Czechoslovak Mathematical Journal

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Pure closures

Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 1, 78–82

Persistent URL: http://dml.cz/dmlcz/101078

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

PURE CLOSURES

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(Received June 29, 1970)

The purpose of this note is to give some sufficient conditions for the existence of ω-pure closures of any submodule of an arbitrary Λ-module B.

First of all we shall give basic definitions. In this paper Λ stands for an associative ring with unity. We shall say that in the category of (all) Λ-modules a purity ω is given if in any Λ-module B, some set of submodules called ω-pure in B is taken (the fact that A is ω-pure in B being denoted by $A \subseteq {}_\omega B$) such that:

P0: Any direct summand of B is ω-pure in B,
P1: $A \subseteq {}_\omega B,\ B \subseteq {}_\omega C \Rightarrow A \subseteq {}_\omega C$,
P2: $A \subseteq B \subseteq C$ [1]), $A \subseteq {}_\omega C \Rightarrow A \subseteq {}_\omega B$,
P3: $A \subseteq {}_\omega B,\ K \subseteq A \Rightarrow A/K \subseteq {}_\omega B/K$,
P4: $K \subseteq A \subseteq B,\ K \subseteq {}_\omega B,\ A/K \subseteq {}_\omega B/K \Rightarrow A \subseteq {}_\omega B$.

Let $\mathscr{E}$ be any set of (left) ideals of Λ, $A \subseteq B$ Λ-modules. We say that A is $\mathscr{E}$-pure in B if for any commutative diagram

$$\begin{array}{ccc} I & \xrightarrow{\ \chi\ } & \Lambda \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\eta} \\ A & \xrightarrow{\ i\ } & B \end{array}$$

where $I \in \mathscr{E}$ and χ, i are canonical injections there exists $\psi : \Lambda \to A$ such that $\chi\psi = \varphi$. It can be shown that all the properties P0–P4 are satisfied in this case. A Λ-module A is called ω-divisible if it is ω-pure in any of its extensions. It is easy to see that any projective module is ω-divisible (for any purity ω). An extension B of A will be called an ω-divisible closure of A if B is ω-divisible and no proper submodule of B containing A is ω-divisible (such a B need not exist and need not be unique). Similarly, a Λ-module C with $A \subseteq C \subseteq B$ will be called an ω-pure closure of A in B if $C \subseteq {}_\omega B$ and no proper submodule of C containing A is ω-pure in B (again, such a C need not exist

[1]) Throughout this paper $A \subseteq B$ means that A is a submodule of B.

and need not be unique). Finally, a Λ-module C is called ω-flat if, for any epimorphism $\varphi : B \to C$, Ker φ is ω-pure in B.

1. Throughout this section let $\mathscr{E}$ be some set of maximal left ideals of Λ and let ω denote the $\mathscr{E}$-purity. For any Λ-module G and any $I \in \mathscr{E}$ we put $G(I) = \{g \in G; \lambda g = 0 \text{ for any } \lambda \in I\}$.

Lemma 1.1. *Let G be a Λ-module, $\hat{G}$ its injective closure, $I \in \mathscr{E}$. Then $G(I) = \hat{G}(I)$.*

Proof. It clearly suffices to show $\hat{G}(I) \subseteq G(I)$. Proving this relation indirectly, let us suppose the existence of $g \in \hat{G}(I) \dot{-} G(I)$ and let us consider the module Λg. In view of $g \neq 0$ and $g = 1g$ there is $\Lambda g \neq 0$. To any $\mu \notin I$ there exists $\varrho \in \Lambda$ and $\sigma \in I$ with $\varrho\mu + \sigma = 1$ for I being maximal. Then $g = \varrho\mu g \notin G$, hence $\mu g \notin G$ which implies $\Lambda g \cap G = 0$ – a contradiction with the essentiallity of G in $\hat{G}$.

Theorem 1.2. *Let G be a Λ-module and $\hat{G}$ its injective closure. If $D \subseteq {}_\omega\hat{G}$, then $D \cap G \subseteq {}_\omega G$.*

Proof. For any $I \in \mathscr{E}$ let us consider the following two diagrams

$$(*)\qquad \begin{array}{ccc} I & \xrightarrow{\chi} & \Lambda \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\eta} \\ D \cap G & \xrightarrow{i} & G \end{array} \qquad\qquad (**)\qquad \begin{array}{ccc} I & \xrightarrow{\chi} & \Lambda \\ {\scriptstyle\vartheta}\downarrow & & \downarrow{\scriptstyle\theta} \\ D & \xrightarrow{j} & \hat{G} \end{array}$$

where χ, i, j are canonical injections, φ, η arbitrary homomorphisms making $(*)$ commutative and ϑ, θ are defined as follows: If $1\eta = g$ then θ is determined by $1\theta = g$ and $\vartheta = \theta/I$. Now the diagram $(**)$ is commutative because for any $\lambda \in I$ it is $\lambda\vartheta = \lambda\theta = \lambda g = \lambda\eta = \lambda\varphi \in D \cap G \subseteq D$. By hypothesis there exists $\varrho : \Lambda \to D$ with $\chi\varrho = \vartheta$. Denoting $1\varrho = d$ we have $\lambda\chi\varrho = \lambda d = \lambda\vartheta = \lambda g$ for any $\lambda \in I$ which implies $\lambda(d - g) = 0$, i.e. $d - g \in \hat{G}(I)$. From Lemma 1.1 we get $d - g \in G(I) \subseteq G$, hence $d \in G$. Now we can define a homomorphism $\psi : \Lambda \to D \cap G$ by putting $1\psi = d$. Then for any $\lambda \in I$ there is $\lambda\chi\psi = \lambda d$ and $\lambda\varphi = \lambda\chi\eta = \lambda g = \lambda d$ so that $\chi\psi = \varphi$ and the proof is finished.

The following example shows that the maximality of ideals from $\mathscr{E}$ is essential.

Example 1.3. For $\Lambda = Z$ (the ring of integers), $G = \{a\} \dotplus \{b\}$, $p^3a = pb = 0$, $N = \{pa + b\}$, $\mathscr{E} = \{(p^2)\}$ we have $\hat{N} \subseteq_\omega \hat{G}$, $N = \hat{N} \cap G$ (for the proof see e.g. [1] § 28, h) and for the commutative diagram

$$\begin{array}{ccc} (p^2) & \xrightarrow{\chi} & Z \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\eta} \\ N & \xrightarrow{i} & G \end{array}$$

where χ, i are canonical injections and $1\eta = a$, $\varphi = \eta \mid (p^2)$ it is $p^2\eta = p^2a = p(pa + b) \in N$, but no $\psi : Z \to N$ with $\chi\psi = \varphi$ exists, because for $1\psi = \alpha(pa + b)$ we have $p^2\psi = 0$ while $p^2\varphi = p^2a \neq 0$. (This example is essentially that from [1] p. 92).

Theorem 1.4. *Let us suppose that the following condition holds:*

$$(1) \qquad N \subseteq {}_{\omega}G \Rightarrow \exists D\,, \quad D \subseteq {}_{\omega}\hat{G}\,, \quad N = D \cap G\,.$$

Then any Λ-module A has an ω-pure closure in any of its extensions if and only if A has an ω-divisible closure.

Proof. a) If A has an ω-pure closure in any of its extensions then, particularly, A has an ω-pure closure A^{ω} in its injective closure. A^{ω} is ω-divisible by 1,7 from [2]. In fact, A^{ω} is an ω-divisible closure of A.

b) Conversely, let B be any extension of A and A^{ω} an ω-divisible closure of A. We can assume $\hat{A} \subseteq \hat{A}^{\omega}$ owing to $A \subseteq A^{\omega}$ and Lemma 11.1 from [3]. Then clearly $\hat{A} \subseteq {}_{\omega}\hat{A}^{\omega}$ and by Theorem 1.2 $\hat{A} \cap A^{\omega} \subseteq {}_{\omega}A^{\omega}$. $\hat{A} \cap A^{\omega}$ contains A and is ω-divisible by 1,8 from [2], hence $\hat{A} \cap A^{\omega} = A^{\omega}$ in view of the minimality of A^{ω}. Thus we have $A^{\omega} \subseteq \hat{A}$ and $\hat{A} = \hat{A}^{\omega}$.

Further, we can assume $\hat{A} \subseteq \hat{B}$. It is $A^{\omega} \subseteq {}_{\omega}\hat{A} \subseteq {}_{\omega}\hat{B}$ so that Theorem 1.2 implies $A^{\omega} \cap B \subseteq {}_{\omega}B$. It remains to show that $A^{\omega} \cap B$ is a minimal Λ-module ω-pure in B and containing A. Let us suppose $A \subseteq A' \subseteq {}_{\omega}A^{\omega} \cap B \subseteq {}_{\omega}B$. By (1) there exists a Λ-module D with $D \subseteq_{\omega} \widehat{A^{\omega} \cap B}$ and $A' = A^{\omega} \cap B \cap D$. It can be assummed that $\widehat{A^{\omega} \cap B} \subseteq \hat{A}$ since $A^{\omega} \cap B \subseteq A^{\omega} \subseteq \hat{A}$. Then $D \subseteq {}_{\omega}A^{\omega} \cap B \subseteq {}_{\omega}\hat{A} = \hat{A}^{\omega}$ and by Theorem 1.2 $D \cap A^{\omega} \subseteq {}_{\omega}A^{\omega}$. The same arguments as above lead to $D \cap A^{\omega} = A^{\omega}$, hence $A' = B \cap A^{\omega} \cap D = B \cap A^{\omega}$.

2. In this section we shall give a sufficient condition for the existence of ω-pure closures.

Theorem 2.1. *Let $\mathscr{E} = \{\Lambda\mu,\ \mu \in M\}$ be any set of maximal principal left ideals of Λ and let ω denote the $\mathscr{E}$-purity. Then any Λ-module has an ω-divisible closure.*

Proof. First of all let us note that

$$(2) \qquad A \subseteq {}_{\omega}B \Leftrightarrow \mu B \cap A = \mu A \quad \text{for any} \quad \mu \in M\,.$$

The proof of this fact we omit because it is given in [2], Prop. 1, 52. Now we shal construct an ω-divisible closure for any Λ-module A. Let us put $D_0 = A$ and if D_n is constructed then D_{n+1} is a submodule of $\hat{A}$ (the injective closure of A) generated by D_n and all $d \in \hat{A}$ satisfying $\mu d \in D_n$ for some $\mu \in M$. Thus $D = \bigcup_{n=0}^{\infty} D_n$ is a submodule of $\hat{A}$ containing A. For $d \in \mu\hat{A} \cap D$, $d = \mu\bar{a}$, $\bar{a} \in \hat{A}$ and $d \in D_n$ we have $\bar{a} \in D_{n+1}$ owing to the definition of D_{n+1}, hence $d \in \mu D$. Thus $D \subseteq {}_\omega\hat{A}$ by (2), which implies the ω-divisibility of D (by 1,7 from [2]). We are going to show the minimality of D. Let us suppose $A \subseteq Q \subseteq D$, Q ω-divisible. We have $D_0 \subseteq Q$. If $D_n \subseteq Q$ and $d \in D_{n+1}$ is an arbitrary generator of D_{n+1} (not belonging to D_n) then there exists $\mu \in M$ with $\mu d \in D_n \subseteq Q$. Since Q is ω-divisible, we have $Q \subseteq {}_\omega D$ and $\mu d \in \mu D \cap \cap Q = \mu Q$ by (2). Then $\mu(d - q) = 0$ for a suitable $q \in Q$. In view of Lemma 1.1 and $A \subseteq Q \subseteq D \subseteq \hat{A}$ we have $d - q \in \hat{A}(\Lambda\mu) = A(\Lambda\mu) \subseteq Q$ and hence $d \in Q$. Thus $D_{n+1} \subseteq Q$ and finally $D = Q$.

Theorem 2.2. *Let $\mathscr{E} = \{\Lambda\mu, \mu \in M\}$ be any set of maximal principal left ideals of Λ such that $\mu\Lambda \subseteq \Lambda\mu$ for any $\mu \in M$. Then the $\mathscr{E}$-purity satisfies the condition* (1).

Proof. Let us assume $N \subseteq {}_\omega G$ and let $N \subseteq D \subseteq \hat{N}$ be the ω-divisible closure constructed in the preceding proof. It is obvious that $N \subseteq D \cap G$. On the other hand it is clear that $D_0 \cap G \subseteq N$. Let us assume we have proved $D_n \cap G \subseteq N$ and let $d \in D_{n+1} \cap G$ be an arbitrary element. Then we can write $d = d' + \sum_{i=1}^{r} \lambda_i d_i$, $d' \in D_n$, $\mu_i d_i \in D_n$ for suitable $\mu_i \in M$. Then $\mu_1\mu_2 \ldots \mu_r d = \mu_1\mu_2 \ldots \mu_r d' + \sum_{i=1}^{r} \mu_1\mu_2 \ldots \ldots \mu_r\lambda_i d_i = \mu_1\mu_2 \ldots \mu_r d' + \sum_{i=1}^{r} \mu_1\mu_2 \ldots \mu_{i-1}\lambda'_i\mu_i d_i$ by hypothesis (λ'_i are suitable elements from Λ) and therefore $\mu_1\mu_2 \ldots \mu_r d \in D_n \cap G \subseteq N$. Hence $\mu_1(\mu_2 \ldots \mu_r d) \in \in \mu_1 G \cap N = \mu_1 N$ in view of $N \subseteq {}_\omega G$ and (2). For a suitable element $t \in N$ we have $\mu_1(\mu_2 \ldots \mu_r d - t) = 0$ which implies $\mu_2 \ldots \mu_r d - t \in \hat{N}(\Lambda\mu_1) = N(\Lambda\mu_1) \subseteq N$ (by Lemma 1.1) so that $\mu_2 \ldots \mu_r d \in N$. Similar arguments for $\mu_2, \ldots, \mu_r$ lead to $d \in N$ which finishes the proof.

3. In this section we shall prove a theorem on the existence of ω-pure closures concerning ω-flat modules. We start with the following

Lemma 3.1. *Let B be an ω-flat Λ-module, $K \subseteq {}_\omega B$, $L \subseteq {}_\omega B$. If $\{K, L\}$ is ω-flat, then $K \cap L \subseteq {}_\omega B$.*

Proof. From $L \subseteq {}_\omega B$ it follows $L \subseteq {}_\omega\{K, L\}$ by P2 and hence $\{K, L\}/L$ is ω-flat by hypothesis and 1,13 from [2]. Then $K/K \cap L \cong \{K, L\}/L$ is ω-flat. The definition of ω-flat modules implies that $K \cap L \subseteq {}_\omega K$. Now it suffices to use P1.

Theorem 3.2. *Let ω be an arbitrary purity such that any submodule of an ω-flat module is ω-flat. Then any submodule of an ω-flat module B has in B the uniquely determined ω-pure closure if and only the following condition is satisfied*:

(3) *For any decreasing chain $B = B_0 \supseteq B_1 \supseteq \ldots \supseteq B_\alpha \supseteq \ldots \supseteq B_\Omega$ of submodules of B satisfying $B_{\alpha+1} \subseteq {}_\omega B_\alpha$ and $B_\alpha = \bigcap_{\gamma<\alpha} B_\gamma$, α a limit ordinal, there is $B_\Omega \subseteq {}_\omega B$.*

Proof. Let B be an ω-flat module, $A \subseteq B$ a submodule and let the condition (3) hold. Using the Zorn's lemma one can easily get the existence of ω-pure closures of A in B. For the proof of unicity it suffices to use Lemma 3.1.

Conversely, let us have an descending chain $B = B_0 \supseteq B_1 \supseteq \ldots \supseteq B_\alpha \supseteq \ldots \supseteq B_\Omega$ of submodules of B satisfying the conditions stated in (3). It is easy to see that we can restrict ourselves to the case $B_\alpha \subseteq {}_\omega B$, $\alpha < \Omega$, where Ω is a limit ordinal. If B_Ω is not ω-pure in B, it has an ω-pure closure $\tilde{B}_\Omega \supsetneqq B_\Omega$. There exists an ordinal $\alpha < \Omega$ with $B_\alpha \cap \tilde{B}_\Omega \subsetneqq \tilde{B}_\Omega$ because the converse leads to the contradiction $\tilde{B}_\Omega = B_\Omega$. By Lemma 3.1 it is $B_\alpha \cap \tilde{B}_\Omega \subseteq {}_\omega B$ – a contradiction with the minimality of $\tilde{B}_\Omega$. Consequently $\tilde{B}_\Omega = B_\Omega \subseteq {}_\omega B$.

Theorem 3.3. *Let r be a radical in the category of Λ-modules and let ω be any purity such that the class of ω-flat Λ-modules coincides with the class of r-semisimple Λ-modules. Then any submodule of an r-semisimple Λ-module B has in B the uniquely determined ω-pure closure.*

Proof. Clearly, the class of ω-flat modules is closed under taking submodules and direct products by 2.12 from [2]. To prove (3) it suffices to show that for α limit, $B_\gamma \subseteq {}_\omega B$, $\gamma < \alpha$ it is $B_\alpha \subseteq {}_\omega B$. However, B/B_α can be selected, in the natural way, in the direct product of B/B_γ, $\gamma < \alpha$ and hence following the arguments mentioned above B/B_α is ω-flat. Thus $B_\alpha \subseteq {}_\omega B$ owing to the definition of ω-flat modules.

Remark. From the above proof it immediately follows that the condition: "The class of ω-flat Λ-modules is closed under taking submodules and direct products" is sufficient for the existence and uniqueness of an ω-pure closure of any submodule of an ω-flat module.

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