Instead of the group W' we have to use its subgroup, generated by the transformations $s_{\alpha\pm\beta}$ and Sg_α. This group will be denoted by $\overline{W}(8)$. For the group $\mathcal{N}(\mathfrak{o}(8))$ we then have

$$\mathcal{N}(\mathfrak{o}(8))/K \sim \overline{W}(8).$$

2.7. Theorem. *The group $\mathcal{N}(\mathfrak{o}(m))$ is given by the matrices*

$$g = \mathbf{a}^{11} \equiv \mathbf{a}_{\alpha\beta} \quad \text{for} \quad m = 2n \quad \text{and} \quad g = \begin{pmatrix} \mathbf{a}^{11} & 0 \\ 0 & \mathbf{a}^{22} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{a}_{\alpha\beta} & 0 \\ 0 & \nu \end{pmatrix}$$
$$\text{for} \quad m = 2n + 1,$$

where

$$\mathbf{a}_{\alpha\beta} = \delta_{\alpha\pi^{-1}(\beta)} \cdot (\varphi\varepsilon)(\alpha), \quad (\varphi\varepsilon)(\alpha) \quad \text{are from (2) and} \quad \nu = \pm 1. \tag{9}$$

In this notation we have $\vartheta(g) = (X_{\alpha\beta})$, where $X_{\alpha\beta} = \varepsilon(\alpha)\, \delta_{\alpha\pi^{-1}(\beta)}$.

Proof. Let us denote $\tilde{\varepsilon}(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon(\alpha) \end{pmatrix}$. Then we shall obtain

$$\mathbf{a}_{\alpha\beta} = \mathbf{d}^{11}(\varphi_\alpha) \cdot \tilde{\varepsilon}(\alpha)\, \delta_{\alpha\pi^{-1}(\beta)} \tag{10}$$

as a product of matrices. The rest is obvious.

2.8. Lemma. *Let us denote*

(11) $$\mathbf{e}''_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{e}_\alpha \end{pmatrix}, \quad \mathbf{e}'''_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{e}_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where $\mathbf{e}_\alpha$ has the same meaning as in (3). *Then $\mathbf{e}''_\alpha$ is a base of a Cartan subalgebra in $\mathfrak{E}(2n)$; $\mathbf{e}'''_\alpha$, $\mathbf{f}$ is a base of a Cartan subalgebra in $\mathfrak{E}(2n+1)$, $\alpha = 1, \ldots, n$.*

2.9. Lemma. *The group W' for the groups $E(2n)$ and $E(2n+1)$ is given respectively by matrices*

(12) $$(X_{\alpha\beta}) \begin{pmatrix} X_{\alpha\beta} & 0 \\ 0 & v \end{pmatrix},$$

where $X_{\alpha\beta}$ and v have the same meaning as in (7). *In the case $E(8)$ we have to use the group $\overline{W}$ as in the case $O(8)$.*

Proof. The lemma can be easily verified by direct calculation.

2.10. Theorem. *The groups $\mathcal{N}(\mathfrak{e}(2n))$ and $\mathcal{N}(\mathfrak{e}(2n+1)$ are given respectively by the matrices*

(13) $$g = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{a}^{11} \end{pmatrix} \quad and \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^{11} & 0 \\ a^{20} & 0 & v \end{pmatrix},$$

where $\mathbf{a}^{11}$ and v have the same meaning as in (9) *and $a^{20} \in \mathbf{R}$.*

Proof. The proof is a direct calculation with use of Theorem 2.7.

2.11. Lemma. *Let us denote by sg the subgroup of W', for W' from* Lemma 2.9, *given by $X_{\alpha\beta} = \varepsilon(\alpha)\,\delta_{\alpha\beta}$. Then sg is an invariant subgroup of W'.*

2.12. Theorem. *Let us denote by $P(\mathfrak{o})$ and $P(\mathfrak{e})$ the group $\vartheta^{-1}(sg)$ for $O(m)$ and $E(m)$ respectively. Then $P(\mathfrak{o})$ is an invariant subgroup of $\mathcal{N}(\mathfrak{o})$ and $P(\mathfrak{e})$ is an invariant subgroup of $\mathcal{N}(\mathfrak{e})$.*

Proof. The theorem is a consequence of Lemmas 2.6 and 2.11.

3. Equivalence of curves

Let G/H be a homogeneous space and let a differentiable immersion $x(t) : I \to G/H$ of an interval I of real numbers in G/H be given. We shall call $x(t)$ a curve in G/H. A lift $g(t)$ of the curve $x(t)$ is a differentiable mapping $g(t) : I \to G$, such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\ \pi\ } & G/H \\ & {\scriptstyle g}\nwarrow \quad \nearrow{\scriptstyle x} & \\ & I & \end{array}$$

Let us denote by ϑ the left invariant Maurer-Cartan form on G. Then every lift $g(t)$ of the curve $x(t)$ defines a 1-form ω on I with values in $\mathfrak{G}$ defined in the following manner (see [5]):

$$\omega = g_* \vartheta \tag{14}$$

where g_* is the differential of g. Let us be given another lift $\tilde{g}(t)$ of the curve $x(t)$, $\tilde{g}(t) = g(t)\, h(t)$, where $h(t) \in H$. Then we have an immersion $h(t) : I \to H$ and the form $h_* \vartheta$ which is usually denoted by $h^{-1}\, \mathrm{d}h$. Let us denote $\tilde{\omega} = \tilde{g}_* \vartheta$. Then we have (see [5])

$$\tilde{\omega} = \mathrm{ad} h^{-1} \omega + h^{-1}\, \mathrm{d}h\,. \tag{15}$$

Let now a faithful representation ϱ of the group G of degree n be given. Let us suppose that we have chosen a frame $\mathscr{R}_0 = \{e_1, \ldots, e_n\}$ in a vector space V^n of dimension n and let us denote $\mathscr{R} = \mathscr{R}_0\, \varrho(g)$ as a product of a row and a matrix. Then the group $\varrho(G)$ acts simply transitively on the set $\mathscr{A}$ of all frames of the type $\mathscr{R} = \mathscr{R}_0\, \varrho(g)$ for some $g \in G$. A frame $\mathscr{R}$ is an n-tuple of vector functions on G with values in V and we have for 1-forms $\mathrm{d}\mathscr{R}$ and $\mathrm{d}(\varrho(g))$ defined on G: $\mathrm{d}\mathscr{R} = \mathscr{R}_0\, \mathrm{d}(\varrho(g))$. Now we may write $\mathrm{d}\mathscr{R} = \mathscr{R}_0\, \varrho(g)\, \varrho(g)^{-1}\, \mathrm{d}(\varrho(g)) = \mathscr{R}\, \varrho(\vartheta)$ since $\varrho(g)^{-1}\, \mathrm{d}(\varrho(g))$ is the Maurer-Cartan form of the group $\varrho(G)$, and ϱ is a faithful representation. We have

$$\mathrm{d}\mathscr{R} = \mathscr{R}\, \varrho(\vartheta)\,.$$

If $g(t) : I \to G$ is a lift of some curve, we have obtained

$$\mathrm{d}\mathscr{R} = \mathscr{R}\, \varrho(\omega)\,, \tag{16}$$

where ω and $\mathrm{d}\mathscr{R}$ are forms on I.

For latter applications we need a slightly different definition of the equivalence of curves.

3.1. Definition. Let the curves $x_i(t_i) : I_i \to G/H$, $i = 1, 2$, be given and let $O \in I_1 \cap I_2$. The curves $x_1(t_1)$ and $x_2(t_2)$ are called equivalent if there exist a difeomorphism φ of I_1 onto I_2 and $g \in G$ such that $\varphi(0) = 0$, $\varphi'(t_1) > 0$ for $t_1 \in I_1$ and the diagram

$$\begin{array}{ccc} & G/H & \\ {}^{gx_1}\nearrow & & \nwarrow^{x_2} \\ I_1 & \xrightarrow{\ \varphi\ } & I_2 \end{array} \tag{17}$$

commutes.

In the sequel we shall solve the equivalence problem from Definition 3.1 for some special curves for $G = O(m)$ and $G = E(m)$.

3.2. Lemma. *Let H be a transitive group of transformations of the set M, let H_1 be an invariant subgroup of H. Let us denote by M_ι, $\iota \in \Omega$ the orbits of the group H_1 on M. Then H/H_1 is a transitive group of transformations of the set $\{M_\iota\}$ of all orbits.*

Let now $\mathbf{E}$ be the unit matrix of the type 2×2 and let us denote $\mathbf{x} = \mathbf{d}^{11}\{\mathbf{E}\delta_{\alpha\pi^{-1}(\beta)}\}$ where π is a permutation of numbers $\{1, \ldots, n\}$. Then we can see from Theorem 2.12 and from (7) and (12) that the matrices

$$(\mathbf{x}) \quad \text{or} \quad \begin{pmatrix} \mathbf{x} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{x} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{x} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

represent respectively the cosets of $\mathcal{N}(\mathfrak{o})$ modulo $P(\mathfrak{o})$ and the cosets of $\mathcal{N}(\mathfrak{e})$ modulo $P(\mathfrak{e})$. The set of matrices (18) we shall denote by L.

Let us suppose that we have lifts $g(t)$ and $\tilde{g}(t)$ of the curve $x(t) : I \to G/H$, where $G = O(m)$ or $G = E(m)$ and $H = \mathcal{N}(\mathfrak{o}(m))$ or $H = \mathcal{N}(\mathfrak{e}(m))$ respectively. Let us write $\tilde{g}(t)\, h(t) = g(t)$, where $h(t) \in H$. If $h(t) \in L$, then $h(t) = \text{const.}$ and we have for the forms $\tilde{\omega}$ and ω of the lifts $\tilde{g}$ and g: $\tilde{\omega} = \operatorname{ad} h\omega$ and then

$$\tilde{\omega}^{10} = \mathbf{x}\omega^{10}, \quad \tilde{\omega}^{11} = \mathbf{x}\omega^{11}\mathbf{x}^{-1}, \quad \omega^{12} = \mathbf{x}\omega^{12}, \quad \tilde{\omega}^{20} = \omega^{20}, \quad \tilde{\omega}^{21} = \omega^{21}\mathbf{x}^{-1}.$$

Let us compute

$$\tilde{\omega}^{11} = \mathbf{x}\omega^{11}\mathbf{x}^{-1}.$$

We shall denote

$$\omega^{11} \equiv (\omega_{\beta\gamma}), \quad \mathbf{x} \equiv (\mathbf{x}_{\alpha\beta}) = \delta_{\alpha\pi^{-1}(\beta)}, \quad \mathbf{x}^{-1}_{\gamma\mu} = \delta_{\gamma\pi(\mu)}.$$

Then we can write

$$(\tilde{\omega}^{11})_{\alpha\mu} = (\mathbf{x}\omega^{11}\mathbf{x}^{-1})_{\alpha\mu} = \mathbf{x}_{\alpha\beta}\omega_{\beta\gamma}\mathbf{x}^{-1}_{\gamma\mu} = \tag{19}$$
$$= \delta_{\alpha\pi^{-1}(\beta)}\omega_{\beta\gamma}\delta_{\gamma\pi(\mu)} = \omega_{\pi(\alpha)\pi(\mu)}.$$

For $\tilde{\omega}^{10}$ we have analogously

$$(\tilde{\omega}^{10})_\alpha = (\mathbf{x}\omega^{10})_\alpha = \omega_{\pi(\alpha)} . \tag{20}$$

Remark. For the sake of simplicity we shall not distinguish between the form $\omega_{\alpha\beta}$ and its coefficient $\mathbf{a}_{\alpha\beta}$ in $\omega_{\alpha\beta} = \mathbf{a}_{\alpha\beta}\, dt$ and analogously for ω_α or ω^{10} and ω^{20}. Let us denote

$$|\omega^{11}|_\alpha = \sum_{\beta=1,\beta\neq\alpha}^{n} (\omega^{11}_{\alpha\beta})^2 , \quad \text{where} \quad (\omega^{11}_{\alpha\beta})^2 = \sum_{p,q=1}^{2} (\omega^{11}_{\alpha\beta;pq})^2 \tag{21}$$

$$|\omega^{10}|_\alpha = (\omega^{11}_{\alpha;1})^2 + (\omega^{10}_{\alpha;2})^2 , \quad |\omega^{12}|_\alpha = (\omega^{12}_{\alpha;1})^2 + (\omega^{12}_{\alpha;2})^2 .$$

In the rest of the paper we shall denote

$$O(m)/\mathcal{N}(\mathfrak{o}(m)) = M_1 \quad \text{and} \quad E(m)/\mathcal{N}(\mathfrak{e}(m)) = N_1 \quad \text{for} \quad m = 2n ,$$
$$O(m)/\mathcal{N}(\mathfrak{o}(m)) = M_2 \quad \text{and} \quad E(m)/\mathcal{N}(\mathfrak{e}(m)) = N_2 \quad \text{for} \quad m = 2n + 1 .$$

3.3. Theorem. *Let $x(t)$ be a curve in M_i or N_i, $i = 1, 2$. Let us suppose that the form ω of some lift $g(t)$ of $x(t)$ satisfies at the point $t = 0$*

$$\begin{aligned} &\textit{for } M_1: && |\omega^{11}|_\alpha \neq |\omega^{11}|_\beta \neq 0 , && \alpha \neq \beta \\ &\textit{for } N_1: && |\omega^{10}|_\alpha \neq |\omega^{10}|_\beta \neq 0 , && \alpha \neq \beta \\ &\textit{for } M_2 \textit{ and } N_2: && |\omega^{12}|_\alpha \neq |\omega^{12}|_\beta , && \alpha \neq \beta , \end{aligned} \tag{22}$$

where $\alpha, \beta = 1, \ldots, n$. Then there exists one $l \in L$ for every lift $g(t)$ of $x(t)$ such that for the form $\tilde{\omega}$ of the lift $g(t)\, l$ it is

$$\begin{aligned} &\textit{for } M_1: && |\omega^{11}|_\alpha > |\omega^{11}|_{\alpha+1} \\ &\textit{for } N_1: && |\omega^{10}|_\alpha > |\omega^{10}|_{\alpha+1} \\ &\textit{for } M_2 \textit{ and } N_2: && |\omega^{12}|_\alpha > |\omega^{12}|_{\alpha+1} \end{aligned} \tag{23}$$

at the point $t = 0$, where $\alpha = 1, \ldots, n - 1$.

Proof. If (22) is satisfied for some lift, it is satisfied for every lift. The construction of the lift $g(t)\, l$ is easy, as we can see from (19) and (20).

As we want to use Lemma 3.2, we must show that the lifts from Theorem 3.3 do not depend on the group $P(\mathfrak{o})$ or $P(\mathfrak{e})$. This is done in the following

3.4. Lemma. $(\omega^{11}_{\alpha\beta})^2$, $|\omega^{10}|_\alpha$, $|\omega^{12}|_\alpha$ *do not depend on the group P.*

Proof. Let $g(t)$ be a lift of some curve, let $\tilde{g}(t) = g(t)\, h(t)$ be another lift of this curve, where $h(t) \in P$. Let us denote $\omega = \omega^{ij}$, $\tilde{\omega} = \tilde{\omega}^{ij}$; i, j are from (1). Then for N_2 it is

$$h(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{d}^{11} & 0 \\ a^{20} & 0 & v \end{pmatrix}$$

where $\mathbf{d}^{11} = \mathbf{d}^{11}\{\sigma_\alpha\}$, $\alpha = 1, \ldots, n$; σ_α are from (2), $a^{20} \in \mathbf{R}$, $v = \pm 1$ and analogously for M_i and N_1. We then have for the forms ω and $\tilde{\omega}$ of the lifts g and $\tilde{g}$:

$$\tilde{\omega}^{10} = (\mathbf{d}^{11})^* \omega^{10} + (\mathbf{d}^{11})^* \omega^{12} a^{20}, \quad \tilde{\omega}^{12} = (\mathbf{d}^{11})^* \omega^{12} v \tag{24}$$
$$\tilde{\omega}^{11} = (\mathbf{d}^{11})^* \omega^{11} \mathbf{d}^{11} + h^{-1} \, \mathrm{d}h \qquad \tilde{\omega}^{20} = v\omega^{20}.$$

Denoting $\omega^{10} = \omega_\alpha$, $\omega^{12} = \vartheta_\alpha$, $\omega^{11} = \omega_{\alpha\beta}$ where $\alpha, \beta = 1, \ldots, n$, we have in the component form

$$\tilde{\omega}_\alpha = \sigma_\alpha^* \omega_\alpha + \sigma_\alpha^* \vartheta_\alpha a^{20}, \qquad \tilde{\vartheta}_\alpha = \sigma_\alpha^* \vartheta_\alpha v \tag{25}$$
$$\tilde{\omega}_{\alpha\beta} = \sigma_\alpha^* \omega_{\alpha\beta} \sigma_\beta \quad \text{for} \quad \alpha \neq \beta, \quad \tilde{\omega}^{20} = v\omega^{20}.$$

From (25) we can see the assertion of the lemma.

In the rest of the paper we shall suppose that the lifts considered satisfy Theorem 3.3. We shall now construct the Frenet lifts of curves.

a) Let us consider the manifold M_1, where $m = 2n$, $n = 2s$.

3.5. Lemma. *Let us denote* $\omega^{11}_{\alpha\beta;pq} \equiv \omega_{pq}$, $\alpha \neq \beta$, $p, q = 1, 2$, $v = \pm 1$. *Then* $(\omega_{11} + v\omega_{22})^2 + (\omega_{12} - v\omega_{21})^2$ *and* $|\det \omega_{pq}|$ *are invariants of the group* P *for* $\alpha, \beta = 1, \ldots, n$.

Proof. The proof is an easy calculation by means of (25).

3.6. Theorem. *Let a curve* $x(t)$ *in* M_1, $m = 4s$, *be given. Let us suppose that the form* ω *of some of its lifts satisfies*

$$|\omega^{11}|_\alpha + |\omega^{11}|_\beta \neq 0; \quad \alpha, \beta = 1, \ldots, n; \quad t = 0, \quad \alpha \neq \beta \tag{26}$$

$$\det \omega^{11}_{\alpha,\alpha+1} \neq 0; \quad \alpha = 1, \ldots, n-1; \quad t = 0 \tag{27}$$

$$(\omega^{11}_{2r-1,2r;11} + v\omega_{2r-1,2r;22})^2 + (\omega^{11}_{2r-1,2r;12} - v\omega^{11}_{2r-1,2r;21})^2 \neq 0; \tag{28}$$
$$v = \pm 1, \quad r = 1, \ldots, s.$$

Then there exist 2^{s+1} *lifts* $g(t)$ *of the curve* $x(t)$ *such that*

$$\omega^{11}_{2r-1,2r;21} = \omega^{11}_{2r-1,2r;12} = 0 \quad \textit{for} \quad r = 1, \ldots, s \tag{29}$$

and at the point $t = 0$ *(and hence in some neighbourhood of* 0*) it is*

$$|\omega|_\alpha > |\omega|_{\alpha+1}, \quad \alpha = 1, \ldots, n-1 \tag{30}$$

$$\omega^{11}_{2r-1,2r;11} > \omega^{11}_{2r-1,2r;22} > 0, \quad r = 1, \ldots, s \tag{31}$$

$$\det \omega_{\alpha,\alpha+1} > 0, \quad \alpha = 1, \ldots, n-1. \tag{32}$$

If $g_1(t)$ and $g_2(t)$ are two such lifts, it is $g_1(t) = g_2(t)\, h$, where $h = \mathbf{d}^{11}\{\varrho_\alpha\}$ and $\varrho_{2r-1} = \varrho_{2r} = \begin{pmatrix} \varepsilon_r & 0 \\ 0 & \varepsilon\varepsilon_r \end{pmatrix}$, $\varepsilon_r = \pm 1$, $\varepsilon = \pm 1$, $r = 1, \ldots, s$.

Remark. The lifts from Theorem 3.6 are defined on the same interval, as (28) is satisfied in spite of the fact that (30), (31) and (32) need not be satisfied there.

Proof of 3.6. Let us write $\omega^{11}_{2r-1,2r;pq} \equiv \omega_{pq}$ and let us suppose that (28) is satisfied. We then have

$$(33) \qquad (\omega_{11} + \nu\omega_{22})^2 + (\omega_{12} - \nu\omega_{21})^2 \neq 0\,, \quad \nu = \pm 1\,.$$

Let a $\sigma(t) = \mathbf{d}^{11}\{\sigma_\alpha\} \in P$ be given. Then for the forms ω and $\tilde{\omega}$ of the lifts $g(t)$ and $\tilde{g}(t) = g(t)\,\sigma(t)$ we have from (25)

$$(34) \qquad (\tilde{\omega}_{pq}) = \sigma^*_{2r-1}\,(\omega_{pq})\,.\,\sigma_{2r}\,.$$

Let us write $\varphi_{2r-1} = \varphi$, $\varphi_{2r} = \psi$, $\varepsilon(2r-1) = \varepsilon$, $\varepsilon(2r) = \eta$. Then we have in the component form

$$(35) \qquad \begin{pmatrix} \tilde{\omega}_{11}, & \tilde{\omega}_{12} \\ \tilde{\omega}_{21}, & \tilde{\omega}_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi, & -\sin\varphi \\ \varepsilon\sin\varphi, & \varepsilon\cos\varphi \end{pmatrix} \begin{pmatrix} \omega_{11}, & \omega_{12} \\ \omega_{21}, & \omega_{22} \end{pmatrix} \begin{pmatrix} \cos\psi, & \eta\sin\psi \\ -\sin\psi, & \eta\cos\psi \end{pmatrix}.$$

After some calculations we obtain

$$(36) \qquad \begin{aligned} \tilde{\omega}_{21} + \tilde{\omega}_{12} &= \sin(\varphi + \psi)(\omega_{11} - \varepsilon\eta\omega_{22}) + \varepsilon\cos(\varphi + \psi)(\omega_{21} + \varepsilon\eta\omega_{12}) \\ \tilde{\omega}_{21} - \tilde{\omega}_{12} &= \sin(\varphi - \psi)(\omega_{11} + \varepsilon\eta\omega_{22}) + \varepsilon\cos(\varphi - \psi)(\omega_{21} - \varepsilon\eta\omega_{12})\,. \end{aligned}$$

Let us set $\tilde{\omega}_{12} = \tilde{\omega}_{21} = 0$ or equivalently

$$(37) \qquad \tilde{\omega}_{21} + \tilde{\omega}_{12} = 0\,, \quad \tilde{\omega}_{21} - \tilde{\omega}_{12} = 0\,.$$

From (28) we see that in each equation in (37) at most one of the coefficients is equal to zero. Hence (37) can be always solved. Let us suppose that $\sigma(t)$ and $\sigma'(t)$ both solve (37). We have for the forms ω and $\tilde{\omega}$ of the lifts $g(t)\,\sigma(t)$ and $g(t)\,\sigma'(t)$

$$(38) \qquad \omega_{12} = \omega_{21} = \tilde{\omega}_{12} = \tilde{\omega}_{21}\,.$$

Let us write $\sigma'(t) = \sigma(t)\,\varkappa(t)$. Then for $\varkappa(t)$we have (36) by means of (38):

$$(39) \qquad \begin{aligned} \sin(\varphi + \psi)(\varepsilon\eta\omega_{22} - \omega_{11}) &= 0 \\ \sin(\varphi - \psi)(\varepsilon\eta\omega_{22} + \omega_{11}) &= 0\,. \end{aligned}$$

The coefficients in (39) are not zero, as we see from (28). We then have $\sin(\varphi + \psi) = \sin(\varphi - \psi) = 0$. Analogously as (36) and by means of (38) we obtain

$$(40) \qquad \begin{aligned} \tilde{\omega}_{11} + \tilde{\omega}_{22} &= (\omega_{11} + \varepsilon\eta\omega_{22})\cos(\varphi - \psi)\,, \quad \tilde{\omega}_{11}\tilde{\omega}_{22} = \varepsilon\eta\omega_{11}\omega_{22}\,, \\ \tilde{\omega}_{11} - \tilde{\omega}_{22} &= (\omega_{11} - \varepsilon\eta\omega_{22})\cos(\varphi + \psi)\,. \end{aligned}$$

Let us set $\tilde{\omega}_{11} > \tilde{\omega}_{22} > 0$ at $t = 0$ or equivalently

$$(41) \qquad \tilde{\omega}_{11} + \tilde{\omega}_{22} > 0\,, \quad \tilde{\omega}_{11} - \tilde{\omega}_{22} > 0\,, \quad \tilde{\omega}_{11}\tilde{\omega}_{22} > 0 \quad \text{at} \quad t = 0\,.$$

(41) will be satisfied, if we set

$$(42) \qquad \begin{aligned} \cos(\varphi - \psi) &= \operatorname{sg}\{\omega_{11} + \operatorname{sg}(\omega_{11}\omega_{22})\,\omega_{22}\}\,, \quad \varepsilon\eta = \operatorname{sg}(\omega_{11}\omega_{22}) \\ \cos(\varphi + \psi) &= \operatorname{sg}\{\omega_{11} - \operatorname{sg}(\omega_{11}\omega_{22})\,\omega_{22}\}\,. \end{aligned}$$

From (42) and (39) we now can see: $\varepsilon\eta$ is uniquely determined and if $\varphi_1, \psi_1; \varphi_2, \psi_2$ are two solutions of (37) and (41), we have $\varphi_1 = \varphi_2$, $\psi_1 = \psi_2$ or $\varphi_1 = \varphi_2 + \pi$, $\psi_1 = \psi_2 + \pi$. From (42) we have

$$\varepsilon(2r - 1)\,\varepsilon(2r) = \operatorname{sg}\det\omega_{2r-1,2r}\,, \quad r = 1, \ldots, s\,.$$

Let us finally set

$$\varepsilon(2r)\,\varepsilon(2r + 1) = \operatorname{sg}\det\omega_{2r,2r+1}\,, \quad r = 1, \ldots, s - 1\,.$$

Then we shall have $\varepsilon(\alpha + 1) = \varepsilon(\alpha)\operatorname{sg}\det\omega_{\alpha,\alpha+1}$, $\alpha = 1, \ldots, n - 1$. There are two solutions for ε: $\varepsilon(1) = 1$, $\varepsilon(1) = -1$ and the theorem is proved.

As we repeat the reasoning from the proof of Theorem 3.6 in the following, we shall proceed more quickly now.

b) Let us suppose M_1 has the dimension $m = 2n$, $n = 2s + 1$.

We can use Theorem 3.6 for the matrix $\omega^{11}_{\alpha\beta}$, where $\alpha, \beta = 3, \ldots, 2s$. Let us suppose that we have such a lift that 3.6 is satisfied for $\omega^{11}_{\alpha\beta}$, $\alpha, \beta = 3, \ldots, 2s$, on the understanding that $\varepsilon(3)$ can be chosen ± 1. This assumption can be made according to Lemma 3.2 as $\sigma_1 \times \sigma_2 \times \sigma_n$ is an invariant subgroup of P. Let us now consider the forms ω^{11}_{12} and ω^{11}_{1n} and let us write $\omega^{11}_{12;pq} \equiv \omega_{pq}$, $\omega^{11}_{1n;pq} \equiv \vartheta_{pq}$, $p, q = 1, 2$. For the transformation of the form ω we obtain

$$(43) \qquad \begin{aligned} \tilde{\omega}_{21} + \tilde{\omega}_{12} &= \sin(\varphi_1 + \varphi_2)\,(\omega_{11} - \varepsilon(1)\,\varepsilon(2)\,\omega_{22}) + \\ &\quad + \varepsilon(1)\cos(\varphi_1 + \varphi_2)\,(\omega_{21} + \varepsilon(1)\,\varepsilon(2)\,\omega_{12}) \\ \tilde{\omega}_{21} - \tilde{\omega}_{12} &= \sin(\varphi_1 - \varphi_2)\,(\omega_{11} + \varepsilon(1)\,\varepsilon(2)\,\omega_{22}) + \\ &\quad + \varepsilon(1)\cos(\varphi_1 - \varphi_2)\,(\omega_{21} - \varepsilon(1)\,\varepsilon(2)\,\omega_{12}) \\ \tilde{\vartheta}_{21} - \tilde{\vartheta}_{12} &= \sin(\varphi_1 - \varphi_n)\,(\vartheta_{11} - \varepsilon(1)\,\varepsilon(n)\,\vartheta_{22}) + \\ &\quad + \varepsilon(1)\cos(\varphi_1 - \varphi_n)\,(\vartheta_{21} + \varepsilon(1)\,\varepsilon(n)\,\vartheta_{12})\,. \end{aligned}$$

Let us set

$$(44) \quad \tilde{\omega}_{12} = \tilde{\omega}_{21} = 0\,, \quad \tilde{\vartheta}_{21} - \tilde{\vartheta}_{12} = 0\,, \quad \tilde{\omega}_{11} > \tilde{\omega}_{22} > 0\,, \quad \det\tilde{\omega}_{1n} > 0\,.$$

We can see that (44) has a solution. We have $\varepsilon(1)\,\varepsilon(2) = \operatorname{sg}\det(\omega_{pq})$, $\varepsilon(1)\,\varepsilon(n) = \operatorname{sg}\det(\vartheta_{pq})$ and for two solutions $\varphi_1, \varphi_2, \varphi_n$; $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_n$ we have $\tilde{\varphi}_1 = \varphi_1 + k\pi$, $\tilde{\varphi}_2 = \varphi_2 + k\pi$, $\tilde{\varphi}_n = \varphi_n + l\pi$. We have proved

3.7. Theorem. *Let $x(t)$ be a curve in M_1, $\dim M_1 = 2n$, $n = 2s + 1$. Let us suppose that the form ω of some lift $g(t)$ of the curve $x(t)$ satisfies (26), (27), (28) and*

$$(\omega^{11}_{1n;11} + \nu\omega^{11}_{1n;22})^2 + (\omega^{11}_{1n;12} - \nu\omega^{11}_{1n;21})^2 \neq 0\,, \quad \nu = \pm 1$$

and let $\det \omega^{11}_{1n} \neq 0$ at $t = 0$. Then there exist 2^{s+2} lifts of $x(t)$ such that (29), (30), (31), (32), $\omega^{11}_{1n;12} = \omega^{11}_{1n;21}$ are satisfied and $\det \omega_{1n} > 0$ at $t = 0$. If $g_1(t)$ and $g_2(t)$ are two such lifts, we have $g_1(t) = g_2(t)\,h$, where $h = \mathbf{d}^{11}\{\varrho_\alpha\}$ and $\varrho_{2r-1} = \varrho_{2r} = \begin{pmatrix}\varepsilon_r & 0\\ 0 & \varepsilon_r\varepsilon\end{pmatrix}$, $\sigma_n = \begin{pmatrix}\varepsilon_n, & 0\\ 0 & \varepsilon_n\varepsilon\end{pmatrix}$, $\varepsilon_r = \pm 1$, $\varepsilon_n = \pm 1$, $\varepsilon = \pm 1$, $r = 1, \ldots, s$.

c) **3.8. Theorem.** *Let $x(t)$ be a curve in M_2, $m = 2n + 1$. Let us suppose that the form ω of some lift $g(t)$ of the curve $x(t)$ satisfies*

(45) $$|\omega^{12}|_\alpha \neq 0$$

and at $t = 0$

(46) $$|\omega^{12}|_\alpha \neq |\omega^{12}|_\beta\,, \quad \alpha \neq \beta\,, \quad \alpha, \beta = 1 \ldots n\,,$$

(47) $$\det \omega^{11}_{\alpha,\alpha+1} \neq 0\,, \quad \alpha = 1, \ldots, n - 1\,.$$

Then there exist four lifts $g(t)$ of $x(t)$ such that

(48) $$\omega^{12}_{\alpha;2} = 0\,, \quad \alpha = 1, \ldots, n$$

and at $t = 0$

(49) $$\omega^{12}_{\alpha;1} > \omega^{12}_{\alpha+1;1} > 0\,, \quad \det \omega^{11}_{\alpha,\alpha+1} > 0\,, \quad \alpha = 1, \ldots, n - 1\,.$$

If g_1 and g_2 are two such lifts, then $g_1(t) = g_2(t)\,h$, where

$$h = \begin{pmatrix}\mathbf{d}^{11}(\varrho_\alpha), & 0\\ 0 & \nu\end{pmatrix} \quad \textit{and} \quad \varrho_\alpha = \begin{pmatrix}\nu & 0\\ 0 & \mu\end{pmatrix}; \quad \nu, \mu = \pm 1\,, \quad \alpha = 1, \ldots, n\,.$$

Proof. The form ω is of the form

$$\omega = \begin{pmatrix}\omega^{11}_{\alpha\beta}, & \omega^{12}_\alpha\\ \omega^{21}_\alpha, & 0\end{pmatrix}, \quad \text{where} \quad \omega^{12} = -(\omega^{21})^*\,.$$

Let us write as in (25) $\omega^{12}_\alpha \equiv \vartheta_\alpha$. We get from (25): $\tilde{\vartheta}_\alpha = \sigma^*_\alpha \vartheta_\alpha \nu$. In the component form we have

(50) $$\tilde{\vartheta}_{\alpha;1} = \nu(\cos\varphi_\alpha \vartheta_{\alpha;1} - \sin\varphi_\alpha \vartheta_{\alpha;2})$$
$$\tilde{\vartheta}_{\alpha;2} = \nu\varepsilon(\alpha)\,(\sin\varphi_\alpha \vartheta_{\alpha;1} + \cos\varphi_\alpha \vartheta_{\alpha;2})\,.$$

Let us set $\tilde{\omega}^{12}_{\alpha;2} = 0$ and at $t = 0$ let us set $\tilde{\omega}^{12}_{\alpha;1} > 0$ and $\det \omega^{11}_{\alpha,\alpha+1} > 0$. This condition can be satisfied and it can be easily shown that every two solutions differ by $\varepsilon(1) = \pm 1$ and $\varepsilon(n + 1) = \cos \varphi_\alpha = v$.

d) **3.9. Theorem.** *Let $x(t)$ be a curve in N_1 where $m = 2n$ and suppose that the form ω of some lift $g(t)$ of $x(t)$ satisfies $|\omega^{10}|_\alpha \neq 0$ and at $t = 0$ let it be $|\omega^{10}|_\alpha \neq |\omega^{10}|_\beta \neq 0$, $\alpha \neq \beta$ for $\alpha, \beta = 1, \ldots, n$ and $\det \omega^{11}_{\alpha,\alpha+1} \neq 0$ for $\alpha = 1, \ldots, n - 1$. Then there exist two lifts $g(t)$ of $x(t)$ such that $\omega^{10}_{\alpha;2} = 0$ for $\alpha = 1, \ldots, n$ and at $t = 0$ it is $\omega^{10}_{\alpha;1} > \omega^{10}_{\alpha+1;1} > 0$, $\det \omega^{11}_{\alpha,\alpha+1} > 0$ for $\alpha = 1, \ldots, n - 1$. If $g_1(t)$ and $g_2(t)$ are two such lifts, we have*

$$g_2(t) = g_1(t)\begin{pmatrix} 1 & 0 \\ 0 & \mathbf{d}^{11} \end{pmatrix}, \quad \textit{where} \quad \mathbf{d}^{11} = \mathbf{d}^{11}\{\varrho_\alpha\} \quad \textit{and} \quad \varrho_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \varepsilon = \pm 1 .$$

Proof. The proof is quite similar to that of the case c).

e) **3.10. Theorem.** *Let $x(t)$ be a curve in N_2, $m = 2n + 1$. Let us suppose that the form ω of some lift of $x(t)$ satisfies $|\omega^{12}|_\alpha \neq 0$ and at $t = 0$ $|\omega^{12}|_\alpha = |\omega^{12}|_\beta$, $\alpha \neq \beta$, $\alpha, \beta = 1, \ldots, n$; $\det \omega^{11}_{\alpha,\alpha+1} \neq 0$, $\alpha = 1, \ldots, n - 1$; $\omega^{20} \neq 0$. Then there exists only one lift $g(t)$ of the curve $x(t)$ such that $\omega^{12}_{\alpha;2} = 0$, $\alpha = 1, \ldots, n$; $\omega^{10}_{1;1} = 0$ and at $t = 0$, $\omega^{20} > 0$, $\omega^{10}_{1;2} > 0$; $\omega^{12}_{\alpha;1} > \omega^{12}_{\alpha+1;1} > 0$, $\det \omega^{12}_{\alpha,\alpha+1} > 0$, $\alpha = 1, \ldots, n - 1$.*

Proof. Let us note that the group

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{E} & 0 \\ a^{20} & 0 & 1 \end{pmatrix},$$

where $\mathbf{E}$ is the unit matrix $2n \times 2n$, is an invariant subgroup of $P(e(2n + 1))$. According Lemma 3,2 we can use Theorem 3.8. v can be chosen so that $\tilde{\omega}^{20} > 0$, as we see from (25). From (24) we have $\tilde{\omega}^{10} = (\mathbf{d}^{11})^* (\omega^{10} + a^{20}\omega^{12})$. Now we have $\tilde{\omega}^{10}_{1;1} = \omega^{10}_{1;1} + a^{20}\omega^{12}_{1;1}$, where $\omega^{12}_{1;1} \neq 0$. We can choose a^{20} so that $\tilde{\omega}^{10}_{1;1} = 0$. The rest is easy.

3.11. Definition. *The lifts from Theorems 3.6–3.10 will be called Frenet lifts.*

In the end we shall define the arc for curves in M_i and N_i.

3.12. Definition. *The parameter s of a curve $x(s)$ in M_i or N_i is called the arc, if the form ω of some lift of $x(s)$ satisfies*

$$(51) \qquad \tfrac{1}{2}\sum_{\alpha=1}^{n} |\omega^{11}|_\alpha = 1 \quad \textit{or} \quad \tfrac{1}{2}\sum_{\alpha=1}^{n} |\omega^{11}|_\alpha + \sum_{\alpha=1}^{n} |\omega^{12}|_\alpha = 1$$

for M_1 *and* N_1 *or for* M_2 *and* N_2 *respectively, where* $|\omega^{11}|_\alpha$ *and* $|\omega^{12}|_\alpha$ *have the same meaning as in* (21).

The arc s has the natural meaning in the case of M_i, it is the arc in the canonical Riemann structure induced on M_i by the Killing quadratic form.

3.13. Definition. *Let* $x(t)$ *be a curve in* M_i *or in* N_i *and let* $g(t)$ *be a Frenet lift of* $x(t)$. *The arc of* $x(t)$ *will be denoted by* s. *Then the functions* $k^{ij}(s)$, *where* $\omega^{ij}(s) = k^{ij}(s)\,\mathrm{d}s$, i, j *are the same as in* (1), *are called the invariants of the curve* $x(t)$.

The matrix of the invariants will be denoted by $\mathbf{k}$. The following theorems are obvious:

3.14. Theorem. *Let* $x(s)$ *and* $x'(s')$ *be two curves respectively in* M_i *or in* N_i, *equivalent in the sense of Definition* 3.1. *Let* s, s' *be their arcs. Then there is one* $g \in O(m)$ *or* $g \in E(m)$ *respectively such that* $x'(s') = g \,.\, x(s)$.

3.15. Theorem. *The curves* $x(t)$ *and* $x'(t')$ *in* M_i *or* N_i *are equivalent in the sense of Definition* 3.1 *if and only if they have the same invariants satisfying condition* (*U*). (*About condition* (*U*) *see Definition* 7.4.)

4. Interpretation in S^{m-1} and E^m

Let an m-dimensional Euclidean vector space V^m and a m-dimensional Euclidean space E^m be given. The Euclidean scalar product of vectors $\boldsymbol{x}, \boldsymbol{y}$ will be denoted by $(\boldsymbol{x}, \boldsymbol{y})$ as usual. We shall now define the manifolds P^m and Q^m in the following manner:

a) P^m, $m = 2n$. A point $\boldsymbol{p} \in P^m$ is a (not ordered) n-tuple $\boldsymbol{p} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$, where $\boldsymbol{p}_\alpha$ for $\alpha = 1, \ldots, n$ is a 2-dimensional linear subspace in V^m and

$$\text{(52)} \qquad \text{if} \quad \boldsymbol{x} \in \boldsymbol{p}_\alpha\,, \quad \boldsymbol{y} \in \boldsymbol{p}_\beta\,, \quad \alpha \neq \beta\,, \quad \text{then} \quad (\boldsymbol{x}, \boldsymbol{y}) = 0\,.$$

b) P^m, $m = 2n + 1$. A point $\boldsymbol{p} \in P^m$ is an n-tuple of 2-dimensional linear subspaces $\boldsymbol{p}_\alpha$, satisfying (52) and a 1-dimensional subspace $\boldsymbol{v}$ perpendicular to every $\boldsymbol{p}_\alpha$. Then $\boldsymbol{p} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{v}\}$.

c) Q^m, $m = 2n$. A point $\boldsymbol{q} \in Q^m$ is a point $A \in E^m$ and an n-tuple of 2-dimensional planes $\boldsymbol{q}_\alpha \subset E^m$, $\alpha = 1, \ldots, n$ such that $A \in \boldsymbol{q}_\alpha$ for every α and if $B_\alpha \in \boldsymbol{q}_\alpha$, $B_\beta \in \boldsymbol{q}_\beta$, $\alpha \neq \beta$, then $(B_\alpha - A, B_\beta - A) = 0$. We shall write $\boldsymbol{q} = \{A, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$.

d) Q^m, $m = 2n + 1$. A point $\boldsymbol{q} \in Q^m$ is a straight line $\boldsymbol{a}$ and an n-tuple of 3-dimensional subspaces $\boldsymbol{q}_\alpha \subset E^m$, $\alpha = 1, \ldots, n$, where $\boldsymbol{a} \subset \boldsymbol{q}_\alpha$ for every α and the following condition is satisfied: If $A \in \boldsymbol{a}$, $\boldsymbol{B}_\alpha \in \boldsymbol{q}_\alpha$, $\boldsymbol{B}_\beta \in \boldsymbol{q}_\beta$, $\alpha \neq \beta$, $(\boldsymbol{B}_\alpha - A) \perp \boldsymbol{a}$, $(\boldsymbol{B}_\beta - A) \perp \boldsymbol{a}$, then $(\boldsymbol{B}_\alpha - A, \boldsymbol{B}_\beta - A) = 0$.

Let $\mathscr{R}_0 = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m)$, $\mathscr{R}_0 = (\boldsymbol{O}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_m)$ be fixed orthonormal frames in V^m and E^m respectively. The formula $\mathscr{R} = \mathscr{R}_0 \,.\, g$ shows us the connection between a frame and its coordinates. For a point or a vector $\boldsymbol{X}$ we shall define (as usual) its coordinates x_0 in $\mathscr{R}_0$ so that $\boldsymbol{X} = \mathscr{R}_0 \,.\, x_0$. The group $O(m)$ acts then on V^m and the group $E(m)$ acts then on E^m in the natural manner: If $\boldsymbol{X}$ is a point or a vector, $\boldsymbol{X} = \mathscr{R}_0 \,.\, x_0$, then $g\boldsymbol{X}$ for $g \in O(m)$ or $g \in E(m)$ respectively is the point $g\boldsymbol{X} = \mathscr{R}_0 \,.\, gx_0$. Now we can easily see that the frame $\mathscr{R} = \mathscr{R} \,.\, g_0$ is composed respectively from the vectors $g\boldsymbol{e}_1, \ldots, g\boldsymbol{e}_m$ or from the point $g\boldsymbol{O}$ and the vectors $g\boldsymbol{e}_1, \ldots, g\boldsymbol{e}_m$. The group $O(m)$ acts now in the natural manner on the manifold P^m and the group $E(m)$ acts in the natural manner on Q^m:

If $g \in O(m)$ resp. $g \in E(m)$ and $\boldsymbol{p} = \{\boldsymbol{p}_\alpha\}$, $\boldsymbol{p} = \{\boldsymbol{p}_\alpha, \boldsymbol{v}\}$ resp. $\boldsymbol{q} = \{A, \boldsymbol{q}_\alpha\}$ or $\boldsymbol{q} = \{\boldsymbol{a}, \boldsymbol{q}\}$, then $g\boldsymbol{p} = \{g\boldsymbol{p}_\alpha\}$, $g\boldsymbol{p} = \{g\boldsymbol{p}_\alpha, g\boldsymbol{v}\}$ resp. $g\boldsymbol{q} = \{gA, g\boldsymbol{q}_\alpha\}$ or $g\boldsymbol{q} = \{g\boldsymbol{a}, g\boldsymbol{q}_\alpha\}$.

4.1. Theorem. *$P^m = O(m)/\mathscr{N}(\mathfrak{o}(m))$ and $Q^m = E(m)/\mathscr{N}(\mathfrak{e}(m))$.*

The formula $\mathscr{R} = \mathscr{R}_0 \,.\, g$ gives us a one to one corespondence between the group $O(m)$ or $E(m)$ and the set $\mathscr{A}$ of all orthonormal frames in V^m or E^m. The projection $\pi : \mathscr{A} \to P^m$ or $\pi : \mathscr{A} \to Q^m$ is natural:

Let $\mathscr{R} = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m\}$ or $\mathscr{R} = \{A, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_m\}$ respectively be given. Then

a) $\pi\mathscr{R} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$, where $\boldsymbol{p}_\alpha = \lambda \boldsymbol{f}_{2\alpha-1} + \mu \boldsymbol{f}_{2\alpha}$ for $m = 2n$,

b) $\pi\mathscr{R} = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n, \boldsymbol{v}\}$, where $\boldsymbol{p}_\alpha = \lambda \boldsymbol{f}_{2\alpha-1} + \mu \boldsymbol{f}_{2\alpha}$, $\boldsymbol{v} = \nu \boldsymbol{f}_{2n+1}$ for $m = 2n + 1$,

c) $\pi\mathscr{R} = \{A, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$, where $\boldsymbol{q}_\alpha = \lambda \boldsymbol{f}_{2\alpha-1} + \mu \boldsymbol{f}_{2\alpha}$ for $m = 2n$,

d) $\pi\mathscr{R} = \{\boldsymbol{a}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$, where $\boldsymbol{a} = A + \lambda \boldsymbol{f}_{2n+1}$,

$\boldsymbol{q}_\alpha = A + \lambda \boldsymbol{f}_{2\alpha-1} + \mu \boldsymbol{f}_{2\alpha} + \nu \boldsymbol{f}_{2n+1}$ for all $\lambda, \mu, \nu \in \mathbf{R}$ and $\alpha = 1, \ldots, n$.

From (16) we now have for the identical representation

$$\mathrm{d}\mathscr{R} = \mathscr{R} \,.\, \omega \,. \tag{53}$$

For curves in P^m and Q^m we now can repeat all what was said in §3, in particular Theorems 3.6–3.10. From Frenet lifts we now get the Frenet frames and the Frenet formulas

$$\mathrm{d}\mathscr{R}/\mathrm{d}s = \mathscr{R} \,.\, \mathbf{k} \,, \tag{54}$$

where $\mathrm{d}\mathscr{R}/\mathrm{d}s$ certainly means $\mathrm{d}\mathscr{R}(\partial/\partial s)$ and $\mathbf{k}$ has the meaning from Definition 3.13.

5. The case of small dimensions

Let us see what results can be obtained from (53) in case of small dimensions of S^{m-1} and E^m.

a) S^{m-1}, $m = 3$. For the curve $x(t)$ and for its Frenet frame $\mathscr{R} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}$ we get

$$(55) \qquad \omega = \begin{pmatrix} 0, & -\omega_{21}, & \omega_1 \\ \omega_{21}, & 0, & 0 \\ -\omega_1, & 0, & 0 \end{pmatrix} \quad \text{and} \quad \begin{aligned} d\boldsymbol{r}_1 &= \boldsymbol{r}_2\omega_{21} - \boldsymbol{r}_3\omega_1\,, \quad \omega_1 = ds \\ d\boldsymbol{r}_2 &= -\boldsymbol{r}_1\omega_{21} \\ d\boldsymbol{r}_3 &= \boldsymbol{r}_1\omega_1 \end{aligned}$$

The manifold P^2 is the unit sphere S^2 in V^3 and (55) are really the Frenet formulas for a spherical curve.

b) S^{m-1}, $m = 4$. Now we have for ω and for the Frenet frame $\mathscr{R} = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_4\}$:

$$(56) \qquad \omega = \begin{pmatrix} 0, & \omega_{12}, & \omega_{13}, & 0 \\ -\omega_{12}, & 0, & 0, & \omega_{24} \\ -\omega_{13}, & 0, & 0, & \omega_{34} \\ 0, & -\omega_{24}, & -\omega_{34}, & 0 \end{pmatrix} \quad \text{and} \quad \begin{aligned} d\boldsymbol{r}_1 &= -\omega_{12}\boldsymbol{r}_2 - \omega_{13}\boldsymbol{r}_3 \\ d\boldsymbol{r}_2 &= \omega_{12}\boldsymbol{r}_1 - \omega_{24}\boldsymbol{r}_4 \\ d\boldsymbol{r}_3 &= \omega_{13}\boldsymbol{r}_1 - \omega_{34}\boldsymbol{r}_4 \\ d\boldsymbol{r}_4 &= \omega_{24}\boldsymbol{r}_2 + \omega_{34}\boldsymbol{r}_3 \end{aligned}$$

$(ds)^2 = (\omega_{13})^2 + (\omega_{24})^2$.

The manifold P^3 has the set of all straight lines of S^3 as a two-fold covering and (56) are the Frenet formulas for a ruled surface in S^3.

c) E^m, $m = 2$. For the form ω and for the Frenet frame $\{A, \boldsymbol{r}_1, \boldsymbol{r}_2\}$ we have

$$(57) \qquad \omega = \begin{pmatrix} 0, & 0, & 0 \\ \omega^{10}, & 0, & -\omega_{21} \\ 0, & \omega_{21}, & 0 \end{pmatrix} \quad \text{and} \quad \begin{aligned} dA &= \omega^{10}\boldsymbol{r}_1 \\ d\boldsymbol{r}_1 &= \omega_{21}\boldsymbol{r}_2 \\ d\boldsymbol{r}_2 &= -\omega_{21}\boldsymbol{r}_1 \end{aligned}$$

Q^2 is the plane and (57) are Frenet formulas for a curve in Q^2. The arc is not defined.

d) E^m, $m = 3$. We have for the frame $\mathscr{R} = \{A, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}$:

$$(58) \qquad \omega = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & -\omega_{21}, & \omega^{12} \\ \omega^{10}, & \omega_{21}, & 0, & 0 \\ \omega^{20}, & -\omega^{12}, & 0, & 0 \end{pmatrix} \quad \text{and} \quad \begin{aligned} dA &= \omega^{10}\boldsymbol{r}_2 + \omega^{20}\boldsymbol{r}_3 \\ d\boldsymbol{r}_1 &= \omega_{21}\boldsymbol{r}_2 - \omega^{12}\boldsymbol{r}_3 \\ d\boldsymbol{r}_2 &= -\omega_{21}\boldsymbol{r}_1 \\ d\boldsymbol{r}_3 &= \omega^{12}\boldsymbol{r}_1 \end{aligned}$$

$ds = \omega^{12}$. Q^3 is the set of all straight lines in E^3, the curve $x(t)$ is given by the points $A + \lambda\boldsymbol{r}_3$, $\lambda \in \mathbf{R}$. (58) are the Frenet formulas for a ruled surface in E^3, the arcs coincide.

Part II

APPLICATIONS ON THE KINEMATIC GEOMETRY OF A ONE PARAMETRIC MOTION IN S^{m-1} AND E^m

6. Definition of motion

Let a homogeneous space G/H be given. A motion in G/H is a differentiable immersion $g(\varphi) : I \to G$; the group G is regarded as a group of transformations of G/H.

6.1. Definition. *Let $g_1(\varphi_1) : I_1 \to G$ and $g_2(\varphi_2) : I_2 \to G$ be motions in G/H. Motions g_1 and g_2 are called equivalent, if they are equivalent in the sense of Definition* 3.1 *as curves in the homogeneous space $G \times G/G$.*

(Here the group $G \times G$ acts on G as a group of left and right translations: If $(g_1, g_2) \in G \times G$, $g \in G$, then $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$.)

The kinematic geometry then seeks for invariants of equivalence classes of motions in the sense of Definition 6.1 and tries to find geometric objects in G/H that completely determine the motion. In the sequel, this will be done for S^{m-1} and E^m.

Let $g(\varphi) : I \to G$ be a curve on G. Let us denote by ϑ, $\bar{\vartheta}$ respectively the left-invariant and right-invariant M.C. form on G (see §3) and let us define the forms $\Omega = g_* \bar{\vartheta}$, $\bar{\Omega} = g_* \vartheta$. Let (X, Y) be the Killing symmetric bilinear form on $\mathfrak{G}$, where $X, Y \in \mathfrak{G}$. The parameter $t = t(\varphi)$ of the curve $g(\varphi)$ is called the canonical parameter (the canonical time) if

$$(\Omega(\partial/\partial t),\ \Omega(\partial/\partial t)) = 1 . \tag{59}$$

6.2. Lemma. *Let $g(\varphi) : I \to G$ be a curve and let us suppose that $(\Omega(\partial/\partial\varphi), \Omega(\partial/\partial\varphi)) \neq 0$ for every $\varphi \in I$. Then there exists one (up to a transformation from $G \times G$) curve $\gamma(t)$ equivalent with $g(\varphi)$ such that t is the canonical parameter.*

Proof. The parameter t is determined uniquely by (59) and by conditions $t(0) = 0$ and $\mathrm{d}\varphi/\mathrm{d}t > 0$ from Definition 3.1. In the rest of the paper we shall suppose that the motion is given as a function of its canonical parameter. We can uniquely write

$$\Omega = \mathbf{R}(t)\,\mathrm{d}t\,, \quad \bar{\Omega} = \bar{\mathbf{R}}(t)\,\mathrm{d}t\,. \tag{60}$$

$\mathbf{R}(t)$ and $\bar{\mathbf{R}}(t)$ will be called the fixed and the moving cone of the motion respectively.

7. Motion in E^m and S^{m-1}

We shall now consider only the groups $O(m)$ and $E(m)$. Let us suppose that we are given a motion $g(t)$ in $O(m)$ or $E(m)$ such that $\mathbf{R}(t)$ is a regular element of $\mathfrak{O}(m)$ or $\mathfrak{E}(m)$ respectively for every $t \in I$. Then the cone $\mathbf{R}(t)$ uniquely determines a one-parametric system of Cartan subalgebras $\mathfrak{h}(t)$ respectively in $\mathfrak{O}(m)$ or $\mathfrak{E}(m)$ such that $\mathbf{R}(t) \in \mathfrak{h}(t)$. The algebras $\mathfrak{h}(t)$ determine the groups $\mathcal{N}(\mathfrak{h}(t))$. The groups $\mathcal{N}(\mathfrak{h}(t))$ determine a curve $\boldsymbol{p}(t) \in P^m$ or $\boldsymbol{q}(t) \in Q^m$ such that the isotropic groups of $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ respectively are $\mathcal{N}(\mathfrak{h}(t))$. The curve $\boldsymbol{p}(t)$ or $\boldsymbol{q}(t)$ is called the central curve (the set of instantaneous centers) of the motion and the system

$$(61) \qquad \boldsymbol{p}(t) = \{\boldsymbol{p}_\alpha(t)\} \quad \text{or} \quad \boldsymbol{p}(t) = \{\boldsymbol{p}_\alpha(t), \boldsymbol{v}(t)\} \quad \text{or}$$
$$\boldsymbol{q}(t) = \{A(t), q_\alpha(t)\} \quad \text{or} \quad \boldsymbol{q}(t) = \{\boldsymbol{a}(t), \boldsymbol{q}_\alpha(t)\}$$

is called the fixed axoid. An analogous construction can be given for the moving cone. We get the moving central curve in P^m or Q^m and the moving axoid in S^{m-1} or E^m respectively.

7.1. Lemma. *Let $\mathfrak{h}(t)$ be a one-parametric system of Cartan subalgebras in $\mathfrak{O}(m)$ or $\mathfrak{E}(m)$. Then there exists respectively $g(t) \in O(m)$ or $g'(t) \in E(m)$ such that* $\operatorname{ad}g(t)\, \mathfrak{o}(m) = \mathfrak{h}(t)$ *or* $\operatorname{ad}g'(t)\, \mathfrak{e}(m) = \mathfrak{h}(t)$. *We have for the central curve $\boldsymbol{p}$ or $\boldsymbol{q}$: $\boldsymbol{p}(t) = \pi\, g(t)$ or $\boldsymbol{q}(t) = \pi\, g'(t)$, respectively.*

Proof. The $g(t)$ exists for $\mathfrak{O}(m)$ by Cartan theorem about inner automorphisms of compact semisimple algebras. The existence of $g'(t)$ can be proved by direct calculation from the existence of $g(t)$. We now have respectively $\mathcal{N}(\mathfrak{h}(t)) = g(t)\, \mathcal{N}(\mathfrak{o}(m))$ or $\mathcal{N}(\mathfrak{h}(t)) = g'(t)\, \mathcal{N}(\mathfrak{e}(m))$ and the assertion follows.

7.2. Theorem. *Let $g(t)$ be a motion in $O(m)$ or $E(m)$. Then its cones $\mathbf{R}(t)$ and $\bar{\mathbf{R}}(t)$ and its axoids $\boldsymbol{p}$ and $\bar{\boldsymbol{p}}$ or $\boldsymbol{q}$ and $\bar{\boldsymbol{q}}$ respectively satisfy*

$$(61) \qquad \operatorname{ad}g(t)\, \bar{\mathbf{R}}(t) = \mathbf{R}(t)$$

$$(62) \qquad g(t)\, \bar{\boldsymbol{p}}(t) = \boldsymbol{p}(t) \quad \text{or} \quad g(t)\, \bar{\boldsymbol{q}}(t) = \boldsymbol{q}(t)\,.$$

Proof. (61) is obvious, (62) is obtained from (61) and from Lemma 7.1.

7.3. Theorem. *Let $g(t)$ be a motion in $O(m)$ or $E(m)$ with the moving cone $\bar{\mathbf{R}}(t)$. Let $\boldsymbol{p}, \bar{\boldsymbol{p}}$ or $\boldsymbol{q}, \bar{\boldsymbol{q}}$ respectively be the axoids of $g(t)$. Let us choose some Frenet lifts γ, γ_1 of $\boldsymbol{p}, \bar{\boldsymbol{p}}$ or $\boldsymbol{q}, \bar{\boldsymbol{q}}$ and let $\omega, \bar{\omega}$ be the canonical forms of the axoids. Then there exists respectively $h(t) \in \mathcal{N}(\mathfrak{o}(m))$ or $h(t) \in \mathcal{N}(\mathfrak{e}(m))$ such that*

$$(63) \qquad \bar{\omega} = \operatorname{ad}h^{-1}\omega - \operatorname{ad}\gamma_1 \bar{\mathbf{R}}\, dt + h^{-1}\, dh\,.$$

Then we get respectively for $\bar{\mathbf{R}}(t)$:

$$(64) \qquad \operatorname{ad}\gamma_1(t)\, \bar{\mathbf{R}}(t) \in \mathfrak{o}(m) \quad \text{or} \quad \operatorname{ad}\gamma_1(t)\, \bar{\mathbf{R}}(t) \in \mathfrak{e}(m)\,.$$

Proof. Let us denote $\mathfrak{o}(m) = \mathfrak{h}_0$ or $\mathfrak{e}(m) = \mathfrak{h}_0$. Then we have $\mathfrak{h}(t) = \operatorname{ad}\gamma\mathfrak{h}_0$, $\bar{\mathfrak{h}}(t) = \operatorname{ad}\gamma_1\mathfrak{h}_0$, $\mathfrak{h}(t) = \operatorname{ad}g(t)\, \bar{\mathfrak{h}}(t)$ and $\mathfrak{h}(t) = \operatorname{ad}g(t)\, \bar{\mathfrak{h}}(t) = \operatorname{ad}(g\gamma_1)\, \mathfrak{h}_0 = \operatorname{ad}\gamma\mathfrak{h}_0$. Then

$\operatorname{ad}(\gamma^{-1}g\gamma_1)\,\mathfrak{h}_0 = \mathfrak{h}_0$ and $\gamma^{-1}g\gamma_1 \in \mathcal{N}(\mathfrak{h}_0)$. Now we see that there is $h(t) \in \mathcal{N}(\mathfrak{h}_0)$ such that $\gamma^{-1}g\gamma_1 = h(t)$ and then $g\gamma_1 = \gamma h$. After differentiating we get $\mathrm{d}g\gamma_1 + g\,\mathrm{d}\gamma_1 = \mathrm{d}\gamma h + \gamma\,\mathrm{d}h$. Then

$$\operatorname{ad}\gamma_1^{-1} \cdot g^{-1}\,\mathrm{d}g + \gamma_1^{-1}\,\mathrm{d}\gamma_1 = \operatorname{ad}h^{-1} \cdot \gamma^{-1}\,\mathrm{d}\gamma + h^{-1}\,\mathrm{d}h$$

and finally,

$$\operatorname{ad}\gamma_1^{-1}\overline{\mathbf{R}}\,\mathrm{d}t + \bar{\omega} = \operatorname{ad}h^{-1}\omega + h^{-1}\,\mathrm{d}h\,.$$

7.4. Definition. *Let $X(t) \in \mathfrak{O}(m)$ or $X(t) \in \mathfrak{E}(m)$ for $t \in I$. Let us denote $X(t) = \mathbf{x}(t)^{ij}$, i, j being the same as in (1). We shall say that $X(t)$ satisfies the condition (U), if it holds:*

a) *for $O(m)$, $m = 4s$:*

$$\mathbf{x}^{11}_{2r-1,2r;21}(t) = \mathbf{x}^{11}_{2r-1,2r;12}(t) = 0\,, \quad r = 1, \ldots, s$$

$$|\mathbf{x}^{11}(0)|_\alpha > |\mathbf{x}^{11}(0)|_{\alpha+1}\,, \quad \det \mathbf{x}^{11}_{\alpha,\alpha+1}(0) > 0\,, \quad \alpha = 1, \ldots, 2s-1$$

$$\mathbf{x}^{11}_{2r-1,2r;11}(0) > 0\,, \quad \mathbf{x}^{11}_{2r-1,2r;22}(0) > 0\,. \quad r = 1, \ldots, s\,,$$

$$\mathbf{x}^{11}_{2r-1,2r+1;11}(0) > 0\,, \quad r = 1, \ldots, s-1\,, \quad \mathbf{x}^{11}_{12;12}(0) > 0\,.$$

b) *for $O(m)$, $m = 2(2s+1)$, $n = 2s+1$:* a) *and*

$$\mathbf{x}^{11}_{1n;12}(t) = \mathbf{x}^{11}_{1n;21}(t)$$

$$\det \mathbf{x}^{11}_{1n}(0) > 0\,, \quad \mathbf{x}^{11}_{1n;11}(0) > 0$$

c) *for $O(m)$, $m = 2n + 1$:*

$$\mathbf{x}^{12}_{\alpha;2}(t) = 0\,, \quad \alpha = 1, \ldots, n$$

$$\mathbf{x}^{12}_{\alpha;1}(0) > \mathbf{x}^{12}_{\alpha+1;1}(0) > 0\,, \quad \det \mathbf{x}^{11}_{\alpha,\alpha+1}(0) > 0\,, \quad \alpha = 1, \ldots, n-1\,; \quad \mathbf{x}^{11}_{12;12}(0) > 0$$

d) *for $E(m)$, $m = 2n$:*

$$\mathbf{x}^{10}_{\alpha;2}(t) = 0\,, \quad \alpha = 1, \ldots, n$$

$$\mathbf{x}^{10}_{\alpha;1}(0) > \mathbf{x}^{10}_{\alpha+1;1}(0) > 0\,, \quad \det \mathbf{x}^{11}_{\alpha,\alpha+1}(0) > 0\,, \quad \alpha = 1, \ldots, n-1\,; \quad \mathbf{x}^{11}_{12;12}(0) > 0$$

e) *for $E(m)$, $m = 2n + 1$:*

$$\mathbf{x}^{12}_{\alpha;2}(t) = \mathbf{x}^{10}_{1;1}(t) = 0\,, \quad \alpha = 1, \ldots, n$$

$$\mathbf{x}^{12}_{\alpha;1}(0) > \mathbf{x}^{12}_{\alpha+1;1}(0) > 0\,, \quad \det \mathbf{x}^{11}_{\alpha,\alpha+1}(0) > 0\,, \quad \alpha = 1, \ldots, n-1\,;$$

$$\mathbf{x}^{20}(0) > 0\,, \quad \mathbf{x}^{10}_{1;2}(0) > 0\,.$$

7.5. Lemma. *Let $X(t)$, $Y(t) \in \mathfrak{O}(m)$, $Z(t) \in \mathfrak{o}(m)$, $h(t) \in \mathcal{N}(\mathfrak{o}(m))$ or $X(t)$, $Y(t) \in \mathfrak{E}(m)$, $Z(t) \in \mathfrak{e}(m)$, $h(t) \in \mathcal{N}(\mathfrak{e}(m))$ be given such that X, Y, Z, h are continuous, X, Y satisfy condition (U) and $X(t) = \operatorname{ad}h(t)\, Y(t) + Z(t)$. Then $h(t) = \pm \mathbf{E}$ in* a) b) c) *and $h(t) = \mathbf{E}$ in* d) e) *of Definition* 7.4, $\mathbf{E}$ *being the unit matrix.*

Proof. The lemma can be proved by direct calculation.

7.6. Theorem. *Let a motion $g(t)$ in $O(m)$ or $E(m)$ be given. Let $\boldsymbol{p}$, $\bar{\boldsymbol{p}}$ or $\boldsymbol{q}$, $\bar{\boldsymbol{q}}$ be its axoids, let γ, γ_1 be the Frenet lifts respectively of $\boldsymbol{p}$, $\bar{\boldsymbol{p}}$ and $\boldsymbol{q}$, $\bar{\boldsymbol{q}}$, satisfying (U). Let us denote ω, $\bar{\omega}$ the canonical forms of γ, γ_1, let $\bar{\mathbf{R}}$ be the moving cone. Then*

$$(65)\qquad \bar{\omega} = \omega - (\operatorname{ad}\gamma_1^{-1}\bar{\mathbf{R}})\,\mathrm{d}t \quad \textit{and} \quad g = \pm\gamma\gamma_1^{-1} \quad \textit{or} \quad g = \gamma\gamma_1^{-1} \quad \textit{respectively}\,.$$

Proof. The theorem is a corollary of Lemma 7.5. Let us denote $\operatorname{ad}\gamma_1^{-1}\bar{\mathbf{R}} = \operatorname{ad}\gamma^{-1}\mathbf{R} = r_\alpha \mathbf{e}_\alpha$ for $O(m)$ and $E(2n)$, $\operatorname{ad}\gamma_1^{-1}\bar{\mathbf{R}} = \operatorname{ad}\gamma^{-1}\mathbf{R} = r_\alpha \mathbf{e}_\alpha + r_0 \mathbf{f}$ for $E(2n + 1)$. Then we get from (65):

$$(66)\qquad \omega^{11}_{\alpha\alpha;12} - \bar{\omega}^{11}_{\alpha\alpha;12} = r_\alpha\,\mathrm{d}t\,,\quad \omega^{20} - \bar{\omega}^{20} = r_0\,\mathrm{d}t\,,$$
$$\omega^{11}_{\alpha\beta} = \bar{\omega}^{11}_{\alpha\beta} \quad \text{for} \quad \alpha \neq \beta\,,\quad \omega^{12}_\alpha = \bar{\omega}^{12}_\alpha\,,\quad \omega^{10}_\alpha = \bar{\omega}^{10}_\alpha$$

(with upper indices as in (1)).

Let us suppose from now on that the metric in $\mathfrak{o}(m)$ and $\mathfrak{e}(m)$ is chosen so that $(\mathbf{e}_\alpha, \mathbf{e}_\alpha) = 1$ or $(\mathbf{e}'_\alpha, \mathbf{e}'_\alpha) = 1$ and $(\mathbf{e}''_\alpha, \mathbf{e}''_\alpha) = 1$ or $(\mathbf{e}'''_\alpha, \mathbf{e}'''_\alpha) = 1$ respectively and let us write for ω satisfying (U)

$$(67)\qquad \omega = \varkappa\,\mathrm{d}t\,.$$

7.7. Theorem. *Functions* t, $r_\alpha(t)$, $r_0(t)$, $\varkappa(t)$, $\bar{\varkappa}(t)$ *are invariants of the motion and it is (upper indices as in* (1))

$$(68)\qquad \varkappa^{11}_{\alpha\alpha;12} - \bar{\varkappa}^{11}_{\alpha\alpha;12} = r_\alpha\,,\quad \varkappa^{20} - \bar{\varkappa}^{20} = r_0\,,\quad \varkappa^{12}_\alpha = \bar{\varkappa}^{12}_\alpha\,,\quad \varkappa^{10}_\alpha = \bar{\varkappa}^{10}_\alpha$$
$$\varkappa^{11}_{\alpha\beta} = \bar{\varkappa}^{11}_{\alpha\beta} \quad \text{for} \quad \alpha \neq \beta\,,\quad \sum_{\alpha=1}^{n} (\varkappa^{11}_{\alpha\alpha;12} - \bar{\varkappa}^{11}_{\alpha\alpha;12}) = 1\,.$$

Proof. We only have to prove the last assertion. This one we get from $(\mathbf{R}, \mathbf{R}) = 1$.

Let us denote $\varkappa = \mathrm{d}s/\mathrm{d}t$, where s is the arc of the fixed axoid.

7.8. Theorem. *Let $g(t)$ be a motion in S^{m-1} or E^m. Let $\mathbf{k}$ and $\bar{\mathbf{k}}$ be the invariants of the fixed and the moving axoid respectively, satisfying condition (U). Then*

$$(69)\qquad \varkappa(s) = [\sum_{\alpha=1}^{n} (\mathbf{k}^{11}_{\alpha\alpha;12}(s) - \bar{\mathbf{k}}^{11}_{\alpha\alpha;12}(s)^2]^{-1/2}\,,\quad \varkappa^{ij}(s) = \mathbf{k}^{ij}(s)\,\varkappa(s)\,,$$
$$r_\alpha(s) = (\mathbf{k}^{11}_{\alpha\alpha;12}(s) - \bar{\mathbf{k}}^{11}_{\alpha\alpha;12}(s))\,\varkappa(s)\,,\quad r_0(s) = (\mathbf{k}^{20}(s) - \bar{\mathbf{k}}^{20}(s))\,\varkappa(s)$$

and conversely

$$(70)\qquad \mathbf{k}^{ij}(t) = \varkappa^{ij}(t)\,\varkappa^{-1}(t)\,,\quad \bar{\mathbf{k}}^{ij}(t) = \bar{\varkappa}^{ij}(t)\,\varkappa^{-1}(t)\,,$$

for $m = 2n + 1$

$$\varkappa(t) = \left(\tfrac{1}{2}\sum_{\alpha=1}^{n} |\varkappa^{11}|_\alpha + \sum_{\alpha=1}^{n} |\varkappa^{12}|_\alpha\right)^{1/2}$$

and for $m = 2n$

$$\varkappa(t) = \left(\tfrac{1}{2}\sum_{\alpha=1}^{n} |\varkappa^{11}|_\alpha\right)^{1/2}.$$

From Theorem 7.8 we see that the system of invariants of the axoids and the system of invariants of the motion are equivalent to each other.

7.9. Theorem. *Let one parametric differentiable systems* $\boldsymbol{p}(s)$ *and* $\bar{\boldsymbol{p}}(s)$ *in* V^m *be given, where* $\boldsymbol{p}(s)$ *and* $\bar{\boldsymbol{p}}(s)$ *are defined as in* §4. *Let us suppose that the invariants of* $\boldsymbol{p}(s)$ *and* $\bar{\boldsymbol{p}}(s)$ *satisfy* (U) *and that in the case* $m = 2n$

$$\boldsymbol{k}^{11}_{\alpha\beta}(s) = \bar{\boldsymbol{k}}^{11}_{\alpha\beta}(s)\,,\quad \alpha \neq \beta\,,\quad |\boldsymbol{k}^{11}_{\alpha\alpha;12} - \bar{\boldsymbol{k}}^{11}_{\alpha\alpha;12}| \neq |\boldsymbol{k}^{11}_{\beta\beta;12} - \bar{\boldsymbol{k}}^{11}_{\beta\beta;12}|\,,\quad \alpha \neq \beta$$

and in the case $m = 2n + 1$

$$\boldsymbol{k}^{11}_{\alpha\beta}(s) = \bar{\boldsymbol{k}}^{11}_{\alpha\beta}(s)\,,\quad \alpha \neq \beta\,,\quad \boldsymbol{k}^{12}(s) = \bar{\boldsymbol{k}}^{12}(s)\,,$$
$$|\boldsymbol{k}^{11}_{\alpha\alpha;12} - \bar{\boldsymbol{k}}^{11}_{\alpha\alpha;12}| \neq |\boldsymbol{k}^{11}_{\beta\beta;12} - \bar{\boldsymbol{k}}^{11}_{\beta\beta;12}|\,,\quad \alpha \neq \beta\,,\quad \boldsymbol{k}^{11}_{\alpha\alpha;12} \neq \bar{\boldsymbol{k}}^{11}_{\alpha\alpha;12}\,.$$

Then there exist only two motions $g(t)$, $-g(t)$ *such that the axoids of* $g(t)$, $-g(t)$ *are* $\boldsymbol{p}(s)$ *and* $\bar{\boldsymbol{p}}(s)$.

Proof. Let γ, γ_1 be the Frenet lifts of $\boldsymbol{p}$, $\bar{\boldsymbol{p}}$, satisfying (U). Let us set $g = \pm\gamma\gamma_1^{-1}$. We shall now find the axoids of the motion $g(t)$. We have $\pi\gamma_1 = \bar{\boldsymbol{p}}$, $\pi\gamma = \boldsymbol{p}$. The moving cone of $g(t)$ is $\bar{\mathbf{R}}\,ds = g^{-1}\,dg = \gamma_1\gamma^{-1}(d\gamma\,.\,\gamma_1^{-1} + \gamma\,d\gamma_1^{-1}) = \gamma_1(\gamma^{-1}\,d\gamma + d\gamma^{-1}\,.\,\gamma_1)\,\gamma_1^{-1} = \mathrm{ad}\gamma_1(\omega - \bar{\omega})$ because of $d\gamma_1^{-1}\,.\,\gamma_1 + \gamma_1^{-1}\,d\gamma_1 = 0$. If we denote $\omega - \bar{\omega} = \mathbf{R}_0\,ds$, then $\mathbf{R}_0 \in \mathfrak{o}(m)$ and we have $\bar{\mathbf{R}} = \mathrm{ad}\gamma_1\mathbf{R}_0$. $\mathbf{R}_0$ is a regular element of $\mathfrak{O}(m)$, as we see from the assumptions of the theorem, and so is $\bar{\mathbf{R}}$. For Cartan subalgebras $\mathfrak{h}(t)$ we then have $\mathfrak{h}(s) = \mathrm{ad}\gamma_1\,\mathfrak{o}(m)$ and $\mathcal{N}(\mathfrak{h}(s)) = \mathcal{N}(\mathrm{ad}\gamma_1\,\mathfrak{o}(m)) = \gamma_1\,\mathcal{N}(\mathfrak{o}(m))$ and finally $\pi\,\mathcal{N}(\mathfrak{h}(s)) = \pi\gamma_1 = \bar{\boldsymbol{p}}$ and $g\boldsymbol{p} = \gamma\gamma_1^{-1}\,.\,\pi\gamma_1 = \pi\gamma = \boldsymbol{p}$ which proves the theorem.

7.10. Theorem. *Let a one parametric differentiable system* $\boldsymbol{q}(s)$ *and* $\bar{\boldsymbol{q}}(s)$ *be given, where* $\boldsymbol{q}(s)$, $\bar{\boldsymbol{q}}(s)$ *are defined as in* §4. *Let us suppose that the invariants of* $\boldsymbol{q}(s)$ *and* $\bar{\boldsymbol{q}}(s)$ *satisfy* (U) *and that* $|\boldsymbol{k}^{11}_{\alpha\alpha;12} - \bar{\boldsymbol{k}}^{11}_{\alpha\alpha;12}| \neq |\boldsymbol{k}^{11}_{\beta\beta;12} - \bar{\boldsymbol{k}}^{11}_{\beta\beta;12}|$, $\alpha \neq \beta$, $\boldsymbol{k}^{11}_{\alpha\alpha;12} \neq \bar{\boldsymbol{k}}^{11}_{\alpha\alpha;12}$, $\boldsymbol{k}^{11}_{\alpha\beta}(s) = \bar{\boldsymbol{k}}^{11}_{\alpha\beta}(s)$, $\alpha \neq \beta$, $\boldsymbol{k}^{10}_{\alpha}(s) = \bar{\boldsymbol{k}}^{10}_{\alpha}(s)$ *for* $m = 2n$ *and* $\boldsymbol{k}^{11}_{\alpha\beta}(s) = \bar{\boldsymbol{k}}^{11}_{\alpha\beta}(s)$, $\alpha \neq \beta$, $\boldsymbol{k}^{10}_{\alpha}(s) = \bar{\boldsymbol{k}}^{10}_{\alpha}(s)$, $\boldsymbol{k}^{20}(s) = \bar{\boldsymbol{k}}^{20}(s)$, $\boldsymbol{k}^{12}_{\alpha}(s) = \bar{\boldsymbol{k}}^{12}_{\alpha}(s)$ *for* $m = 2n + 1$. *Then there exists only one motion* $g(t) \in E(m)$ *with the axoids* $\boldsymbol{q}(t)$ *and* $\bar{\boldsymbol{q}}(t)$.

Proof. The proof is the same as above.

7.11. Theorem. *The motion in $O(m)$ or $E(m)$ is determined uniquely up to the equivalence by the system of its invariants from Theorem 7.7.*

7.12. Theorem. *The motion in P^m or Q^m is determined uniquely by its central curves. (The central curves must satisfy assumptions of Theorem 7.9 or of Theorem 7.10.)*

Theorems 7.11 and 7.12 are equivalent with Theorems 7.9 and 7.10 respectively.

8. Geometrical meaning of some invariants

Let us consider the case of a motion in E^m, where $m = 2n$, the other cases being analogous. Let $\boldsymbol{q}_\alpha = \boldsymbol{A}(s) + \lambda \boldsymbol{r}_{\alpha;1} + \mu \boldsymbol{r}_{\alpha;2}$, $\lambda, \mu \in \mathbf{R}$ be one of the generating planes of the axoid $\boldsymbol{q}$ at the point s. The tangent space of $\boldsymbol{q}_\alpha$ at the point $\boldsymbol{X} = \boldsymbol{A} + \lambda \boldsymbol{r}_{\alpha;1} + \mu \boldsymbol{r}_{\alpha;2}$ has the base $\{\boldsymbol{A}, \boldsymbol{r}_{\alpha;1}, \boldsymbol{r}_{\alpha;2}, \boldsymbol{r}_{\alpha;3}\}$, where

$$(71) \qquad \boldsymbol{r}_{\alpha;3} = \sum_{\beta=1,\beta\neq\alpha}^{n} \{\boldsymbol{k}_{\beta;1}^{10} \boldsymbol{r}_{\beta;1} + \lambda(\boldsymbol{k}_{\alpha\beta;11}^{11} \boldsymbol{r}_{\beta;1} + \boldsymbol{k}_{\alpha\beta;12}^{11} \boldsymbol{r}_{\beta;2}) + \mu(\boldsymbol{k}_{\alpha\beta;21}^{11} \boldsymbol{r}_{\beta;1} + \boldsymbol{k}_{\alpha\beta;22}^{11} \boldsymbol{r}_{\beta;2})\} .$$

From (71) we see that the invariants $\boldsymbol{k}_{\beta;1}^{10}$ and $\boldsymbol{k}_{\alpha\varrho;pq}^{11}$, $\alpha \neq \beta$ give the tangent spaces of $\boldsymbol{q}$ at the points of the generating plane $\boldsymbol{p}_\alpha$. They are hence analogous to the parameter of distribution of a ruled surface. From (62) and (66) we see: The tangent spaces of every generating plane of the fixed axoid coincide with that of the moving axoid at every moment. Let us denote by $\sigma(s)$ the Euclidean arc of the curve $\boldsymbol{A}(s)$. Then we have $\mathrm{d}\sigma/\mathrm{d}s = \{\sum_{\alpha=1}^{n} (\boldsymbol{k}_{\alpha;1}^{10})^2\}^{1/2}$. From (62) and (66) we now see that the curve $\bar{\boldsymbol{A}}(s)$ rolls on the curve $\boldsymbol{A}(s)$ without slipping.

8.1. Definition. Let a motion $g(t)$ be given. Let $\mathbf{R}(t)$ be its fixed cone. Then the motion $\exp \tau\, \mathbf{R}(t_0)$, $\tau \in \mathbf{R}$ is called the instantaneous motion of $g(t)$ at $t = t_0$.

The instantaneous motion is characterized by

Proposition. The motions $g(t)$ at $t = t_0$ and $\exp \tau\, \mathbf{R}(t_0)$ at every $\tau \in \mathbf{R}$ have the same field of tangent vectors of trajectories of points.

The following proposition gives the geometrical definition of axoids.

8.2. Proposition. *Let $g(t)$ be a motion in $O(m)$ or $E(m)$. Then the generating subspaces of $\boldsymbol{p}(t_0)$ or $\boldsymbol{q}(t_0)$ respectively generate all invariant subspaces of the instantaneous motion of $g(t)$ at $t = t_0$.*

8.3. Proposition. *The functions r_α are velocities (in the canonical time) of rotation in the generating subspaces $\boldsymbol{p}_\alpha$ or $\boldsymbol{q}_\alpha$ respectively, r_0 is the parameter of the instantaneous helicoidal motion in the case of E^{2n+1}.*

8.4. Theorem. *The motion in P^m or Q^m is a roulette motion.*

Proof. Let us denote by $\boldsymbol{p}(t)$, $\bar{\boldsymbol{p}}(t)$ the central curves and by $\gamma(t)$, $\gamma_1(t)$ the Frenet lifts of $\boldsymbol{p}(t)$, $\bar{\boldsymbol{p}}(t)$. Then we have $g(t) = \gamma(t)\,\gamma_1^{-1}(t)$, $\pi\,\gamma(t) = \boldsymbol{p}(t)$, $\pi\,\gamma_1(t) = \bar{\boldsymbol{p}}(t)$. Let us write $\pi\varkappa = \pi\bar{\varkappa} = X$, where $\varkappa$, $\bar{\varkappa}$ are the same as in (67). Let $t = t_0$ be fixed. Then

$$\gamma_1'(t_0) = L_{\gamma_1(t_0)}\bar{\omega}\,, \quad \gamma'(t_0) = L_{\gamma(t_0)}\omega\,, \quad \bar{\boldsymbol{p}}'(t_0) = \pi\,\gamma_1'(t_0) = L_{\gamma_1(t_0)}X\,, \quad \boldsymbol{p}'(t_0) = L_{\gamma(t_0)}X\,,$$

$$(72) \qquad g(t_0)_*\,\bar{\boldsymbol{p}}'(t_0) = L_{\gamma(t_0)\gamma_1^{-1}(t_0)}\cdot L_{\gamma_1(t_0)}X = L_{\gamma(t_0)}X = \boldsymbol{p}'(t_0)\,.$$

From (72) we see that the tangent vectors at the coinciding points of the central curves coincide. The arcs are the same, as we see from (68). (Compare with [8].)

In the end we shall find the geometric interpretation for the invariants t, r_α, $\varkappa$ of a motion (See Theorem 7.7). Let $\mathscr{R}_0 \equiv \boldsymbol{E}_{\alpha\beta}$ be the canonical base of the algebra $\mathfrak{GL}(m, \mathbf{R})$ or $\mathfrak{GL}(m+1, \mathbf{R})$. Let us denote $\mathscr{A}$ the set of frames $\mathscr{R} = g^*\,\mathscr{R}_0(g^*)^{-1}$, where respectively $g \in O(m)$ or $g \in E(m)$. We have from (16)

$$\mathrm{d}\mathscr{R} = [\omega^*, \mathscr{R}]\,.$$

Let us suppose that we are given a motion $g(t)$, its fixed cone $\mathbf{R}(t)$ and the Frenet lift $\gamma(t)$ of its fixed axoid. Then $\mathscr{R}_{\alpha\alpha}$ gives the canonical base in $\mathfrak{h}(t)$ and we have

$$\mathbf{R} = r_\alpha(\mathscr{R}_{\alpha\alpha;12} - \mathscr{R}_{\alpha\alpha;21})\,, \quad (\mathrm{d}\mathscr{R}_{\alpha\beta}) = [(\omega_{\alpha\beta})^*, (\mathscr{R}_{\alpha\beta})]$$

where $\omega_{\alpha\beta} = \varkappa_{\alpha\beta}\,\mathrm{d}t$. We have obtained

8.5. Theorem. *The functions t, r_α, $\varkappa$ are invariants of the fixed cone (invariants of the group* ad$G$*).*

Remark. Theorem 8.5 is valid for any cone in $\mathfrak{O}(m)$ or $\mathfrak{E}(m)$ consisting from regular elements.

9. Cylic motion

9.1. Definition. Let $\mathfrak{G}$ be a Lie algebra. Let $X, Y \in \mathfrak{G}$, $[X, Y] \neq 0$ be given. Motions equivalent with $g(\varphi) = \exp\varphi X \exp\varphi Y$ are called cyclic motions.

Our definition of cyclic motion is a generalisation of the cyclic motion in E^2 and of the helicoidal motion in E^3.

9.2. Theorem. *Let us suppose that we are given a motion $g(t)$ with $\mathbf{R}(t)$ consisting of regular elements. Let us suppose that the lifts of the fixed and the moving axoids satisfy the assumptions of Theorems* 3.6–3.10. *Then $g(t)$ is cyclic if and only if it has all invariants constant.*

Proof. Let $g(\varphi)$ be a cylic motion. We may choose such a representant of $g(\varphi)$ in its equivalence class that $X + Y \in \mathfrak{o}(m)$ or $X + Y \in \mathfrak{c}(m)$ respectively and that X, Y satisfy condition (U). Let us write $g(\varphi) = g_1(\varphi)\, g_2(\varphi)$. We now have

$$\mathbf{R}(\varphi) = \lambda \mathrm{d}/\mathrm{d}\varphi(g_1 \cdot g_2) \cdot (g_1 \cdot g_2)^{-1} = \lambda \operatorname{ad} g_1(X + Y), \quad \lambda \in \mathbf{R}$$

and $\mathfrak{h}(\varphi) = \operatorname{ad} g_1\, \mathfrak{o}(m)$ or $\mathfrak{h}(\varphi) = \operatorname{ad} g_1\, \mathfrak{c}(m)$ respectively. The fixed axoid is $\boldsymbol{p}(\varphi) = \pi g_1$ or $\boldsymbol{q}(\varphi) = \pi g_1$ and g_1 is its lift. But $g_1(\varphi)$ is the Frenet lift of the fixed axoid, as $g_1^{-1}\, \mathrm{d}g_1 = X\, \mathrm{d}\varphi$, and X satisfies (U). As X is constant and $\mathrm{d}\varphi/\mathrm{d}t$ is constant, the invariants are constant, too. The rest is obvious.

Remark. If $(X + Y, X + Y) = 1$, then the coordinates of X and Y are invariants of the motion.

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