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ON MATRICES HAVING AN INVARIANT CONE

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1. INTRODUCTION

The well known theorems of Perron and Frobenius have been generalized to operators in a partially ordered Banach space (cf. [7]). This has motivated several authors to consider linear operators (or matrices) in a finite dimensional space which leave a cone invariant (cf. [1] and [16]). Our purpose is to continue the extensions of Perron-Frobenius theory to the more general case of a matrix nonnegative with respect to a cone. We assume a familiarity with the papers of BIRKHOFF [1] and VANDERGRAFT [16].

Throughout we shall use *iff* for *if and only if*, and on occasion we use $\forall$ and $\exists$ for *for all* and *there exists* respectively. For cones K we let K^0 denote the interior of K, ∂K its boundary, and if F is a face of K (definition 2 below) F^Δ denotes the relative interior of F. Finally, if $A \geqq 0$ then $\varrho(A)$ denote the Perron root of A, that is, the eigenvalue of A which is the spectral radius.

2. CONES AND PARTIAL ORDERS

Definition 1. A set K in a real vector space V of dimension n is said to be *a cone* iff

(i) K is a nonempty closed subset of V,
(ii) $K + K \subseteq K$,
(iii) $\alpha K \subseteq K$ for all $\alpha \geqq 0$,
(iv) $K \cap (-K) = \{0\}$.

If in addition K satisfies

(v) $K - K = V$,

then K is *a full cone*. In general we shall use K to denote a full cone, but we shall omit the word full.

As is well known a cone K determines a partial order in V. For this partial order we use the notation

$x \geqq 0$ iff $x \in K$ (x is nonnegative),
$x > 0$ iff $x \geqq 0$ and $x \neq 0$ (x is positive),
$x \gg 0$ iff $x \in K^0$ (x is strictly positive).

Definition 2. Let K be a cone. By a face F of K is meant a subset of K which satisfies (i), (ii), (iii), (iv) above and the following condition:

$$0 \leqq y \leqq x \quad \text{and} \quad x \in F \quad \text{implies} \quad y \in F .$$

This definition of face is due to HANS SCHNEIDER. In what follows we may regard vectors in V as column vectors in R^n and vectors in the dual space may be regarded as row vectors. Thus if $x \in R^n$, if A is an $n \times n$ matrix, and if $f \in (R^n)^*$, then fAx and fx are just the usual products of matrices.

Finally, we set

$$K^* = \{f \in V^* \mid fx \geqq 0, \text{ all } x \in K\} .$$

If $S \subseteq K$, we shall denote by $\Phi(S)$ the intersection of all faces containing S. Clearly, $\Phi(S)$ is a face. It is called the face generated by S.

The set of all $n \times n$ matrices

$$C = \{A \mid AK \subseteq K\}$$

is easily seen to be a cone in the space of all $n \times n$ matrices. With respect to C we have two additional refinements of the order relation.

Definition 3. Let $A \in C$.

(i) A is irreducible [16] iff A leaves invariant no face of K except $\{0\}$ and K itself.
(ii) A is primitive, denoted by A (>0, iff

$$\forall x \in \partial K \setminus \{0\} \; \exists n \;\; A^n x \gg 0 .$$

It is well known [7] that for $f \in K^*$, $f \gg 0$ (in the partial order induced by K^*) iff $fx > 0$ for all $x > 0$. An analogous result holds for $A \gg 0$.

Proposition 1. $A \gg 0$ *iff* $Ax \gg 0$ *for all* $x > 0$.

Proof. Let us first observe that if $f \in V^*$ and $x \in V$, then the operation defined by

$$(f, x)\, A = fAx$$

is a linear functional on the set of $n \times n$ matrices. In particular, if $f \in K^*$, $x \in K$, then $(f, x) \in C^*$.

Suppose first that $Ax \gg 0$ does not hold for all $x > 0$. Since $Ay \gg 0$ for some $y > 0$ implies $AK^0 \subseteq K^0$, there is an $x \in \partial K \setminus \{0\}$ for which $Ax \in \partial K$. But then there is a linear functional $f > 0$ for which the hyperplane $fy = 0$ contains Ax. Let $l = (f, x) \in C^*$, l is not the zero functional. We have $lA = fAx = 0$, so $A \notin C^0$. Thus $A \in C^0$ implies $Ax \gg 0$ for all $x > 0$.

Conversely, suppose $A \gg 0$ for all $x > 0$. The mapping $(A, x) \to Ax$ is jointly continuous in A and x. Let $\|.\|$ be a norm on V and let

$$S = K \cap \{x \mid \|x\| = 1\}.$$

For each $x \in S$ there are open neighborhoods $U_x(A)$ and $N(x)$ of A and x respectively such that $U_x(A)\, N(x) \subseteq K^0$ since $Ax \gg 0$. However S is compact. We may therefore extract a finite subcover $N(x_1), \ldots, N(x_m)$ of it and take the corresponding neighborhoods $U_{x_1}(A), \ldots, U_{x_m}(A)$ of A. Let

$$U = \bigcap_{j=1}^{m} U_{x_j}(A).$$

U is an open neighborhood of A. Let $B \in U$. If $x \in S$, then $x \in N(x_i)$ for some i. Since $B \in U_{x_i}(A)$, we have $Bx \gg 0$. Thus $BS \subseteq K^0$. If $x \in K \setminus \{0\}$, then $\|x\|^{-1} x \in S$. Thus

$$Bx = \|x\|\, B(\|x\|^{-1} x) \gg 0$$

and so $U \subseteq C^0$. Hence $A \in C^0$ and the proposition is proved.

3. PRIMITIVE MATRICES

Krein and Rutman [7 Definition 6.1] have introduced the concept of a strongly positive operator. However, in the matrix case it is the generalization of primitivity, so we employ this latter term in definition 3.

Proposition 2. *A is primitive iff* $\exists n\ \forall x > 0,\ A^n x \gg 0$.

Proof. Since the condition is clearly a strengthening of definition 3, we need prove only that if A is primitive then n is independent of x.

Let $B = \{x \in V \mid x^T x = 1\}$ and let $Q = K \cap B$. Q is compact, and A restricted to Q remains continuous. For each $x \in Q$, there is an integer $n(x)$ and a set $U(x)$ open in the relative topology of Q such that

$$A^{n(x)}\, U(x) \subseteq K^0.$$

The collection $\{U(x) \mid x \in Q\}$ is an open cover from which we may extract a finite subcover, say $U(x_1), \ldots, U(x_m)$ with corresponding exponents $n(x_1), \ldots, n(x_m)$. Let

$n = \max\{n(x_1), \ldots, n(x_m)\}$. For any $x \in Q$, $\exists x_i$ such that $x \in U(x_i)$. Thus

$$A^n x = A^{n-n(x_i)}(A^{n(x_i)}x) \in K^0 .$$

If $x \in K^0$, then $Ax \in K^0$, so $A^n x \in K^0$. If $x \in \partial K \setminus \{0\}$, then $(x^T x)^{-1/2} x \in Q$. Thus

$$(x^T x)^{-1/2} A^n x = A((x^T x)^{-1/2} x) \in K^0 ,$$

whence $A^n x \in K^0$, and the theorem is proved.

It is clear that if A is primitive, then A is irreducible. Let us remark in passing that if A is irreducible the spectral radius $\varrho(A) > 0$, if $\dim V > 1$.

Theorem 1. $A (> 0$ *iff* A *leaves no subset of* ∂K *other than* $\{0\}$ *invariant.*

Proof. Let $A (> 0$. Then $\exists n$ for which

$$(*) \qquad A^n(K \setminus \{0\}) \subseteq K^0$$

by proposition 2. If $S \subseteq \partial K$ is invariant under A, then

$$A^n S \subseteq S \subseteq \partial K .$$

Hence by (*) $S = \{0\}$.

Conversely suppose A leaves no nonzero subset of ∂K invariant. This implies that

$$\ker A \cap \partial K = \{0\} .$$

Let $x \in \partial K \setminus \{0\}$, and consider the sequence

$$x_0 = x, \quad x_1 = Ax, \ldots, x_n = A^n x, \ldots$$

If there is no n such that $A^n x \gg 0$, then the set $S = \{x_0, x_1, \ldots\}$ satisfies

$$S \subseteq \partial K \setminus \{0\} , \quad AS \subseteq S .$$

However, this is impossible, so there is an $n = n(x)$ such that $A^n x \gg 0$. Hence A is primitive.

When K is the nonnegative orthant the relation between $A (> 0$ and A^k irreducible is well known (see Pták [12]). Our analog to this theorem is

Theorem 2. *If* K *is a polyhedral cone with the positive basis* $\{x^1, \ldots, x^p\}$, *then the following are equivalent*:

(1) $A (> 0$;
(2) A^k *is irreducible for* $k = 1, 2, \ldots$;
(3) *the matrices* $A, A^2, \ldots, A^q$ *are irreducible, where* $q = 2^p - 1$.

Proof. To show (1) implies (2) assume (1) hold but (2) is false. Then $AK^0 \subseteq K^0$. Assume for some k that A^k has an invariant face F. There is an m such that $A^m \gg 0$. Then we can find an r for which $rk > m$ and $A^{rk}F \subseteq F \subseteq \partial K$. On the other hand

$$A^{rk}(F \setminus \{0\}) = A^{rk-m}(A^m(F \setminus \{0\})) \subseteq A^{rk-m}K^0 \subseteq K^0 .$$

This contradiction establishes the implication.

(2) obviously implies (3).

Suppose (3) holds but A is not primitive. Then by theorem 1 there is a set $S \subseteq \partial K$ such that $AS \subseteq S$. We assume that S is maximal; that is, S is the union of all the proper faces F such that $AF \subseteq \partial K$. Since K is polyhedral, S is the union of finitely many faces. Let $F_1 \subseteq S$. Then AF_1 is a cone. If $\Phi(AF_1) = K$, there are vectors $x_1, \ldots, x_r \in F_1$ and scalars $\alpha_1, \ldots, \alpha_r > 0$ such that $A(\alpha_1 x_1 + \ldots + \alpha_r x_r) \gg 0$. This contradicts $AF_1 \subseteq \partial K$, whence $F_2 = \Phi(AF_1)$ is a face contained in S, an $F_2 \neq F_1$. Continuing in this fashion we obtain a sequence

$$(*) \qquad F_k \supset AF_{k-1} \supset \ldots \supset A^k F_1 .$$

But there are only finitely many faces so there is an F such that $A^m F \subset F$. Since the face F_1 was arbitrary we may take $F_1 = F$, and the sequence (*) becomes

$$F_k \supset AF_{k-1} \supset \ldots \supset A^k F_1 = A^k F_k$$

where all the inclusions are proper by irreducibility. But K has at most 2^p faces, so $k \leqq 2^p - 1$. This contradicts the irreducibility of $A, A^2, \ldots, A^q$, and so (3) implies (1).

For general cones (2) does not imply (1). If in R^3 we take

$$K = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \,\middle|\, (x_2^2 + x_2^3)^{1/2} \leqq x_1 \right\}$$

and let A be a rotation of the cone through an irrational multiple of 2π, then A^k is irreducible for all k. However $A(\partial K) = \partial K$, so A is not primitive. If instead we take A to be a rotation through the angle $2\pi/N$, then $A^1, \ldots, A^{N-1}$ are irreducible while A^N is reducible.

4. IMPRIMITIVE MATRICES

Definition 4. Let $A \geqq 0$ be irreducible. A is called *imprimitive* iff there is a set $S \subseteq \partial K$, $S \neq \{0\}$, such that $AS \subseteq S$.

Note that by theorem 1 any irreducible matrix is either primitive or imprimitive.

Proposition 3. *Let A be irreducible. A is imprimitive iff there is a maximal nonzero invariant subset $S \subseteq \partial K$. If A is imprimitive, then S is closed.*

Proof. If such an S exists, then A is clearly imprimitive. If A is imprimitive, let $\{S_\alpha\}$ be the collection of all invariant sets of $A (S_\alpha \subseteq \partial K$ of course), and define

$$S = \bigcup_\alpha S_\alpha .$$

S is obviously the maximal invariant subset of ∂K. Let y be a limit point. Then there is a sequence $\{x_n\} \subseteq S$ such that $x_n \to y$ as $n \to \infty$. By continuity for $k = 0, 1, 2, \ldots$,

$$A^k x_n \to A^k y$$

as $n \to \infty$. Since for all n and all k $A^k x_n \in \partial K$ and ∂K is closed, then $A^k y \in K$ for all k. Thus

$$S \bigcup \{A^k y \mid k = 0, 1, 2, \ldots\}$$

is an invariant subset of A. By the maximality of S, $A^k y \in S$, $k = 0, 1, \ldots$ So S is closed.

In the remainder of this section S will denote the maximal invariant subset of A whenever A is imprimitive. We shall also let $T = \partial K \setminus S$. Note that T may be empty.

Theorem 3. *Let A be imprimitive and let F be a face of K.*

(i) $F^\Delta \cap T \neq \emptyset$ *implies* $F^\Delta \subseteq T$.
(ii) $F^\Delta \cap S \neq \emptyset$ *implies* $F \subseteq S$.

Consequently, if T consists of finitely many open faces, and in particular if K is polyhedral, then there is a k such that

$$A^k T \subseteq K^0 .$$

Proof. Let $x \in F^\Delta \cap T$ and $y \in F^\Delta$. Then there are $\alpha > 0$, $k > 0$ such that $0 \leqq \alpha x \leqq y$ and $0 \ll A^k x$. Then

$$0 \ll \alpha A^k x \leqq A^k y ,$$

whence $A^k y \gg 0$.

Now let $x \in F^\Delta \cap S$. Then $\Phi(x) = F$. If $y \in F$, there is an $\alpha > 0$ such that $0 \leqq \alpha y \leqq x$. Thus $0 \leqq \alpha A^k y \leqq A^k x$ for $k = 0, 1, 2, \ldots$ But $A^k x \in S$, whence $A^k y \in \partial K$. Thus $S \cup F$ is an invariant subset of ∂K, and by the maximality of S, $F \subseteq S$.

Finally, if

$$T = \bigcap_{i=1}^{p} F_i^\Delta ,$$

choose $x_i \in F_i^\Delta$, $i = 1, \ldots, p$. We can find k_i for which

$$A^{k_i} x_i \gg 0 .$$

Let $k = \max \{k_1, \ldots, k_p\}$. Then $A^k T \subseteq K^0$.

We know that if A is imprimitive, then for each $y \in T$ there is a k such that $A^k y \gg 0$. Theorem 3 shows that if K is polyhedral, then the k may be chosen independently of y. Whether k can be taken indepedently of y for arbitrary cones remains an open question.

If A is imprimitive and $n = 2$, it is clear that $S = \partial K$.

Theorem 4. *Let $n = 3$, and let A be imprimitive. If $S \setminus \{0\}$ is arcwise connected, then $S = \partial K$.*

Proof. To the contrary, let us suppose that $S \neq \partial K$. For $x \in K$ we let $(x) = \{y \in K \mid y = \alpha x\}$, the ray determined by x. Let $B = \{x \mid x^T x = 1\}$. Then the curve $\sigma = B \cap S$ is rectifiable with endpoints x_1 and x_2, say. We define a distance function ϱ on the rays of S as follows: if $t_1, t_2 \in \sigma$, then $\varrho(t_1, t_2)$ is the arc length of the segment of σ determined by t_1 and t_2; if $x, y \in S$ there are unique vectors $t_1 \in (x) \cap \sigma$, $t_2 \in (y) \cap \sigma$ and we set $\varrho((x), (y)) = \varrho(t_1, t_2)$. Note that ϱ is well defined since there is only one segment of σ joining t_1 and t_2.

A is irreducible, so that $Ax = 0$ for $x \in K$ only if $x = 0$. Since ϱ is jointly continuous in t_1 and t_2, then the function $\varrho(x, Ax)$ is continuous on the compact set σ, and therefore assumes its infimum ϱ_0 at some point $x_0 \in \sigma$.

Suppose $\varrho_0 > 0$. Then as x traverses σ from x_1 to x_2, Ax determines a connected segment of σ. Hence $\varrho_0 > 0$ implies that Ax moves from Ax_1 to x_2, otherwise there would be a $y \in \sigma$ such that $\varrho(y, Ay) = 0 < \varrho_0$. But then $Ax_2 = \lambda x_2$, a contradiction. Hence $\varrho_0 = 0$. But then $0 = \varrho(x_0, Ax_0)$, so $\lambda x_0 = Ax_0$, $\lambda > 0$. This contradicts the hypothesis that A is irreducible. So $S = \partial K$.

To see that some condition on A is needed, let

$$K = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; (x_1^2 + x_2^2)^{1/2} \leqq x_3 \right\}.$$

Let

$$v^1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v^3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We see that $A \geqq 0$, $v^i \in \partial K$ for all i, and

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ 0 \\ x_3 \end{bmatrix}.$$

A is irreducible since it has but one eigenvector $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in K, and $w \in K^0$. The eigenvector w corresponds to $\lambda = 1$. Since $Av^1 = v^3$, and $Av^3 = v^1$, A is imprimitive. $Av^2 \in K^0$, so $S \neq \partial K$. In fact

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in K \mid x_2 = 0,\ x_1^2 = x_3^2 \right\},$$

and $S \setminus \{0\}$ is not arcwise connected.

The proof of theorem 4 depends upon the topology of 3-space, and it does not seem to carry over to higher dimensional spaces. We have not been able to resolve the problem of when $S = \partial K$ in general, but if A is invertible, we have

Theorem 5. *Let A be irreducible and invertible. Then A is imprimitive with $S = \partial K$ iff $A^{-1} > 0$. Further, if $A^{-1} > 0$, then A^{-1} is also imprimitive.*

Proof. Suppose $A^{-1} > 0$. Then since A and A^{-1} are both homeomorphisms, we have $AK^0 \subseteq K^0$ and $A^{-1}K^0 \subseteq K^0$. Thus $A(\partial K) \subseteq \partial K$ and $A^{-1}(\partial K) \subseteq \partial K$, from which it follows that $A(\partial K) = \partial K = A^{-1}(\partial K)$. Therefore, A is imprimitive. However, A^{-1} can have but one eigenvector in K, and it is in K^0. Thus A^{-1} is irreducible and therefore imprimitive.

Conversely, suppose A is imprimitive with $S = \partial K$. By continuity $A^{-1} > 0$ will follow from $A^{-1}K^0 \subseteq K$. Suppose this is false. There exists a $y \in K^0$ such that $A^{-1}y \in V \setminus K$. Since A is irreducible, there is an $x \gg 0$ for which $Ax = \varrho x$, $\varrho = \varrho(A) > 0$. Then for all α, $0 \leqq \alpha \leqq 1$ we put

$$w_\alpha = \alpha y + (1 - \alpha) x \in K^0 .$$

Further we have $A^{-1}w_0 = \varrho^{-1}w_0 = \varrho^{-1}x_0 \gg 0$ and $A^{-1}w_1 = A^{-1}y \in V \setminus K$. Thus there is a $\beta > 0$ for which $w = w_\beta$ satisfies $A^{-1}w = z \in \partial K$. But then $Az \gg 0$ contrary to the hypothesis that $S = \partial K$. Therefore, $A^{-1}K^0 \subseteq K$, and the theorem is proved.

5. OTHER ASPECTS OF NONNEGATIVITY

Another useful strengthening of the notion of nonnegativity (cf. [8], [10], and [11]) is contained in the following

Definition 5. A matrix $A \geqq 0$ is called *u_0-positive* iff $\exists u > 0$, $\forall x > 0$, $\exists \alpha$, $\beta > 0$, $\exists k > 0$ an integer such that

$$\alpha u \leqq A^k x \leqq \beta u .$$

If $u > 0$ is any vector for which the conditions in definition 5 are satisfied, then we say that A is u_0-positive for u.

Proposition 4. If A is u_0-positive for u and $u \gg 0$, then A is primitive. If A is u_0-positive and irreducible, then $u \gg 0$.

Proof. If A is irreducible, then there is an $x \gg 0$ such that $Ax = \varrho x$. However for suitable α, β, k we have

$$\alpha u \leqq A^k x \leqq \beta u\,, \quad \alpha u \leqq A^k x \leqq \beta u\,.$$

But $x \gg 0$ implies $u \gg 0$. So for $u \gg 0$ and for each $y \in K \setminus \{0\}$ there are α, k such that

$$0 \ll \alpha u \leqq A^k y\,,$$

whence A is primitive.

It is obvious that if A is primitive, then A is irreducible and u_0-positive. However, there need be no relationship between irreducibility and u_0-positivity for the same cone K (cf., however, [16]).

First let K be the nonnegative orthant and let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly A is irreducible but not primitive. Hence by proposition 4 A cannot be u_0-positive. Again, let K be the nonnegative orthant but take

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then A is reducible. However, A is u_0-positive for $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The relations among irreducibility, u_0-positivity, and primitivity in finite dimensional spaces can be derived from the next theorem.

Theorem 6. *Let A be u_0-positive for u. Then there is an integer q for which*

$$A^q(K \setminus \{0\}) \subseteq (\Phi(u))^\Delta\,.$$

Proof. By proposition 4 we need be concerned only with the case $u \in \partial K$. We have of course that $u \in (\Phi(u))^\Delta$. Note that for any $x \in K$, $Ax \in \partial K$. For if $Ax \in K^0$, then for all p, $A^p x \gg 0$. But for some integer r and $\alpha, \beta > 0$,

$$0 < \alpha u \ll A^r x \leqq \beta u\,,$$

whence $u \gg 0$ contrary to hypothesis. Let $x_0 > 0$ be an eigenvector of A belonging to ϱ. Then from

$$0 < \alpha u \leqq A^p x_0 \leqq \beta u\,, \quad 0 < \alpha u \leqq \varrho^p x_0 \leqq \beta u$$

we infer that $x_0 \in \Phi(u)^\Delta$. Thus

$$0 \leqq \alpha A u \leqq \varrho^p A x_0 = \varrho^{p+1} x_0 \leqq \beta A u .$$

Since $x_0 \in \Phi(u)^\Delta$, we have $Au \in \Phi(u)^\Delta$. Therefore, $A^r u \in \Phi(u)^\Delta$ for all r, and so if $A^p x \in \Phi(u)^\Delta$, then $A^q x \in \Phi(u)^\Delta$ for all $q \geqq p$. Also if $y \in \Phi(u)$, then from

$$0 \leqq \gamma_1 y \leqq u , \quad 0 \leqq \gamma_1 A y \leqq A u \leqq \gamma_2 u$$

we infer that $Ay \in \Phi(u)$. Thus $\Phi(u)$ is an invariant face of A and $\Phi(u) - \Phi(u)$ is an invariant subspace of A. Consequently, for a suitably chosen basis of V we have that

$$y \in \Phi(u) \quad \text{implies} \quad y = \begin{bmatrix} y_1 \\ 0 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} A_1 & B_0 \\ 0 & A_2 \end{bmatrix}.$$

On the one hand, A restricted to $\Phi(u)$ is A_1. So A_1 is primitive on $\Phi(u)$ and there is a k such that for any $y \in \Phi(u)$, $y \neq 0$, $A^k y \in \Phi(u)^\Delta$.

On the other hand if $y > 0$ then there is some m such that $A^m y \in \Phi(u)^\Delta$ since A is u_0-positive. Thus

$$A^m y = \begin{bmatrix} A_1^m & B_m \\ 0 & A_2^m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_1^m y_1 + B_m y_2 \\ A_2^m y_2 \end{bmatrix} = \begin{bmatrix} y' \\ 0 \end{bmatrix}.$$

Therefore, A_2 is nilpotent of some order m_0, and if $y > 0$, $m \geqq m_0$ then $A^m y \in \Phi(u)$. Let $q = k m_0$. Then for any $y > 0$

$$A^q y \in \Phi(u)^\Delta .$$

Corollary. *Let A be u_0-positive for $u > 0$. Then for any $y \in V$, $\exists \gamma > 0$,*

$$\gamma A^q y \leqq u ,$$

where q is as in theorem 6.

In the representation used in the proof of theorem 6, we observed that A_1 was primitive. Hence by theorem 6.3 of [7] $\varrho(A_1)$ is larger than the modulus of any other eigenvalue of A_1, and therefore of A as A_2 is nilpotent. Since it is clear that any eigenvector of A lying in K must lie in $\Phi(u)$ we have established

Proposition 5. *If A is u_0-positive, then $\varrho > |\lambda|$ for any other eigenvalue λ of A, and the Perron vector x_0 is the only eigenvector of A in K.*

This proposition is known as well for operators leaving invariant a cone in a Banach space (cf. [8], [10], [11]).

In partially ordered Banach spaces other generalizations of irreducible matrices have been studied. We shall close this section by examining three of these in the context of a finite dimensional space.

Definition 6. (a) $A \geqq 0$ is called *semi-nonsupporting* iff

$$\forall x > 0 \, \forall f > 0 \, \exists p = p(x, f) , \quad f A^p x > 0 .$$

(b) $A \geqq 0$ is called *nonsupporting* iff

$$\forall x > 0 \, \forall f > 0 \, \exists p = p(x, f) \, \forall n \geqq p , \quad f A^n x > 0 .$$

Definition 6 is due to IKUKO SAWASHIMA [13]. She further introduces the notions of nonsupporting vectors and strictly nonsupporting operators. In the finite dimensional case these become elements of K^0 and primitive matrices, respectively. MAREK [10] also treats both nonsupporting operators and quasipositive operators. In finite dimensional spaces Vandergraft [16] has shown that the classes of quasi-positive matrices and irreducible matrices coincide.

The fundamental result about semi-nonsupporting matrices is

Sawashima's Theorem. *A is semi-nonsupporting iff $\varrho > 0$ and the row and column eigenspaces are one-dimensional spaces determined by vectors $x_0 \in K^0$ and $f_0 \in (K^*)^0$.*

Lemma 1. *If A is semi-nonsupporting, then A is irreducible.*

Proof. Suppose A is reducible. Then there is a proper face F of K for which $AF \subseteq F$. Let $f \in K^*$ be so chosen that

$$\{y \mid fy = 0, \; y \in K\} \supseteq F .$$

If $x \in F$, then for any p, $fA^p x = 0$. Hence A is not semi-nonsupporting.

We shall shortly see that the converse is also true.

Examples. We know the following implications: u_0-positive and irreducible $\Leftrightarrow$ primitive $\Rightarrow$ nonsupporting $\Rightarrow$ semi-nonsupporting $\Rightarrow$ irreducible. We shall now show that two of the arrows cannot be reversed. Let K be the cone in the example following theorem 4.

(a) Let

$$A = \begin{bmatrix} \cos \Theta & \sin \Theta & 0 \\ -\sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where Θ is not a rational multiple of π. Let $f \in \partial K^* \setminus \{0\}$, $x \in \partial K \setminus \{0\}$. If

$$H(f) = \{y \mid fy = 0\}$$

then $H(f) \cap K$ is a line segment in K. By the choice of Θ there is an integer p such that $n \geqq p$ implies $A^n x \notin H(f) \cap K$. So $fA^n x > 0$. Thus A is nonsupporting but not primitive. It is worth noting that if K is polyhedral, then primitive and nonsupporting equivalent. This is an immediate consequence of the spectral properties of irreducible matrices which we shall publish elsewhere.

(b) If A is of the same form as in (a) but $\Theta = 2\pi/r$, r and integer greater than one, then A is semi-nonsupporting. However, given $x \in \partial K \setminus \{0\}$, there is an f such that $fx = 0$. Thus

$$fA^p x > 0 \quad \text{if} \quad p \neq qr ,$$

$$fA^p x = 0 \quad \text{if} \quad p = qr .$$

Consequently A is not nonsupporting.

V. JA. STETSENKO in his paper [15] has used the following as his definition of irreducibility:

$$C : \alpha > 0 , \quad x_0 > 0 , \quad \alpha x_0 \geqq A x_0 \quad \text{implies} \quad \forall f > 0 , \quad f x_0 > 0 .$$

Proposition 6. *A matrix A is irreducible iff it satisfies condition C.*

Proof. Suppose condition C is satisfied and F is a face of K which A leaves invariant. Let $x_0 \in F^\Delta$. Since $Ax_0 \in F$, there is an $\alpha > 0$ such that $\alpha x_0 \geqq Ax_0$, whence by C

$$\forall f > 0 , \quad f x_0 > 0 .$$

Therefore, $x_0 \gg 0$; i.e., $F = K$, unless $x_0 = 0$. Thus A leaves no prover face invariant.

Conversely, suppose A is irreducible. Let α and x_0 satisfy

$$\alpha > 0 , \quad x_0 > 0 , \quad \alpha x_0 \geqq A x_0 .$$

For any $y \in \Phi(x_0)$ there is a $\beta > 0$ for which $\beta x_0 \geqq y \geqq 0$. Thus

$$\alpha\beta x_0 \geqq \beta A x_0 \geqq A y ,$$

and therefore $Ay \in \Phi(x_0)$; i.e., $\Phi(x_0)$ is an invariant face of K. Since $x_0 \neq 0$, $\Phi(x_0) = K$ and so $x_0 \gg 0$. It follows that for any $f > 0$, $fx > 0$.

In his paper Stetsenko also states two theorems which we shall paraphrase here for finite dimensional spaces.

Theorem 7. *A is irreducible iff A is semi-nonsupporting.*

Theorem 8. *A is irreducible iff A^* is irreducible with respect to K^* (regarded now as column vectors, not row vectors).*

The proof of theorem 7 follows from lemma 1, Sawashima's theorem, and theorem 4.2 of [16].

Proof of theorem 8. Suppose A is reducible. Let F be a proper invariant face of A. Define

$$F^* = \{f \in K^* \mid fx = 0,\ x \in F\}.$$

It is easily seen that F^* is a proper face of K^*. Further for $x \in F$

$$(fA)\,x = f(Ax) = 0$$

since $AF \subseteq F$. Therefore $A^*F^* \subseteq F^*$. Thus

A reducible implies A^* reducible, or

A^* irreducible implies A irreducible.

Hence

$(A^*)^*$ irreducible implies A^* irreducible, or

A irreducible implies A^* irreducible.

6. SPLITTINGS OF MATRICES

In this section we shall use the results on matrices nonnegative with respect to a cone to obtain a generalization of the theory of M-matrices. While our definition of an M-matrix requires A to be nonsingular, we note in passing that some authors use a different definition which permits singular M-matrices. For a synopsis of the theory of M-matrices see FIEDLER and PTÁK [4] and [5]. In our generalization we shall use the concept of a splitting of a matrix which concept finds application in the iterative solution of systems of equations (cf. [17]). Also our definition of an M-matrix yields a largeı class of matrices when K is the nonnegative orthant than the usual definition.

Definition 7. (a) A matrix A admits a *regular splitting* iff $A = B - C$ where $B^{-1} \geqq 0$, $C \geqq 0$.

(b) A admits a *completely regular splitting* iff $A = B - C$ with $B > 0$, $B^{-1} > 0$, $C \geqq 0$.

(c) A is an *M-matrix* iff A admits a completely regular splitting and $A^{-1} > 0$.

A key result for the proposed extension is the following lemma due to H. SCHNEIDER [14].

Lemma 2. *Suppose $S \geqq 0$ and either $RK^0 \supseteq K^0$ or $RK^0 \cap K^0 = \emptyset$. If $T = R - S$, then the following are equivalent.*

(1) *R is nonsingular, $R^{-1} > 0$, and $\varrho(R^{-1}S) < 1$;*

(2) *T is nonsingular and $T^{-1}K^0 \subseteq K^0$;*

(3) *$TK^0 \cap K^0 \neq \emptyset$.*

This result contains a generalization of theorem (3.13) of [17]. This same result has been generalized in a different way by O. L. MANGASARIAN in [9] for K the non-negative orthant.

Theorem. *Let A, M, and N be $n \times n$ real matrices, let $A = M - N$, let A and M be nonsingular, and let*

$$M'y \geqq 0 \quad \textit{imply} \quad N'y \geqq 0\,,$$

$$A'y \geqq 0 \quad \textit{imply} \quad N'y \geqq 0\,,$$

where the prime denotes transpose. Then $\varrho(M^{-1}N) < 1$.

Mangasarian proved this theorem using the theorems of the alternative. Using instead the fact that $K^{**} = K$ we can generalize this result to arbitrary cones.

Theorem 9. *Let K be a cone. Let $A = M - N$, let A and M be nonsingular, and let*

$$fM \in K^* \quad \textit{imply} \quad fN \in K^*\,, \quad fA \in K^* \quad \textit{imply} \quad fN \in K^*\,.$$

Then $M^{-1}N \geqq 0$ and $\varrho(M^{-1}N) < 1$.

Proof. Let $g \in K^*$. Since M is $1 - 1$, there is an $f \in V^*$ such that $g = fM$. Therefore, $fN \in K^*$. Consequently,

$$gM^{-1}N = (fM)\,M^{-1}N = fN \in K^*\,.$$

Thus $K^*M^{-1}N \subseteq K^*$, whence $M^{-1}N \geqq 0$. Similarly $A^{-1}N \geqq 0$.

The argument given by VARGA on pages 88 and 89 of [17] now applies and the remainder of the theorem follows.

Another sufficient condition for $A^{-1} > 0$ is containted in the next theorem, which is a generalization of lemma 0 of HOUSEHOLDER [6].

Theorem 10. *Suppose $A = B - C$ is a completely regular splitting. If for any $x > 0$ there is an $f > 0$ such that $fAx > 0$, then $A^{-1} > 0$.*

Proof. $B^{-1}C \geqq 0$ so let $\varrho = \varrho(B^{-1}C)$ and let $y > 0$ be an eigenvector belonging to ϱ. If $\varrho = 0$, then $A^{-1} > 0$ by lemma 2. Let us therefore assume that $\varrho > 0$. Thus $\varrho y > 0$. From $B^{-1}Cy = \varrho y$ it follows that $(\varrho B - C)\,y = 0$. If $\varrho \geqq 1$, then $\varrho B \geqq B$, so $\varrho B - C \geqq B - C$. Thus

$$0 = (\varrho B - C)\,y \geqq (B - C)\,y\,.$$

If f is the functional guaranteed by the hypothesis, we have

$$0 = f(\varrho B - C)\,y = (\varrho - 1)\,fBy + f(B - C)\,y\,.$$

But $\varrho - 1 \geqq 0$ and $fBy \geqq 0$ since $B \geqq 0, f > 0$. Also

$$f(B - C)\,y = fAy > 0\,.$$

Thus

$$0 = f(\varrho B - C)\,y > 0\,,$$

a contradiction. Therefore $\varrho < 1$, and $A^{-1} > 0$ by lemma 2.

For a converse we have

Proposition 7. *Suppose $A^{-1} > 0$. Then there is an $f \gg 0$ such that for all $x > 0$, $fAx > 0$. Moreover, if $A^{-1} (> 0$, then f can be taken as the eigenvector of A^{-1} in $(K^*)^0$.*

Proof. Let $f_1 \gg 0$. Then for all $x > 0, fx > 0$. Since $A^{-1} > 0$, we have

$$A^{-1}(K \setminus \{0\}) \subseteq K \setminus \{0\}\,.$$

Thus $f_1A^{-1}x > 0$ for all $x > 0$, so take $f = f_1A^{-1} \gg 0$. Then

$$fAx = f_1A^{-1}Ax = f_1x > 0$$

for $x > 0$.

Finally, if $A^{-1} (> 0$, its eigenvector $f \gg 0$ satisfies

$$f = \varrho^{-1}fA$$

where $\varrho = \varrho(A^{-1}) > 0$. Thus $0 < fx = \varrho^{-1}fAx$ for $x > 0$.

The next result and some of its consequences are patterned after known results in the theory of M-matrices. In particular see section 4 of [4].

Proposition 8. *Let A and A_1 satisfy the following conditions:*

(1) $A = B - C$ is a regular splitting,
(2) $A_1 = B_1 - C_1$ is a completely regular splitting,
(3) $A_1 \geqq A$,
(4) $A^{-1} > 0$.

Then A_1^{-1} exists and $A^{-1} \geqq A_1^{-1} \geqq 0$.

Proof. Let $U = I - B_1^{-1}A_1 = B_1^{-1}C_1 \geqq 0, \quad V = I - B_1^{-1}A$. Then

$$V = I - B_1^{-1}A \geqq I - B_1^{-1}A_1 = U \geqq 0\,.$$
$$(I - V)^{-1} = (B_1^{-1}A)^{-1} = A^{-1}B_1 \geqq 0\,,$$

so V is convergent. Since $0 \leqq U^k \leqq V^k$ for $k = 1, 2, \ldots$, it follows that

$$A^{-1}B_1 = I + V + V^2 + \ldots \geqq I + U + U^2 + \ldots = (B_1^{-1}A_1)^{-1} \geqq 0\,.$$

So $A^{-1}B_1 \geqq A_1^{-1}B_1$. However, $B_1^{-1} > 0$, so $A^{-1} \geqq A_1^{-1} \geqq 0$.

Corollary. *If* $A = B - C$ *is a regular splitting,* $D^{-1} > 0$, $D \geqq B$, *and* $A^{-1} > 0$, *then* $(D^{-1}C) < 1$.

Proposition 9. *Let* $A = B - C$ *be a regular splitting. Then the following are equivalent:*

(1) $A^{-1} > 0$,

(2) *the real parts of the eigenvalues of* $B^{-1}A$ *are positive,*

(3) *the real eigenvalues of* $B^{-1}A$ *are positive.*

Proof. If $A^{-1} > 0$, then $\varrho(B^{-1}C) < 1$. The eigenvalues of $B^{-1}A$ are of the form $1 - \lambda$ for λ an eigenvalue of $B^{-1}C$. But then $|\lambda| < 1$, so $|\text{Re}\,\lambda| < 1$, and so $\text{Re}\,(1 - \lambda) > 0$.

That (2) implies (3) is obvious.

If the real eigenvalues of $B^{-1}A$ are positive, then in particular $1 - \varrho(B^{-1}C) > 0$. So $1 > \varrho(B^{-1}C)$, and $A^{-1} > 0$ by lemma 2.

However the situation regarding the eigenvalues of an M-matrix A is not so simple as in the standard case. If

$$K = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, 0 \leqq \tfrac{1}{2}x_1 \leqq x_2 \leqq 2x_1 \right\}, \quad A = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & -1 \end{bmatrix},$$

then $A = A^{-1} > 0$ is an M-matrix with respect to $K(C = 0)$. The eigenvalues of A are 1 and -1, so A is even irreducible.

Notation. If A is a matrix, then $\Sigma(A)$ will denote the set of eigenvalues of A.

Proposition 10. *Let* A *be an M-matrix. If* $(B - \alpha I)^{-1} > 0$ *for all* $\alpha \leqq 0$, *then the real eigenvalues of* A *are positive. Further, if there is a* $\beta > 0$ *for which* $\beta I > B$, *then the real parts of the eigenvalues of* A *are positive.*

Proof. Let $\alpha \leqq 0$. Then

$$A_1 = A - \alpha I = (B - \alpha I) - C \geqq B - C = A\,.$$

Further A_1 admits a completely regular splitting, so by proposition 8 it is an M-matrix. Thus $\alpha \notin \Sigma(A)$.

Since $(B - \alpha I)^{-1} > 0$, $\alpha \notin \Sigma(B)$. Let $\beta > 0$ be such that $\beta I > B$. Then

$$\beta I - A = \beta I - B + C > 0\,.$$

Thus $\varrho(\beta I - A) = \beta - \lambda$, where $\lambda \in \Sigma(A)$ and λ real hence positive. If $\xi \in \Sigma(A)$, then $\beta - \xi \in \Sigma(\beta I - A)$ and

$$|\beta - \xi| \leqq \beta - \lambda < \beta\,.$$

So

$$\beta > |\beta - \xi| = [(\beta - \text{Re}\,\xi)^2 + (\text{Im}\,\xi)^2]^{1/2} \geqq |\beta - \text{Re}\,\xi| \,.$$

Hence $\text{Re}\,\xi > 0$.

Theorem 11. *Let $A = B - C$ be a completely regular splitting and let A be nonsingular. Suppose for every nonsingular $A_1 = B_1 - C_1$, where B_1, C_1 is a regular splitting, we have the following condition:*

$$A_1 > A \quad \textit{implies} \quad A_1^{-1} > 0 \,.$$

Then $A^{-1} > 0$.

Proof. Let $A(\varepsilon) = B + \varepsilon I - C$. For all sufficiently small $\varepsilon > 0$ we have that $[A(\varepsilon)]^{-1}$ and $(B + \varepsilon I)^{-1}$ exist. Clearly $B + \varepsilon I > 0$. On the other hand since $B^{-1} > 0$ we know that B is an open map so $B > 0$ implies $BK^0 = K^0$. If $x \gg 0$

$$(B + \varepsilon I)\, x = Bx + \varepsilon x \gg 0 \,,$$

whence by lemma 2 $(B + \varepsilon I)^{-1} > 0$. Finally

$$A(\varepsilon) = B + \varepsilon I - C \geqq A = B - C$$

so $A(\varepsilon)$ satisfies the hypothesis. Thus $[A(\varepsilon)]^{-1} > 0$. Clearly, $A(\varepsilon) \to A$, so that since A^{-1} exists, $[A(\varepsilon)]^{-1} \to A^{-1}$ as $\varepsilon \to 0$. Since the cone of nonnegative matrices is closed, it follows that $A^{-1} > 0$.

Proposition 11. *If $A = B - C$ is a completely regular splitting and if $B^{-1}C$ or CB^{-1} has an eigenvector $x \gg 0$ corresponding to an eigenvalue $\lambda < 1$, then $A^{-1} > 0$.*

Proof. Since $B > 0$, $B^{-1} > 0$, we know that $BK^0 = B^{-1}K^0 = K^0$. Hence $B^{-1}AK^0 \cap K^0 \neq \emptyset$ iff $AK^0 \cap K^0 \neq \emptyset$ iff $AB^{-1}K^0 \cap K^0 \neq \emptyset$. Now let $x \gg 0$ be the eigenvector of $B^{-1}C$ belonging to $\lambda \leqq 1$. (The same proof works for CB^{-1}.)

$$B^{-1}Ax = (I - B^{-1}C)\, x = (1 - \lambda)\, x \gg 0$$

since $1 - \lambda > 0$. Thus $A^{-1} > 0$ and $\varrho(B^{-1}C) < 1$.

This result is very close to a theorem of Collatz which we now establish for establish for arbitrary cones (cf. Wielandt [17] page 33).

Theorem 12. *If $A \geqq 0$, $x \gg 0$, and $\sigma x \leqq Ax \leqq \tau x$, then*

$$\sigma \leqq \varrho(A) \leqq \tau \,.$$

Proof. Let $f > 0$ satisfy $fA = \varrho f$, $\varrho = \varrho(A)$. Thus

$$f(\sigma x) \leqq fAx \leqq f(\tau x)$$
$$\sigma(fx) \leqq \varrho(fx) \leqq \tau(fx) \,.$$

But $f > 0$ and $x \gg 0$, so $fx > 0$. Therefore

$$\sigma \leqq \varrho \leqq \tau .$$

Corollary. *If $A \geqq 0$ and there is an $x \gg 0$ such that $Ax = \mu x$, then $\varrho(A) = \mu$.*

Theorem 13. *If $A = B - C$ is a completely regular splitting with $B \geqq I \geqq C$, if $A_1 = B_1 - C_1$ is an M-matrix, and if $A_2 = BB_1 - CC_1$, then A_2 is an M-matrix.*

Proof. $B \geqq I \geqq 0$ implies $BB_1 \geqq B_1 \geqq 0$. Further $I \geqq C \geqq 0$ implies $C_1 \geqq \geqq CC_1 \geqq 0$. Consequently

$$A_2 = BB_1 - CC_1 \geqq B_1 - C_1 = A_1 .$$

Also, $BB_1 - CC_1$ is a completely regular splitting. Thus by proposition 8

$$A_1^{-1} \geqq A_2^{-1} \geqq 0 .$$

Therefore A_2 is an M-matrix.

Ky Fan in [3] gives a definition of multiplication of M-matrices for which the product of two M-matrices is again an M-matrix. Since in the present situation the decomposition $A = B - C$ need not be unique, we shall define our multiplication for the ordered pairs (B, C). We shall call $M = (B, C)$ an *M-matrix pair* iff $B - C$ is an M-matrix. For two M-matrix pairs $M_1 = (B_1, C_1)$ nad $M_2 = (B_2, C_2)$ we define

$$M_1 \circ M_2 = (B_1B_2, C_1C_2) .$$

We would like $M_1 \circ M_2$ to be an M-matrix pair, but the best we have been able to do is

Proposition 12. *If $M_1 = (B_1, C_1)$ and $M_2 = (B_2, C_2)$ are M-matrix pairs, and if B_1, B_2, C_1, C_2 all commute, then $N = M_1 \circ M_2$ is also an M-matrix pair.*

Proof. Clearly $B_1B_2 - C_1C_2$ is a completely regular splitting. Let us estimate $\varrho(B_1^{-1}B_2^{-1}C_1C_2)$. By hypothesis

$$\varrho(B_1^{-1}C_1) < 1 \quad \text{and} \quad \varrho(B_2^{-1}C_2) < 1 .$$

However, B_1, B_2, C_1, C_2 commute, so

$$B_2^{-1}B_1^{-1}C_2C_1 = (B_2^{-1}C_2)(B_1^{-1}C_1) = (B_1^{-1}C_1)(B_2^{-1}C_2)$$

and (see [2] for the relevant results) for a suitable ordering of the eigenvalues $\{\lambda_i\} = = \Sigma(B_2^{-1}C_2)$ and $\{\mu_i\} = \Sigma(B_1^{-1}C_1)$ we have

$$\Sigma(B_2^{-1}C_2B_1^{-1}C_1) = \{\lambda_i\mu_i\} .$$

Thus

$$\sup_i |\lambda_i \mu_i| \leqq (\sup_i |\lambda_i|) (\sup_i |\mu_i|) < 1 .$$

Therefore, $(B_1 B_2 - C_1 C_2)^{-1}$ exists and is positive.

Finally, let us classify those M-matrices for which $A^{-1} \gg 0$.

Theorem 14. *Let A be an M-matrix with the completely regular splitting $B - C$. Then $A^{-1} \gg 0$ iff $B^{-1}C$ is irreducible.*

Proof. $A^{-1} = (I - B^{-1}C)^{-1} B^{-1}$, or $A^{-1}B = (I - B^{-1}C)^{-1}$. But since $BK^0 = B^{-1}K^0 = K^0$, $B(\partial K) = B^{-1}(\partial K) = \partial K$ we have that $A^{-1} \gg 0$ iff $A^{-1}B \gg 0$. Thus $A^{-1} \gg 0$ iff

$$0 \ll (I - B^{-1}C)^{-1} = I + B^{-1}C + (B^{-1}C)^2 + \dots$$

Also $A^{-1} \gg 0$ iff for all $f > 0$ and $x > 0$, $fA^{-1}x > 0$. Thus $A^{-1} \gg 0$ iff

$$\forall f > 0 \, \forall x > 0 , \quad f(I - B^{-1}C)^{-1} x = \sum_{k=0}^{\infty} f(B^{-1}C)^k x > 0 .$$

However, since for all k, $f(B^{-1}C)^k x \geqq 0$, then

$$\sum_{k=0}^{\infty} f(B^{-1}C)^k x > 0 \quad \text{iff} \quad \exists m = m(f, x) , \quad f(B^{-1}C)^m x > 0 .$$

This last condition is precisely the definition of seminonsupporting, which we know is equivalent to irreducibility. The theorem is proved.

Corollary. *Let $A = B - C$ be as in theorem 14, and let $BC = CB$. If $C (> 0$, then $A^{-1} \gg 0$.*

Proof. For suitable m, $(B^{-1}C)^m = (B^{-1})^m C^m \gg 0$, so that $B^{-1}C$ is irreducible.

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