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REPRESENTATIONS OF PRESHEAVES OF CLOSURE SPACE

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INTRODUCTION

Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a presheaf of closure spaces over a topological space X, P its covering space and $\mathscr{S}' = \{A_U; \tilde{\rho}_{UV}; X\}$ its natural representation. That is to say, we know that every $a \in S_U$ can be regarded in a natural way as a section $a(x)$ over U in P. Denoting the assignement $a \to a(x)$ by p_U, then to the set S_U there corresponds a set $A_U = p_U(S_U)$ of sections over U. Moreover, let us denote by $\tilde{\rho}_{UV} : A_U \to A_V$ the map defined as follows: $a(x) \in A_U$, $a(x) \to \tilde{\rho}_{UV}(a) = a(x)/V = [\varrho_{UV}(a)] (x)$. Then $\mathscr{S}' = \{A_U; \tilde{\rho}_{UV}; X\}$ is a presheaf of sets over X. If $\mathscr{S}$ satisfies convenient natural requirements, every p_U is injective and thus $\mathscr{S}'$ is a natural representation of $\mathscr{S}$ with help of the presheaf of certain sets of sections in P.

Let us denote by $\mathscr{B}(X)$ the set of all open subsets in X. We say that a nonempty family $\mathscr{K}$ of subsets of a set L is cofilter base if the folowing holds: $M, N \in \mathscr{K} \Rightarrow M \cup N \subset K$ for some $K \in \mathscr{K}$.

We say that for the presheaf $\mathscr{S} = \{S_U; \varrho_{UV}; X\}$ there is given a cofiltration $\varkappa = \{\mathscr{K}^U_a; U \in \mathscr{B}(X); a \in S_U\}$, if for every $U \in \mathscr{B}(X)$, $a \in S_U$ there is given a cofilter base $\mathscr{K}^U_a$ in U such that for $V \in \mathscr{B}(U)$ we have: For every $K \in \mathscr{K}^V_{\varrho_{UV}(a)}$ there exists $L \in \mathscr{K}^U_a$ such that $K \subset L$.

Let be given a closure t in P. Further let for every $U \in \mathscr{B}(X)$ q_U be some method which enables us to form a closure $q_U(t)$ in A_U from t. Then $\mathscr{S}' = \{(A_U, q_U(t)); \tilde{\rho}_{UV}; X\}$ is a presheaf of sets A_U with the closures $q_U(t)$. Now, we can try to find a closure t in P, so that the all natural maps $p_U : (S_U, \tau_U) \to (A_U, q_U(t))$ would be homeomorphisms. Every such a closure will be called normal.

Here we restrist ourselves to the case when the method q_U is the closure of uniform convergence on some cofiltration $\varkappa$. Now, two following questions can be studied:

1. If there is given a cofiltration $\varkappa$ and if $\underset{U}{k}(t)$ is the closure of uniform convergence on $\varkappa$, we study when there exists a normal closure t in P.

2. We study if there exists in P a closure t generating a representation i.e. for which the following is satisfied:

(a) There exists a cofiltration $\varkappa$ for which t is normal.

(b) If for $U \in \mathscr{B}(X)$ $\Gamma(U, t)$ is the set of all continuous sections over U, then $\Gamma(U, t) = A_U$ for all $U \in \mathscr{B}(X)$. That is to say, if t generates such a representation, then $\mathscr{S}' = \{(A_U, k_U(t)); \tilde{\rho}_{UV}; X\}$ is not only a set representation, but even a topological representation of $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$, i.e. $\mathscr{S}$ can be represented even topologically using the presheaf of all continuous section sin its covering space ($k_U(t)$ is the closure of uniform convergence on $\varkappa$).

Agreements and notations

The set of all open subsets of a topological space X will be denoted by $\mathscr{B}(X)$.

0.1. Definition. A presheaf of sets over a topological space X is a system

$$\mathscr{S} = \{S_U; \varrho_{UV}; X\} \tag{0.2}$$

where S_U for $U \in \mathscr{B}(X)$ are sets and ϱ_{UV} for $U, V \in \mathscr{B}(X)$, $V \subset U$ is a map $\varrho_{UV} : S_U \to$ $\to S_V$, such that the following holds:

(1) If $W \subset V \subset U$ then $\varrho_{UW} = \varrho_{VW} \circ \varrho_{UV}$,

(2) $\varrho_{UU} = i_U$ – identity map of S_U.

We say the presheaf $\mathscr{S}$ is the presheaf of closure spaces (semiuniform spaces) if in every set S_U there is given a closure τ_U (a semiuniformity η_U) and the maps ϱ_{UV} are continuous (uniformly continuous) maps of the closure (semiuniform) spaces

$$\varrho_{UV} : (S_U, \tau_U) \to (S_V, \tau_V) \left((S_U, \eta_U) \to (S_V, \eta_V)\right).$$

Where it is not important to repeat the all data, we will speak only about a presheaf and write simply $\mathscr{S} = \{(S_U, \tau_U); X\}$.

If $\mathscr{S}$ is a presheaf over X from (0.2), we can to every $x \in X$ assign the system

$$\mathscr{S}_x = \{S_U; \varrho_{UV}; U, V \in \mathscr{B}(X);\ x \in U, V\}. \tag{0.3}$$

Because ϱ_{UV} satisfy the conditions (1), (2) from (0.1), we can form the set $\mathscr{F}_x =$ $= \varinjlim \mathscr{S}_x$ – the inductive limit of $\mathscr{S}_x$ (see [2], p. 744).

0.4. Definition. The set $\mathscr{F}_x$ is called stalk over the point x. For every $U \in \mathscr{B}(X)$ containing x there is a natural map of the set S_U into $\mathscr{F}_x$.

0.5. Notation. Let $x \in U \in \mathscr{B}(X)$. The natural map of S_U into $\mathscr{F}_x$ will be denoted by ξ_{Ux}. If $a \in S_U$, then the element $\xi_{Ux}(a) \in \mathscr{F}_x$ will be called germ of a over x.

Let us put $P = \bigcup\{\mathcal{F}_x \mid x \in X\}$. Further let ψ be the map of P onto X constructed as follows: If $\alpha \in P$, then there is the unique $x \in X$ such that $\alpha \in \mathcal{F}_x$. Let us set $\psi(\alpha) = x$.

0.6. Notation. The set P is called covering space of the presheaf $\mathcal{S}$. The map ψ is called projection.

0.7. Remark. Clearly there is $\mathcal{F}_x = \psi^{-1}(x)$. Thus for the stalk over x will be more frequently used the symbol $\psi^{-1}(x)$. The capital P will in the next denote only the covering space of $\mathcal{S}$.

0.8. Definition. Let $U \in \mathcal{B}(X)$. Every map $r : U \to P$ for which $\psi \circ r = i_U$ is the identity map of U is called section over U.

If $U \in \mathcal{B}(X)$, $a \in S_U$, then we can to every $x \in U$ assign an element $h_a(x) \in P$ as follows: $h_a(x) = \xi_{Ux}(a)$ (see (0.5)).

According to our way of introduction of ξ_{Ux} and ψ, there is $(\psi \circ h_a)(x) = \psi(\xi_{Ux}(a)) = x$. Thus the map h_a is a section over U. If we assign in this way to every $a \in S_U$ the section h_a over U, we get a certain sets of sections over U.

0.9. Notation. The map which to every $a \in S_U$ assigns the section h_a over U will be denoted by p_U and called natural map. The set of sections $\{p_U(a) \mid a \in S_U\}$ will be denoted by A_U. Instead of h_a we will write briefly $a(x)$, where x denotes the variable taking values from U. Thus $a(x)$ is the section over U such that

$$p_U(a) = a(x)\,. \tag{0.10}$$

The element $\xi_{Uy}(a)$ will be denoted in accordance with our agreements by $a(y)$. We will say that the section $a(x)$ goes through the point $y \in X$. Thus we have

$$\xi_{Uy}(a) = a(y)\,. \tag{0.11}$$

0.12. Remark. If $U, V \in \mathcal{B}(X)$, $V \subset U$, and if $a(x) \in A_U$ is a section over U from (0.9), we can to $a(x)$ assign a section from A_V as follows: $a(x) \to a(x)/V$, i.e. the restriction of $a(x)$ to V. Let us denote this map by $\tilde{\rho}_{UV}$. Then $\tilde{\rho}_{UV}(a(x)) \in A_V$ and we get the commutative diagram

$$\begin{array}{ccc} S_U & \xrightarrow{p_U} & A_U \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \tilde{\rho}_{UV}} \\ S_V & \xrightarrow{p_V} & A_V \end{array} \tag{0.13}$$

0.14. Remark. For $y \in U$ there exists a natural map $\tilde{\xi}_{Uy} : A_U \to \psi^{-1}(y)$. Namely, if there is $a(x) \in A_U$, then we set $\tilde{\xi}_{Uy}(a(x)) = a(y)$. According to (0.10, 11) we have

$\tilde{\xi}_{Uy}(p_U(a)) = \xi_{Uy}(a) = a(y)$. Thus the following diagram is commutative:

$$(0.15) \qquad \begin{array}{ccc} S_U & \xrightarrow{p_U} & A_U \\ & {\scriptstyle \xi_{Uy}} \searrow \quad \swarrow {\scriptstyle \tilde{\xi}_{Uy}} & \\ & \mathscr{F}_y & \end{array} .$$

0.16. Remark. The natural map p_U from (0.10) need not be injective. This can be reached by adding this assumption:

0.17. Assumption. *Let $U \in \mathscr{B}(X)$, $a, b \in S_U$ and let $\mathscr{V}$ be an open cover of U. If $\varrho_{UV}(a) = \varrho_{UV}(b)$ for all $V \in \mathscr{V}$, then $a = b$.*

This assumption implies the injectivity of p_U (see [5]). In the next we suppose steadily, that this assumption is satisfied.

0.18. Definition. Let $\alpha \in P$, $y = \psi(\alpha)$. Then there exists $U \in \mathscr{B}(X)$ such that for some $a \in S_U$ we have $\xi_{Uy}(a) = a(y) = \alpha$. Every such a will be called generating element for α. Similarly the section $a(x) \in A_U$ will be called generating section for α.

0.19. Notation. Let $a \in S_U$, let $A \subset U$ be an arbitrary subset. Let us denote

$$\xi_{UA}(a) = \bigcup\{\xi_{Uy}(a) \mid y \in A\}$$

and further, more generally, if $M \subset S_U$ is an arbitrary subset,

$$\xi_{UA}(M) = \bigcup\{\xi_{UA}(a) \mid a \in M\} .$$

Thus $\xi_{UA}(a)$ and $\xi_{UA}(M)$ are subsets of P. By (0.5) for $y \in A$ there is

$$\xi_{UA}(M) \cap \psi^{-1}(y) = \xi_{Uy}(M) .$$

In the same way (with respect to (0.9,14)) we proceed if $M \subset A_U$. Thus, for example, if $A \subset U$, $M \subset \psi^{-1}(A)$:

$$\xi_{UA}^{-1}(M) = \{a;\ a \in S_U,\ a(y) \in M,\ y \in A\} ,$$

$$\tilde{\xi}_{UA}^{-1}(M) = \{a(x);\ a(x) \in A_U,\ a(y) \in M,\ y \in A\} .$$

0.20. Notation. The set $\xi_{UA}(a)$ from (0.19) will be denoted by $\mathrm{gr}_A\, a$ and called graph of the section $a(x)$ (resp. of the element a) over A. By (0.9, 11) we have $\xi_{UA}(a) = \bigcup\{a(y) \mid y \in A\}$.

0.21. Notation. Let (X, t) he a closure space, M its subset.

A. If $M \subset X$, then every filter-base of t-neighborhoods of M will be denoted by $\Delta(M; t)$.

B. By the symbol $\mathrm{ind}_M\, t$ will be denoted the closure in M induced by restriction of t to M. If $x \in M$ and $\Delta(x; t)$ is a filter-base of t-neighborhoods of x, then the filter of $\mathrm{ind}_M\, t$-neighborhoods of x will be denoted briefly by $M \cap \Delta(x; t)$. The relation "the closure u is finer than v" will be denoted briefly by $u \leqq v$.

C. If $\mathscr{F}$ is such a filter in X that for every $F \in \mathscr{F}$ there is $M \subset F$, we say that $\mathscr{F}$ is filter round M. If $\mathscr{F}$ and $\mathscr{G}$ are two filter-bases and $\mathscr{F}$ majorises $\mathscr{G}$, we write briefly $\mathscr{F} \leqq \mathscr{G}$. If $\mathscr{F} \leqq \mathscr{G}$, $\mathscr{G} \leqq \mathscr{F}$, we write $\mathscr{F} \sim \mathscr{G}$.

D. For $U \in \mathscr{B}(X)$, $M \subset U$ let us set

$$\mathrm{B}(M; U) = \{V \mid V \in \mathscr{B}(U), M \subset V\}.$$

0.22. Notation. In the set X let us have a nonempty family Ω or closures. The coarsest (finest) closure in X, finer (coarser) than every closure from Ω will be denoted by $\underleftarrow{\lim}\, \Omega$ (resp. $\underrightarrow{\lim}\, \Omega$).

0.23. Remark. Let $\{(X_a, \tau_a) \mid a \in A\}$ be a nonempty family of closure spaces, let X be a set and for every $a \in A$ let φ_a be a map $\varphi_a : (X_a; \tau_a) \to X$ (resp. $\varphi_a : X \to (X_a, \tau_a)$). Then if $\tau = \underrightarrow{\lim}\, \tau_a$ (resp. $\tau = \underleftarrow{\lim}\, \tau_a$) is the closure in X, generated by the closures $\{\tau_a \mid a \in A\}$ inductively (projectively), then (0.22) is inkeeping with this notation.

0.24. Remark. The map $f : (Q, u) \to (X, \underleftarrow{\lim}\, \tau_a)$ is continuous iff for every a the map $\varphi_a \circ f : (Q, u) \to (X_a, \tau_a)$ is continuous. A similar remark is true for $\underrightarrow{\lim}\, \tau_a$.

0.25. Notation. If $\varphi : M \to N$ is a map, let $\bar{\varphi} : M \times M \to N \times N$ be the map defined as follows: $(x, y) \in M \times M \Rightarrow \bar{\varphi}(x, y) = (\varphi(x), \varphi(y))$.

0.26. Agreement. When speaking about a compact space in a topological space X, we suppose that X is a Hausdorff space.

0.27. Agreement. Let $U \in \mathscr{B}(X)$. The set of all open covers (of all finite open covers) of U will be denoted by $\Pi_U(\Pi_U^0)$.

0.28. Notation. Let X, Y be two sets, let $f : X \to Y$ be a map and let $\mathscr{F}$ be a filter-base in X. Then the filter-base $\{f(F) \mid F \in \mathscr{F}\}$ in Y will be denoted by $f(\mathscr{F})$.

0.29. Notation. For a semiuniform space (X, η) let us denote by $\mathscr{D}(X; \eta)$ the filter base of η-neighborhoods of the diagonal in $X \times X$.

0.30. Notation. For a set X let us denote by d the discrete topology in X, and by h the coarsest topology in X, where the only open sets are X and $\emptyset$.

0.31. Definition. We say, that the presheaf $\mathscr{S} = \{S_U; \varrho_{UV}; X\}$ is projective, if the following condition holds: "If $U = \bigcup V_\alpha$, $U, V_\alpha \in \mathscr{B}(X)$ and if there exist the elements $a_\alpha \in S_{V_\alpha}$ such that for $V_\alpha \cap V_\beta$ there is $\varrho_{V_\alpha V_\alpha \cap V_\beta}(a_\alpha) = \varrho_{V_\beta V_\alpha \cap V_\beta}(a_\beta)$, then there exists $a \in S_U$ such that $\varrho_{UV_\alpha}(a) = a_\alpha$ for all α.

Because we assume that (0.17) holds, there exists the unique such $a \in S_U$.

0.32. Definition. We say that the presheaf $\mathscr{S}$ is the presheaf with the unique continuation, if the following conditions are satisfied:

1. X is locally connected,

2. if $U \in \mathscr{B}(X)$ is connected, $a, b \in S_U$, and $\xi_{Uy}(a) = \xi_{Uy}(b)$ for some $y \in U$, then $a = b$.

0.33. Remark. Let P be the covering space of the presheaf $\mathscr{S} = \{S_U; X\}$. If $U \in \mathscr{B}(X)$, then by (0.9) every $a(x) \in A_U$ is a section over U. Let us set $\Omega' = \{t \mid t$ is a closure in P such that for every $U \in \mathscr{B}(X)$ every $a(x) \in A_U$ is a continuous map of U into the closure space $(P, t)\}$.

0.34. Definition, notation. The closure $\underline{\lim}\ \Omega'$ is called sheaf topology and denoted by t_s (see [5]). If t is a closure in P and $U \in \mathscr{B}(X)$, then the set of all continuous sections over U is denoted by $\Gamma(U, t)$.

0.35. Remark. If the presheaf is projective, then $\Gamma(U, t_s) = A_U$ (see [5]). t_s is the finest of the all closures t in P, for which $A_U \subset \Gamma(U, t)$ for all $U \in \mathscr{B}(X)$.

0.36. Notation. Let (X_a, u_a) be closure spaces. The space (X, u) will be called topoogical sum of the spaces (X_a, u_a) if $X = \bigcup X_a$ and $u = \underline{\lim}\ u_\alpha$ (see [1], § 8).

0.37. Notation. A nonempty family $\mathscr{K}$ of subsets of the set L will be called cofilter base (cofilter) if the following holds: $M, N \in \mathscr{K} \Rightarrow M \cup N \subset K$ for some $K \in \mathscr{K}$. ($M, N \in \mathscr{K} \Rightarrow M \cup N \in \mathscr{K}$).

0.38. Definition. A closure collection $\mu = \{\tau_U;\ U \in B(X)\}$ of a presheaf $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ is defined to be projective if the following holds: „If $U \in \mathscr{B}(X)$ and $\mathscr{V} \in \Pi_U$ (see 0.27) is an open cover of U, then the closure $\tau_U(\mathscr{V})$, defined in S_U projectively by the maps $\{\varrho_{UV} : S_U \to (S_V, \tau_V);\ V \in \mathscr{V}\}$, coincides with τ_U." (See [8], 1.1.4.)

CHAPTER 1

UNIFORM CONVERGENCE ON COFILTRATION

1. Introduction of notions

Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a presheaf of closure spaces, P its covering space, ψ the corresponding projection of P onto X, A_U the set of sections over U in the covering space, corresponding to S_U under the natural map p_U.

1.1.1. Assumption. For $\mathscr{S}$ let us have a cofiltration $\varkappa = \{\mathscr{K}_a^U \mid U \in \mathscr{B}(X),\ a \in S_U\}$, i.e. for every $U \in \mathscr{B}(X)$ and every $a \in S_U$ let us have a cofilter $\mathscr{K}_a^U$ (see (0.37)) of sets in U such that for every $V \in \mathscr{B}(U)$ the following holds: "If $K \in \mathscr{K}^V_{\varrho_{UV}(a)}$, then there is $L \in \mathscr{K}_a^U$ such that $K \subset L$" – see [8], (1.2.15).

Let be given a closure t in P. We provide every set A_U with the closure $k(t)$ of uniform convergence on the cofilters $\{\mathscr{K}_a^U \mid a \in S_U\}$. Then we try to find the closure t in P in such away, so that the all natural maps

$$p_U^t : (S_U, \tau_U) \to (A_U, k(t)) \tag{1.1.2}$$

are homeomorphisms.

1.1.3. Definition. A closure t in P will be called normal if the all p_U^t are homeomorphisms.

1.1.4. Notation. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$. Let us take the set $\mathrm{gr}_K\, a$ (see (0.20)) and its t-neighborhood O, i.e. $O \in \Delta(\mathrm{gr}_K\, a; t)$. We set

$$\begin{aligned} B(a(x); K, O) &= \{b(x) \mid b(x) \in A_U,\ b(y) \in O \text{ for } y \in K\} = \\ &= \xi_{UK}^{-1}(O \cap \psi^{-1}(K)), \end{aligned} \tag{1.1.5}$$

see (0.19),

$$\mathscr{B}(a(x)) = \{B(a(x); K, O) \mid \mathscr{K} \in \mathscr{K}_a^U,\ O \in \Delta(\mathrm{gr}_K\, a; t)\}\,. \tag{1.1.6}$$

It is clear, that $\mathscr{B}(a(x))$ is a subbase of a filter in A_U round $a(x)$. Therefore we can take it for subbase of the filter of neighborhoods of $a(x)$ with respect to some closure in A_U.

1.1.7. Definition. The closure in A_U generated by the subbases (1.1.6) will be called uniform convergence closure on the cofiltration $\varkappa$ from 1.1.1. Its dependence on the closure t can be expressed by introducing for it the symbol $k(t)$.

From 1.1.5 we can see, that the form of the sets $B(a(x); K, 0)$ is not influenced by the whole filter $\Delta(\mathrm{gr}_K\, a; t)$, but only by the induced filter $\Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$ of neighborhoods of $\mathrm{gr}_K\, a$ in the space $(\psi^{-1}(K); \mathrm{ind}_{\psi^{-1}(K)}\, t)$. From this we get

1.1.8. Proposition. *The closure $k(t)$ in the sets A_U depends only on the closure, which is induced by the closure t in the sets $\psi^{-1}(K)$ for $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$.*

1.1.9. Corollary. *All the closures t in P can be divided into classes with respect to the equivalence*

$$t_1 \sim t_2 \quad \textit{iff} \quad \mathrm{ind}_{\psi^{-1}(K)}\, t_1 = \mathrm{ind}_{\psi^{-1}(K)}\, t_2 \tag{1.1.10}$$

From $t_1 \sim t_2$ follows the equality $k(t_1) = k(t_2)$ in all A_U. The example (3.4.12) shows, that from $t_1 \sim t_2$ does not follow the equality $t_1 = t_2$ even for the topologies t_1, t_2. From $k(t_1) = k(t_2)$ does not follow the relation $t_1 \sim t_2$ – see example (3.4.13).

1.1.11. Condition. Let the closure t in P have the property: "If $U, V \in \mathscr{B}(X)$, $V \subset U$, $a \in S_U$, $K \subset \mathscr{K}^V_{\varrho_{UV}(a)}$, $L \in \mathscr{K}_a^U$, $K \subset L$, then $\Delta(\mathrm{gr}_L\, a; t) \cap \psi^{-1}(K) \leqq \Delta(\mathrm{gr}_K\, a; t)$."

1.1.12. Condition. Let the cofilters $\mathscr{K}_a^U$ have the property: "For $U, V \in \mathscr{B}(X)$, $V \subset U$, $a \in S_U$ there is

$$K^V_{\varrho_{UV}(a)} = \{K \mid K \in \mathscr{K}_a^U,\ K \subset V\}\,.\text{"} \tag{1.1.13}$$

1.1.14. Remark. *For $U, V \in \mathscr{B}(X)$, $V \subset U$ the map $\tilde{\rho}_{UV} : (A_U, k(t)) \to (A_V, k(t))$ need not generally be continuous for any closure t in P.*

Naturally, such a closure can not be normal. But when the closure t satisfies (1.1.11), we can see from (1.1.5) and (1.1.1) that the all $\tilde{\rho}_{UV}$ are continuous for this t. If the cofilters $\mathscr{K}_a^U$ satisfy (1.1.12), we can easily see, that the all $\tilde{\rho}_{UV}$ are continuous for any t.

2. Homeomorphie of p_U^t

First of all we are going to check condition for the continuity of p_U^t. The following trivial proposition we state in order we could paraphrase it in the next assertion into the form, which will be more convenient for us.

1.2.1. Proposition. *Let $U \in \mathscr{B}(X)$. Then p_U^t is continuous iff the following condition holds: "If $a \in S_U$, $B(a(x); K, O) \in \Delta(a(x); k(t))$, then there is $W \in \Delta(a; \tau_U)$ such that $p_U^t(W) \subset B(a(x); K, O)$."*

According to (0.14, 15) we paraphrase 1.2.1 in this way:

1.2.2. Proposition. *Let $U \in \mathscr{B}(X)$. The map p_U^t is continuous iff the following condition holds: "If $a \in S_U$, $B(a(x); K, O) \in \Delta(a(x); k(t))$, then there is $W \in \Delta(a; \tau_U)$ such that*

$$\xi_{UK}(W) \subset O \cap \psi^{-1}(K) \text{."} \tag{1.2.3}$$

Now we can already find without difficulties, what a neighborhood O of the set $\mathrm{gr}_K\, a$ in $(\psi^{-1}(K))$; $\mathrm{ind}_{\psi^{-1}(K)}\, t)$ must contain in order that the all p_U^t would be continuous. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$, $O \in \Delta(\mathrm{gr}_K\, a; t)$. We suppose, that the all p_U^t are continuous. The continuity of p_U^t at the point a gives by 1.2.2 that 1.2.3 holds for some $W^U \in \Delta(a; \tau_U)$. If $V \in \mathrm{B}(K, U)$ – see 0.21 –, again from the continuity of p_V^t at the point $\varrho_{UV}(a)$ follows (by 1.2.2) that $\xi_{VK}(W^V) \subset O \cap \psi^{-1}(K)$ for some $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$. (We suppose that 1.1.12 holds.) Thus for the continuity of all p_U^t the following condition is necessary:

1.2.4. Condition. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$, $O \in \Delta(\mathrm{gr}_K\, a; t)$. Then for every $V \in \mathrm{B}(K; U)$ there exists $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$ such that

$$\bigcup\{\xi_{VK}(W^V) \mid V \in \mathrm{B}(K; U)\} \subset O \cap \psi^{-1}(K) . \tag{1.2.5}$$

From 1.2.1. is qute clear that the condition 1.2.4. is also sufficient for the continuity of all p_U^t. Thus we get

1.2.6. Proposition. *All p_U^t are continuous iff* 1.2.4 *holds.*

1.2.7. Proposition. *Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$, $O \in \Delta(\mathrm{gr}_K\, a; t)$ and let there exists $U' \in \mathrm{B}(K; U)$ such that for every $V \in \mathrm{B}(K; U')$ there exists $W_1^V \in \Delta(\varrho_{UV}(a); \tau_V)$ such that $\bigcup\{\xi_{VK}(W_1^V) \mid V \in \mathrm{B}(K; U')\} \subset O \cap \psi^{-1}(K)$. Then for every $V \in \mathrm{B}(K; U)$ there exists $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$ such that $\bigcup\{\xi_{VK}(W^V) \mid V \in \mathrm{B}(K; U)\} \subset O \cap \psi^{-1}(K)$.*

Proof. We use the continuity of ϱ_{UV}. For $V \in \mathrm{B}(K; U)$ we set $W^V = \varrho_{VU' \cap V}^{-1}(W_1^{U' \cap V})$. Then the neighborhoods W^V have the requered property.

Proposition 1.2.7 says, that if the condition 1.2.4 holds locally, i.e. in the form from 1.2.7, then it holds in the original form from 1.2.4.

1.2.8. Notation. Let $U \in \mathscr{B}(X)$, $a \in S_U$ and let $K \subset U$ be an arbitrary set. Every map η of the set $\mathrm{B}(K; U)$ into the cartesian product $\prod_{K \subset V \subset U} \Delta(\varrho_{UV}(a); \tau_V)$ such that for $V \in \mathrm{B}(K; U)$ we have $\eta(V) = W^V \in \Delta(\varrho_{UV}(a); \tau_V)$ will be called choice.

1.2.9. Definition. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$, K an arbitrary set, η a choice. The chain round a from U to K is the family

$$R(U, a, K, W^V) = \{W^V \mid V \in \mathrm{B}(K; U),\ W^V = \eta(V)\} . \tag{1.2.10}$$

1.2.11. Agreement. Let $K \subset X$, $M \subset \psi^{-1}(K)$ be such sets, that for some $U \in \mathscr{B}(X)$ and some $a \in S_U$ there is $K \in \mathscr{K}_a^U$ and $M = \mathrm{gr}_K\, a$.

With such 4-tuples (K, M, U, a) we will often meet later. In order, that we need not describe them again, we will say briefly, that (K, M, U, a) is a 4-tuple from 1.2.11.

1.2.12. Notation. Let (K, M, U, a) be from 1.2.11. Let us choose a chain $R(U, a, K, W^V)$ and let us set

$$(1.2.13) \qquad F = S(R(U, \alpha, K, W^V)) = \bigcup_{V \in B(K;U)} \xi_{VK}(W^V) \subset \psi^{-1}(K)\,,$$

$$(1.2.14) \qquad \mathscr{F}_M(U, a, K) = \{F \mid F = S(R(U, a, K, W^V)),\ R(U, a, K, W^V) \text{ is a chain round } a \text{ from } U \text{ to } K\}\,.$$

We have clearly

1.2.15. Proposition. *If (K, M, U, a) is from 1.2.11, then $\mathscr{F}_M(U, a, K)$ is a filter base in $\psi^{-1}(K)$ round M.*

1.2.16. Lemma. *Let $U, V \in \mathscr{B}(X)$, $a \in S_U$, $b \in S_V$ and let $K \subset U \cap V$ be an arbitrary set that $\mathrm{gr}_K\, a = \mathrm{gr}_K\, b$. Then for some $\tilde{V} \in B(K; U \cap V)$ we have $\varrho_{U\tilde{V}}(a) = \varrho_{U\tilde{V}}(b)$.*

Proof. Because of $\mathrm{gr}_K\, a = \mathrm{gr}_K\, b$, for every $x \in K$ there exists an open neighborhood $V_x \subset U \cap V$ such that $\varrho_{UV_x}(a) = \varrho_{VV_x}(b)$. Let us set $\tilde{V} = U\{V_x \mid x \in K\}$. Then $\tilde{V} \in B(K; U \cap V)$ and for the elements $a' = \varrho_{U\tilde{V}}(a)$, $b' = \varrho_{V\tilde{V}}(b)$ we have: For every V_x, $x \in K$ there is $\varrho_{\tilde{V}V_x}(a') = \varrho_{\tilde{V}V}x(b')$ (by 0.11). Because of $\tilde{V} = \bigcup V_x$, there is by 0.17 $a' = b'$.

1.2.17. Corollary. *Let (K, M, U, a), (K, M, V, b) be two 4-tuples from 1.2.11. Then the filter bases $\mathscr{F}_M(U, a, K)$ and $\mathscr{F}_M(V, b, K)$ are equivalent.*

Proof. Let $F_1 = S(R(U, a, K, W_1^{V_1})) \in \mathscr{F}_M(U, a, K)$. For U and V let us choose $\tilde{V}$ such that the assertion from 1.2.16 holds.

For $V_2 \in B(K; V)$ let us set $W_2^{V_2} = \varrho_{V_2 V_2 \cap \tilde{V}}^{-1}(W_1^{V_2 \cap \tilde{V}})$. Then we have $F_2 = S(R(V, b, K, W_2^{V_2})) \in \mathscr{F}_M(V, b, K)$ and $F_2 \subset F_1$, which finishes the proof.

1.2.18. Corollary. *Now we can write simply $\mathscr{F}_M$ or $\mathscr{F}\ \mathrm{gr}_K\, a$ instead of $\mathscr{F}_M(U, a, K)$ and use always the more convenient base $\mathscr{F}_M(U, a, K)$ which we do in the following.*

1.2.19. Proposition. *All the maps p_U^t are continuous iff for every $U \in \mathscr{B}(X)$, $a \in S_U$, $\mathscr{K} \in \mathscr{K}_a^U$ the base $\mathscr{F}\ \mathrm{gr}_K\, a$ majorises the base $\Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$ (briefly $\mathscr{F}\ \mathrm{gr}_K\, a \leqq \leqq \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$).*

Proof. If we take into consideration the corollary 1.2.18 and 1.2.12–15, we see, that we have only paraphrased Proposition 1.2.6.

Now, for $U \in \mathscr{B}(X)$, $a \in S_U, \mathscr{K} \in \mathscr{K}_a^U$, $F \in \mathscr{F}\, \mathrm{gr}_K\, a$ let us take the set $\xi_{UK}^{-1}(F) = \{b(x) \mid b(x) \in A_U, \mathrm{gr}_K\, b \subset F\}$ (see 0.19) and let us set

$$(1.2.20) \qquad \{\xi_{UK}^{-1}(F) \mid K \in \mathscr{K}_a^U,\ F \in \mathscr{F}\, \mathrm{gr}_K\, a\} .$$

We have obviously

1.2.21. Proposition. *The family* 1.2.20 *is a filter subbase in A_U round $a(x)$.*

1.2.22. Definition. The closure in the sets A_U generated by the filter-subbases 1.2.20 will be denoted by k. The natural map

$$(1.2.23) \qquad p_U : (S_U, \tau_U) \to (A_U, k)$$

will be denoted by p_U^k.

1.2.24. Corollary. Let $\mu = \{\tau_U\}$ be a closure collection of our presheaf. From 1.2.13 we can see, that for $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$ the form of the base $\mathscr{F}\, \mathrm{gr}_K\, a$ depends on the form of the collection μ, and thus k from 1.2.22 depends also on μ. Therefore in cases if necessary, we will denote the closure k by the symbol k_μ, understanding under it the closure k from 1.2.22, which is formed from μ.

1.2.25. Corollary. A. *All the maps p_U^k are continuous.*
B. *All the maps p_U^t are continuous iff we have $k \leqq k(t)$ in all A_U.*

Proof. A. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$, $F = S(R(U, a, K, W^V)) \in \mathscr{F}\, \mathrm{gr}_K\, a$. Then $D = \xi_{UK}^{-1}(F) \in \Delta(a(x); k)$ and $p_U(W^U) \subset D$, thus p_U^k is continuous.
B. If all the p_U^t are continuous, then by 1.2.19 (comparing 1.2.20 and 1.1.6) it follows, that in all A_U there is $k \leqq k(t)$. Conversely, if in all A_U $k \leqq k(t)$, then A implies the continuity of all p_U^t.

1.2.26. Proposition. *Let for the closure t in P all the maps $(p_U^t)^{-1}$ be continuous. Then $k(t) \leqq k$ in every A_U.*

Proof. By 1.2.25A the all p_U^k are continuous. From the continuity of all $(p_U^t)^{-1}$ follows the continuity of the map $i = p_U^k \circ (p_U^t)^{-1}$, which is the identity map $i_U : (A_U, k(t)) \to (A_U, k)$.

1.2.27. Corollary. *If the closure t is P is normal* (see 1.1.3), *then $k(t) = k$ in all A_U.*

Proof. If t is normal, then every p_U^t is continuous and thus by 1.2.25B we get $k \leqq k(t)$ in every A_U. Because every $(p_U^t)^{-1}$ is continuous, 1.2.27 follows from 1.2.26.

1.2.28. Definition. A closure t in P is called seminormal, if $k(t) = k$ in all A_U.
From 1.2.27 follows immediately:

1.2.29. Theorem. *Every normal closure is seminormal. A necessary condition for the existence of a normal closure in P is the existence of a seminormal one.*

1.2.30. Theorem. *A necessary condition for the existence of a normal closure in P is the continuity of all* $(p_U^k)^{-1}$.

Proof. Let t be normal, i.e. all the p_U^t are homeomorphisms. By 1.2.27 the all p_U^k are homeomorphisms and therefore the all $(p_U^k)^{-1}$ are continuous.

1.2.31. Theorem. A. *Let all the* $(p_U^k)^{-1}$ *be continuous. Then there exists a normal closure in P iff there exists a seminormal closure* t_1 *in P. If this condition is satisfied, then* t_1 *is normal.*

B. *If there exists a seminormal closure* t_1 *in P, then there exists a normal closure t in P iff all the* $(p_U^k)^{-1}$ *are continuous. If this condition is satisfied, then* t_1 *is normal.*

Proof. A. If all the $(p_U^k)^{-1}$ are continuous, then by 1.2.25A the all p_U^k are homeomorphisms. If t_1 is seminormal, then $p_U^k = p_U^{t_1}$ for all $U \in \mathscr{B}(X)$, hence t_1 is normal. Conversely, if t_1 is normal, then by 1.2.29 is seminormal.

B. If t is normal, then by 1.2.27 all the $(p_U^k)^{-1}$ are continuous. Conversely, from the continuity of all $(p_U^k)^{-1}$ follows by 1.2.25 A that all p_U^k are homeomorphisms. Then the seminormality of t_1 implies that t_1 is normal.

1.2.32. Theorem. *The necessary condition for the existence of a normal closure in P is the simultaneous validity of both necessary conditions* 1.2.29, 30. *Conversely, the simultaneous validity of both these conditions provides the existence of a normal closure in P.*

Proof. Follows immediately from 1.2.31 A.

So that we could decide, whether there exists a normal closure in P, we must study the simultaneous validity of 1.2.29, 30. In the following paragraph we study the second condition.

3. Continuity of $(p_U^k)^{-1}$

For sake of completeness we state the necessary and sufficient condition for the continuity of all $(p_U^k)^{-1}$.

1.3.1. Theorem. *All the* $(p_U^k)^{-1}$ *are continuous iff the following condition holds: "If* $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, *then there exist* $K_1, \ldots, K_n \in \mathscr{K}_a^U$ *and* $F_i \in \in \mathscr{F}\ \mathrm{gr}_{K_i}\ a$, $i = 1, \ldots, n$ *such that*

$$\bigcap_{i=1}^{n} \xi_{UK_i}^{-1}(F_i) \subset p_U(W)\,. \text{"} \tag{1.3.2}$$

Proof. By the definition 1.2.22, 1.3.1 is only the verification of the definition of the continuity of all $(p_U^k)^{-1}$.

We will study when the condition in 1.3.1 holds. For that we need to stronger Condition 1.1.1 for the cofilters $\mathscr{K}_a^U$.

1.3.3. Assumption. From now up to the end of the chapter we assume that 1.1.12 holds, i.e.: "If $U, V \in \mathscr{B}(X)$, $V \subset U$, $a \in S_U$, then

$$K^V_{\varrho_{UV}(a)} = \{K \mid K \in \mathscr{K}^U_a,\ K \subset V\}\,. \text{"} \tag{1.3.4}$$

Let $\mu = \{\tau_U\}$ be a closure collection of our presheaf and let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$. Let us put

$$P(a, K) = \{\xi_{UK}^{-1}(F) \mid F \in \mathscr{F} \operatorname{gr}_K a\}\,, \tag{1.3.5}$$

$$\mathscr{P}(a) = \{P(a, K) \mid K \in \mathscr{K}_a^U\}\,. \tag{1.3.6}$$

Then $\mathscr{P}(a)$ is a filter-subbase in S_U round a. Thus the sets $\mathscr{P}(a)$ form a closure τ_U^0 in S_U. If we form the closure τ_U^0 in all S_U, then by 1.3.3, 1.2.18, 13 we have obviously

1.3.7. Proposition. *Let us denote $\mu^0 = \{\tau_U^0\}$. Then μ^0 is a closure collection coarser than μ.*

1.3.8. Theorem. *Let $\mathscr{S}$ be a presheaf over X, μ and ν its closure collections, $\mu \leqq \nu$. Let all the $(p_U^{k_\nu})^{-1}$ be continuous (see* 1.2.24*). Then $\mu^0 \leqq \nu$.*

Proof. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $\mu = \{\tau_U\}$, $\nu = \{\tau_U^1\}$, $W \in \Delta(a; \tau_U^1)$. From the continuity of $(p_U^{k_\nu})^{-1}$ and from 1.3.1 follows the existence of the sets $K_1, \ldots, K_n \in \mathscr{K}_a^U$ and the chains $R(U, a, \operatorname{gr}_{K_i} a, W_1^{V_i})$, $i = 1, \ldots, n$ such that

$$S_1 = \bigcap_{i=1}^{n} \xi_{UK_i}^{-1}[\bigcup_{V_i \in \beta(K_i; U)} \xi_{V_i K_i}(W_1^{V_i})] \subset p_U(W)\,. \tag{1.3.9}$$

Here for the all $W_1^{V_i}$ in 1.3.9 there is $W_1^{V_i} \in \Delta(\varrho_{UV_i}(a); \tau_{V_i}^1)$. Because $\mu \leqq \nu$, we can for every $W_1^{V_i}$ from 1.3.9 find some $W^{V_i} \in \Delta(\varrho_{UV_i}(a); \tau_{V_i})$ such that $W^{V_i} \subset W_1^{V_i}$. Then we have

$$S_2 = \bigcap_{i=1}^{n} \xi_{UK_i}^{-1}[\bigcup_{V_i \in \beta(K_i; U)} \xi_{V_i K_i}(W^{V_i})] \subset S_1\,. \tag{1.3.10}$$

If we denote by F_i the sets in the brackets, then $F_i \in \mathscr{F} \operatorname{gr}_{K_i} a$, $i = 1, \ldots, n$ and by 1.3.5–7 we have $p_U^{-1}(S_2) \in \Delta(a; \tau_U^0)$ and $p_U^{-1}(S_2) \subset W$.

1.3.11. Theorem. *For every collection $\mu = \{\tau_U\}$ there exists the unique collection $\bar{\mu} = \{\bar{\tau}_U\}$ coarser than μ with these properties:*

(a) $k_\mu = k_{\bar{\mu}}$ *in all* A_U,

(b) *the all* $(p_U^{k_{\bar{\mu}}})^{-1} : (A_U, k_{\bar{\mu}}) \to (S_U, \bar{\tau}_U)$ *are continuous*,

(c) *if* ν *is a coarser collection than* μ *and all* $(p_U^{k_\nu})^{-1}$ *are continuous, then* $\bar{\mu} \leqq \nu$. *And simultaneously we have* $\bar{\mu} = \mu^0$.

Proof. Let us set $\bar{\mu} = \mu^0$. By 1.3.7 we have $\mu \leqq \mu^0$. (c) holds according to 1.3.8.

Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$. Let us form the base $\mathscr{F}$ $\mathrm{gr}_K\, a$ from the collection μ^0 and let us denote it by $\mathscr{F}^1$ $\mathrm{gr}_K\, a$. Because $\mu \leqq \mu^0$, $\mathscr{F}^1$ $\mathrm{gr}_K\, a$ is coarser than the base $\mathscr{F}$ $\mathrm{gr}_K\, a$ formed from the collection μ.

Conversely, let $F \in \mathscr{F}$ $\mathrm{gr}_K\, a$. Let us set $W^V = \xi_{VK}^{-1}(F)$ for $V \in \mathrm{B}(K; U)$. Then $S = S(R(U, a, K, W^V)) \in \mathscr{F}^1$ $\mathrm{gr}_K\, a$ and $S \subset F$, which proves the equalence of $\mathscr{F}$ $\mathrm{gr}_K\, a$ and $\mathscr{F}^1$ $\mathrm{gr}_K\, a$. This proves (a). Let $U \in \mathscr{B}(X)$, $a \in S_U$. Let us choose $D = \xi_{UK}^{-1}(F) \in \Delta(a; \tau_U\}$ for some $K \in \mathscr{K}_a^U$ and some $F \in \mathscr{F}^1$ $\mathrm{gr}_K\, a$. Then $\tilde{D} = \xi_{UK}^{-1}(F) \in \Delta(a(x); k_{\mu^0})$ and $p_U(\tilde{D}) \subset D$, which proves (b).

1.3.12. Definition. Because the collection μ^0 from 1.3.7 has the properties described in 1.3.11, we call it cofilter-modification of the collection μ with respect to the given cofiltration $\varkappa = \{\mathscr{K}_a^U \mid U \in \mathscr{B}(X),\ a \in S_U\}$.

1.3.13. Corollary. *By* 1.3.11 μ^0 *is the finest of the all collections* ν *which are coarser than* μ *and for which the all* $(p_U^{k_\nu})^{-1}$ *are continuous. These three properties are equivalent*:

(a) *The all* $(p_U^{k_\mu})^{-1}$ *are continuous*,

(b) $\mu = \mu^0$,

(c) *for the collection* μ *the condition from the theorem* 1.3.1 *holds.*

1.3.14. Remark. Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a presheaf over a locally compact space with the unique continuation, (see 0.32) and with projective closure collection (see 0.38, or [8]), 1.1.4, for which 1.2.13, 18 from [8] holds. By [8], 1.2.20 for $\mathscr{S}$ there exists its natural cofiltration $\varkappa$. Let $k(t)$ be the closure under the uniform convergence on $\varkappa$ and let k be the closure from 1.2.22 formed for $\varkappa$. Then the all $(p_U^k)^{-1}$ are continuous. For such a presheaf and its natural cofiltration we have $\mu = \mu^0$.

Proof. Let $U \in \mathscr{B}(X)$ be connected, $a \in S_U$, $W \in \Delta(a; \tau_U)$. We can assume (see 1.2.13 in [8]) that $K = M(W) \in \mathscr{K}_a^U$. By [8], 1.2.1–3, for every $V \in \mathrm{B}(K; U)$ there exists $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$ such that

$$\varrho_{UV}^{-1}(W^V) \subset W. \tag{1.3.15}$$

Then $S = S(R(U, a, K, W^V)) \in \mathscr{F}$ $\mathrm{gr}_K\, a$ and $L = \xi_{UK}^{-1}(S) \subset p_U(W)$, $L \in \Delta(a(x); k)$. Thus $(p_U^k)^{-1}(L) \subset W$, which finishes the proof for connected U. For an arbitrary

$U \in \mathscr{B}(X)$ let us notice the commutative diagram for every component V of U:

$$(1.3.16)\qquad \begin{array}{ccc} (S_U, \tau_U) & \xleftarrow{(p_U{}^k)^{-1}} & (A_U, k) \\ \downarrow{\scriptstyle \varrho_{UV}} & & \downarrow{\scriptstyle \tilde{\rho}_{UV}} \\ (S_V, \tau_V) & \xleftarrow{(p_V{}^k)^{-1}} & (A_V, k) \end{array} .$$

The projectivity of $\mu = \{\tau_U\}$ and the local connectedness of X (see 0.32) imply that $(p_U^k)^{-1}$ is continuous iff $\varrho_{UV} \circ (p_U^k)^{-1}$ is continuous for all components V of U. By 1.3.16 there is $\varrho_{UV} \circ (p_U^k)^{-1} = (p_V^k)^{-1} \circ \tilde{\rho}_{UV}$. From [8] 1.2.20 and from 1.2.20–22 follows the continuity of all $\tilde{\rho}_{UV}$ which with the first part finishes the proof.

1.3.17. Remark. Let all the (S_U, τ_U) be compact topological spaces and let all the (A_U, k) be Hausdorff topological spaces. By 0.17 all the p_U^k are injective, and by 1.2.25 also continuous and thus homeomorphic.

Now, we are going to show that in some cases the modification can be also constructed in an other way than in 1.3.5–7.

1.3.18. Notation. Let $U \in \mathscr{B}(X)$, $K \subset U$ be an arbitrary set and u_U resp u_K a closure in U resp in K, induced from X. Let r_{UK} be a map: $a(x) \in A_U, \Rightarrow r_{UK}(a(x)) = a(x)/K$ of the set A_U onto $\Gamma_U^K = r_{UK}(A_U)$. For all $V \in \mathrm{B}(K; U)$ there exists a natural map $\tilde{\delta}_{VK} : S_V \times K \to \psi^{-1}(K)$ defined as follows: $a \in S_V$, $y \in K \Rightarrow \tilde{\delta}_{VK}(a, y) = a(y) = \xi_{V_y}(a)$. Let us denote $\tilde{\Delta}_U = \{\tilde{\delta}_{VK} \mid V \in \mathrm{B}(K, U)\}$. We can define a natural closure t_K^U in $\psi^{-1}(K)$, with respect to the set $\tilde{\Delta}_U$ of the maps $\tilde{\delta}_{VK}$, namely the finest of the all closures t, such that all the $\tilde{\delta}_{VK} \in \tilde{\Delta}_U$, $\tilde{\delta}_{VK} : (S_V \times K, \tau_V \times u_K) \to (\psi^{-1}(K); t)$ are continuous, i.e.

$$(1.3.19)\qquad t_U^K = \varinjlim \{\tau_V \times u_K \mid V \in \mathrm{B}(K; U)\} .$$

(See 0.21.D) For $U, V \in \mathscr{B}(X)$, $V \in \mathrm{B}(K; U)$ let us define the map $R_{UV} : S_U \times K \to S_V \times K$ as follows: $a \in S_U$, $x \in K \Rightarrow R_{UV}(a, x) = (\varrho_{UV}(a), x)$. Clearly the map $R_{UV} : (S_U \times K, \tau_U \times u_K) \to (S_V \times K, \tau_V \times u_K)$ is continuous.

Let $U, U' \in \mathscr{B}(X)$, $K \subset U \cap U'$. We will prove that $t_K^U = t_K^{U'}$. Let t be such a closure in $\psi^{-1}(K)$ that the all maps $\tilde{\delta}_{VK} \in \tilde{\Delta}_U$ are continuous. Let $\tilde{\delta}_{V'K} \in \tilde{\Delta}_{U'}$ for some $V' \in \mathrm{B}(K; U')$. Then we have $\tilde{\delta}_{V'K} = \tilde{\delta}_{V' \cap UK} \circ R_{V'V' \cap U}$. From the fact that the map $\tilde{\delta}_{V' \cap UK}$ is from $\tilde{\Delta}_U$ follows, that it is continuous. Therefore the continuity of $R_{V'V' \cap U}$ implies the continuity of $\tilde{\delta}_{V'K}$. Thus $t_K^{U'} \leqq t_K^U$, which proves the equality $t_K^{U'} = t_K^U$. Now we can instead of t_K^U write briefly t_K.

Let $U \in \mathscr{B}(X)$, $a \in S_U$ and let $K \subset U$ be compact. Similarly as in 1.2.8–18 we can construct a base $\mathscr{F}\ \mathrm{gr}_K\, a$ round $\mathrm{gr}_K\, a$ in $\psi^{-1}(K)$. From 1.3.19 we get $\Delta(\mathrm{gr}_K\, a, t_K) \leqq \mathscr{F}\ \mathrm{gr}_K\, a$. Conversely, let $O \in \Delta(\mathrm{gr}_K\, a; t_K)$. We may assume that $O = U\{O_\alpha \mid \alpha \in \mathrm{gr}_K\, a\}$ (where $O_\alpha \in \Delta(\alpha; t_K)$ for all $\alpha \in \mathrm{gr}_K\, a$) such that the following holds: "If

$\alpha \in \mathrm{gr}_K\, a$, $y = \psi(\alpha)$, then

$$O_\alpha = \bigcup_{V \in \mathrm{B}(K;U)} \xi_{V\varphi_\alpha(V)}(W^V_\alpha)\,, \tag{1.3.20}$$

where for every $V \in \mathrm{B}(K; U)$ we have $\varphi_\alpha(V) \in \mathrm{B}(y; V)$. Now, let $V \in \mathrm{B}(K; U)$. For every $\alpha \in \mathrm{gr}_K\, a$ we take $\varphi_\alpha(V)$ from 1.3.20. A family $\Lambda = \{\varphi_\alpha(V) \mid \alpha \in \mathrm{gr}_K\, a\}$ covers K. Thus $K \subset \bigcup\{\varphi_\alpha(V) \mid \alpha \in A\}$ for some finite A. We may assume that $A = \{\alpha_1, \alpha_2\}$. Then $W^V = W^V_{\alpha_1} \cap W^V_{\alpha_2} \in \Delta(\varrho_{UV}(a); \tau_V)$. We construct W^V for every $V \in \mathrm{B}(K; U)$ in this way. Then $S = S(R(U, a, K, W^V)) \in \mathscr{F}\, \mathrm{gr}_K\, a$ and $S \subset O$. We have proved

1.3.21. Proposition. *Let $U \in \mathscr{B}(X)$, $a \in S_U$ and let $K \subset U$ be compact. Then the bases $\Delta(\mathrm{gr}_K\, a; t_K)$ and $\mathscr{F}\, \mathrm{gr}_K\, a$ are equivalent.*

For $U \in \mathscr{B}(X)$, $K \subset U$, $b \in \Gamma^K_U$, $O \in \Delta(\mathrm{gr}_K\, b, t_K)$ let us set $D(b; O) = \{c \mid c \in \Gamma^K_U$, $c(y) \in O$ for all $y \in K\}$, $\mathscr{D}(b) = \{D(b; O) \mid O \in \Delta(\mathrm{gr}_K\, b; t_K)\}$. Then $\mathscr{D}(b)$ is a filter base in Γ^K_U round b. The closure in Γ^K_U generated by these bases will be denoted by v_K.

If for all $U \in \mathscr{B}(X)$ the following condition

$$\begin{aligned} &\text{(a)}\ \mathscr{K}^U_a = \mathscr{K}^U_b = \mathscr{K}^U \quad \text{for all} \quad a, b \in S_U\,, \\ &\text{(b)}\ K \in \mathscr{K}^U \Rightarrow K \quad \text{is compact} \end{aligned} \tag{1.3.22}$$

is satisfied, let us set

$$m_U = \varinjlim\, \{v_K \mid K \in \mathscr{K}^U\}\,, \tag{1.3.23}$$

with respect to the set of maps $\{r_{UK} \mid K \in \mathscr{K}^U\}$. (See 1.3.18).

Comparing 1.3.23 and 1.3.5–6, we get immediately from 1.3.21 $m_U = \tau^0_U$ for all $U \in \mathscr{B}(X)$. Thus if 1.3.22 holds, then the cofilter modification μ^0 of μ can be constructed also using 1.3.23.

1.3.24. Examples. (1) Let for every $U \in \mathscr{B}(X)$ and every $a \in S_U$ $\mathscr{K}^U_a$ be the cofilter of all finite sets in U. It can be easily seen, that μ^0 is projective (see [8], 1.1.4 or 0.38). Thus we have $\mu' \leqq \mu^0$ (see [8], 1.1.6–7).

(2) Let $\mathscr{S}$ be a presheaf over X, where for $U \in \mathscr{B}(X)$ S_U is some set of continuous functions on U and τ_U the closure under the uniform convergence in S_U. It can be easily seen, that for every $U \in \mathscr{B}(X)$ τ^0_U is the closure under the pointwise convergence in S_U. (The cofiltration is the same as in (1)).

4. Seminormal closure

Here we study whether the necessary condition 1.2.29 for the existence of the normal closure holds.

1.4.1. Definition. We say that the closure t in P is a generating one, if the following condition holds: "If $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}^U_a$ then the filter bases $\Delta(\mathrm{gr}_K\, a; t) \cap \cap\, \psi^{-1}(K)$ and $\mathscr{F}\, \mathrm{gr}_K\, a$ are equivalent."

Comparing 1.2.20 and 1.1.5, we get

1.4.2. Proposition. *Every generating closure is seminormal.*

1.4.3. Remark. The seminormal closure need not be a generating one – see example 3.4.13.

1.4.4. Proposition. *For the existence of a generating closure is P the foloswing condition is necessary*: "*If* $U \in \mathscr{B}(X)$, $a \in S_U$, $K, L \in \mathscr{K}_a^U$, $K \subset L$, *then* $\mathscr{F}\ \mathrm{gr}_K\, a \sim \sim \mathscr{F}\ \mathrm{gr}_L\, a \cap \psi^{-1}(K)$ (*see* 0.21 C).

Proof. Let t be a generating closure, $U \in \mathscr{B}(X)$, $K, L \in \mathscr{K}_a^U$, $K \subset L$, $F_2 \in \mathscr{F}\ \mathrm{gr}_L\, a$. Then $O \cap \psi^{-1}(L) \subset F_2$ for some $O \in \Delta(\mathrm{gr}_L\, a; t)$. But there is also $O \in \Delta(\mathrm{gr}_K\, a; t)$. Hence $F_1 \subset O \cap \psi^{-1}(K)$ for some $F_1 \in \mathscr{F}\ \mathrm{gr}_K\, a$. Therefore $F_1 \subset F_2 \cap \psi^{-1}(K)$ and thus $\mathscr{F}\ \mathrm{gr}_K\, a \leqq \mathscr{F}\ \mathrm{gr}_L\, a \cap \psi^{-1}(K)$. Conversely, if $F_1 = S(R(U, a, K, W^V)) \in \mathscr{F}\ \mathrm{gr}_K\, a$, then $F_2 = S(R(U, a, L, W^V)) \in \mathscr{F}\ \mathrm{gr}_L\, a$ and $F_2 \cap \psi^{-1}(K) \subset F_1$, which finishes the proof.

1.4.5. Agreement. Let the following holds up to the end of the paragraph: "Let $U \in \mathscr{B}(X)$, $a \in S_U$;

(a) every $\mathscr{K} \in \mathscr{K}_a^U$ is compact,

(b) every $x \in U$ has a base β of the filter of neighborhoods such that $\beta \subset \mathscr{K}_a^U$."

From 1.4.5 follows immediately

1.4.6. Corollary. *Let* 1.4.5 *hold. Then*

A. *X is locally compact,*

B. *if* $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$, *then* $K \subset \mathrm{int}\, L$ *for some* $L \in \mathscr{K}_a^U$.

1.4.7. Assumption. Let the following hold up to the end of the paragraph: "Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K, L \in \mathscr{K}_a^U$, $K \subset L$. Then $\mathscr{F}\ \mathrm{gr}_K\, a \sim \mathscr{F}\ \mathrm{gr}_L\, a \cap \psi^{-1}(K)$."

1.4.8. Agreement. Let $\alpha \in P$, $y = \psi(\alpha)$. In some S_U there exists a generating element a for α. By 1.4.5b there exist $K \in \mathscr{K}_a^U$ and $M \in \mathscr{B}(U)$ such that $y \in M \subset K$. Such 4-tuples will often appear later. In order that we need not describe them steadily, we will only say that (α, U, a, M) is a 4-tuple from 1.4.8. Sometimes we also write (α, U, a, K), where $K \in \mathscr{K}_a^U$ has the property that $y = \psi(\alpha) \in \mathrm{int}\, K$.

1.4.9. Notation. For a 4-tuple from 1.4.8 let us denote

$$G(U, K, y) = \{M \mid M \in \mathscr{K}_a^U,\ y = \psi(\alpha) \in \mathrm{int}\, M \subset M \subset K\}\,. \tag{1.4.10}$$

1.4.11. Proposition. *For* (α, U, a, K) *from* 1.4.8 *the set*

$$L(\alpha; U, a, K) = \bigcup_{M \in G(U,K,y)} \mathscr{F}\ \mathrm{gr}_M\, a(U, a, M) \quad \text{(see 1.2.14)} \tag{1.4.12}$$

is a filter base in P round α.

Proof. Let $F_1, F_2 \in L(\alpha; U, a, K)$, $y = \psi(\alpha)$. Then $F_i \in \mathscr{F}\ \mathrm{gr}_{M_i}(U, a, M_i)$ for some $M_i \in G(U, K, y)$, $i = 1, 2$. There exists $M_3 \in G(U, K, y)$, $M_3 \subset M_1 \cap M_2$. By 1.4.7 we find a set $\tilde{F}_i \in \mathscr{F}\ \mathrm{gr}_{M_3} a(U, a, M_3)$ for F_i, such that $\tilde{F}_i \subset F_i$, $i = 1, 2$. Then $F_3 = \tilde{F}_1 \cap \tilde{F}_2 \subset F_1 \cap F_2$ and $F_3 \in \mathscr{F}\ \mathrm{gr}_{M_3} a(U, a, M_3) \subset L(\alpha; U, a, K)$.

1.4.13. Proposition. *Let (α, U_1, a_1, K_1), (α, U_2, a_2, K_2) be two 4-tuples from 1.4.8. Then $L_1 = L(\alpha; U_1, a_1, K_1) \sim L_2 = L(\alpha; U_2, a_2, K_2)$.*

Proof. Let $F_i \in L_i$. Then $F_i \in \mathscr{F}\ \mathrm{gr}_{M_i} a_i(U_i, a_i, M_i)$ for some $M_i \in G(U_i, K_i, y)$, $i = 1, 2$. By 1.4.5b and 1.3.3 there exist $U_3 \in \mathscr{B}(X)$ and $M_3 \in \mathscr{K}^R_{\varrho_{PR}(a_1)} \cap \mathscr{K}^R_{\varrho_{QR}(a_2)}$ (we have put $P = U_1$, $Q = U_2$, $R = U_3$) for which there is $y = \psi(\alpha) \in \mathrm{int}\, M_3 \subset$ $\subset M_3 \subset R \subset M_1 \cap M_2$ such that $\varrho_{PR}(a_1) = \varrho_{QR}(a_2)$. By 1.4.7 we can for F_i find $\tilde{F}_i \in \mathscr{F}\ \mathrm{gr}_{M_3} a_i(U_i, a_i, M_3)$ such that $\tilde{F}_i \subset F_i$, $i = 1, 2$. Because $\mathrm{gr}_{M_3} a_1 = \mathrm{gr}_{M_3} a_2$, it is possible following 1.2.17 to find $F \in \mathscr{F}\ \mathrm{gr}_{M_3} a_2(U_2, a_2, M_3) \subset L_2$ such that $F \subset \tilde{F}_1 \cap \tilde{F}_2 \subset F_1 \cap F_2 \subset F_1$. The proof is completed.

1.4.14. Corollary. All the bases $L(\alpha; U, a, K)$ can briefly denoted by $L(\alpha)$. We can use that $L(\alpha; U, a, K)$ which seems to be most convenient. This we will do in the following.

1.4.15. Definition. The closure formed by the bases $L(\alpha)$ in P will be denoted by t^+.

1.4.16. Corollary. *Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}^U_a$, $\mathrm{int}\, K \neq \emptyset$. If $\alpha \in \mathrm{gr}_K a$, $\psi(\alpha) \in$ $\in \mathrm{int}\, K$, then every set $F \in \mathscr{F}\ \mathrm{gr}_K a$ is a t^+-neighborhood of α.*

Proof. By 1.4.14 we have $F \in \mathscr{F}\ \mathrm{gr}_K a(U, a, K) \subset L(\alpha)$.

1.4.17. Theorem. *The closure t^+ in P is a generating one.*

Proof. A. $\Delta(\mathrm{gr}_K a; t^+) \leqq \mathscr{F}\ \mathrm{gr}_K a$: Let $F \in \mathscr{F}\ \mathrm{gr}_K a$ where $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}^U_a$. By 1.4.6B we can find $M \in \mathscr{K}^U_a$ such that $K \subset \mathrm{int}\, M$. If $F = S(R(U, a, K, W^V))$, then $\tilde{F} = S(R(U, a, M, W^V)) \in \mathscr{F}\ \mathrm{gr}_M a$ and by 1.4.16 we get $\tilde{F} \in (\mathrm{gr}_K a; t^+)$, because $\tilde{F} \in \Delta(\alpha; t^+)$ for any $\alpha \in \mathrm{gr}_K a$. Here we have $\tilde{F} \cap \psi^{-1}(K) \subset F$, which proves A.

B. $\mathscr{F}\ \mathrm{gr}_K a \leqq \Delta(\mathrm{gr}_K a; t^+)$: Let $O \in \Delta(\mathrm{gr}_K a; t^+)$. For any $\alpha \in \mathrm{gr}_K a$ we can find $O_\alpha \in \Delta(\alpha; t^+)$ such that (a) $\bigcup\{O_\alpha \mid \alpha \in \mathrm{gr}_K a\} \subset O$, (b) by 1.4.15 there exists a set $M_x \in \mathscr{K}^U_a$ such that $x = \psi(\alpha) \in \mathrm{int}\, M_x$ and $O_\alpha = F_\alpha \in \mathscr{F}\ \mathrm{gr}_{M_x} a$. Because the family $\{\mathrm{int}\, M_x \mid x \in K\}$ covers K, its finite subfamily covers K, too. For the sake of simplicity let us assume that it consists of only two sets $M_i = M_{x_i}$, $i = 1, 2$. Let us write $F_{\alpha_i} =$ $= F_i$, where $\psi(\alpha_i) = x_i$, $i = 1, 2$. Thus $K \subset M_1 \cup M_2 \subset U$ and $\mathrm{gr}_K a \subset F_1 \cup$ $\cup F_2 \subset O$. Here we have $F_i \in \mathscr{F}\ \mathrm{gr}_{M_i} a$, $i = 1, 2$. By 1.2.17, $F_i = S(R(U, a, M_i, W_i^V))$, $i = 1, 2$. Let us set $F' = S(R(U, a, M_1 \cup M_2, W_1^V \cap W_V^2))$. By 1.4.7 we can find $\tilde{F} \in \mathscr{F}\ \mathrm{gr}_K a$ such that $\tilde{F} \subset F' \cap \psi^{-1}(K)$. Because of $F' \cap \psi^{-1}(K) \subset (F_1 \cup$ $\cup F_2) \cap \psi^{-1}(K)$, we get $\tilde{F} \subset (F_1 \cup F_2) \cap \psi^{-1}(K) \subset O \cap \psi^{-1}(K)$ which proves B.

By 1.4.4 the condition 1.4.7 is necessary for the existence of a generating closure. But if we have moreover 1.4.5, we can form the closure t^+ from 1.4.15, which is a generating one. Thus we get

1.4.18. Proposition. *Let* 1.4.5 *hold. Then the condition* 1.4.7 *is necessary and sufficient for the existence of a generating closure. If it is fulfilled,* t^+ *is a generating closure.*

1.4.2 implies:

1.4.19. Proposition. t^+ *is seminormal.*

5. Appendix

The bases $\mathscr{F}\ \mathrm{gr}_K\ a$ from 1.2.18 have with respect to 1.2.13 rather complicated form, which can be simplified for some presheaves.

Let (K, M, U, a) be from 1.2.11. Let us set

$$C(M; U, a) = \{\xi_{UK}(W) \mid W \in \Delta(a, \tau_U)\}\ . \tag{1.5.1}$$

1.5.2. Definition. A presheaf will be called simple with respect to the given cofiltration $\varkappa = \{\mathscr{K}_a^U \mid U \in \mathscr{B}(X), a \in S_U\}$ if the following holds: "If (K, M, U, a), (K, M, V, b) are from 1.2.11, then $C(M; U, a) \sim C(M; V, b)$."

In case of simple presheaves we may write simply C_M or $C\ \mathrm{gr}_K\ a$ instead of $C(M, U, a)$. If the presheaf is simple, it is clear that $C\ \mathrm{gr}_K\ a \sim \mathscr{F}\ \mathrm{gr}_K\ a$ for all $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$. For the simple presheaf the all objects constructed from the bases $\mathscr{F}\ \mathrm{gr}_K\ a$ will be simpler, namely the closure k, Theorem 1.3.1 etc. For example, 1.3.1 for simple presheaves is as follows:

1.5.3. Theorem. *Let* $\mathscr{S}$ *be a simple presheaf. All the maps* $(p_U^k)^{-1}$ *are continuous iff the following holds: "If* $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, *then there exists* $K \in \mathscr{K}_a^U$ *and* $W' \in \Delta(a; \tau_U)$ *such that* $\xi_{UK}^{-1}\xi_{UK}(W') \subset p_U(W)$."

1.5.4. Remark. Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a subpresheaf of a presheaf $\mathscr{S}' = \{(S'_U, \tau'_U); \varrho'_{UV}; X\}$, i.e. $S_U \subset S'_U$, $\varrho_{UV} = \varrho'_{UV}/S_U$, $\tau_U = \mathrm{ind}_{S_U} \tau'_U$ for $U, V \in \mathscr{B}(X)$. Let us denote by P' resp. P the covering space of $\mathscr{S}'$ resp. of $\mathscr{S}$. Then we have $P \subset P'$. If t' is a closure in P', then $t = \mathrm{ind}_P\, t$ is a closure in P. Let q be a method which enables us from the closure t' to construct a closure $q(t')$ in the sets A'_U and let the all natural maps $(S'_U, \tau'_U) \to (A'_U, q(t'))$ be homeomorphisms. Let $q(t)$ be a closure in A_U constructed from t with use of q.

If q has the property

$$\mathrm{ind}_{A_U}\, q(t') = q(t) \tag{1.5.5}$$

i.e. the diagram

$$(1.5.6)\qquad \begin{array}{ccc} t' & \longrightarrow & t \\ \downarrow & & \downarrow \\ q(t') & \longrightarrow & q(t) \end{array}$$

is commutative, where the horizontal arrows are the corresponding restrictions and the vertical arrows represent the formation of the closure $q(t)$ and $q(t')$, then t is normal. Thus from the normal closure t' for P' we get a normal closure t for P. If we succed in embedding of the presheaf in question into the presheaf for which there exists a normal closure, then if 1.5.5 holds, then there exists a normal closure for P. For example the closure $t = \text{ind}_P\, t'$ is normal.

CHAPTER 2

UNIFORM CONVERGENCE ON COMPACT SETS

The symbols $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$, P, t, ψ, A_U, p_U have the same meaning as in Chapter 1. The results of Chapter 1 are used here in the case, where for every $U \in \mathscr{B}(X)$ and $a \in S_U$ $\mathscr{K}_a^U$ is the cofilter of all compact sets in U. The special way of choosing of the cofiltration allows us to derive a series of further relations which we did not study in Chapter 1.

1. Introduction of notions and existence of a normal closure

For every $U \in \mathscr{B}(X)$ and every $a \in S_U$ let

$$(2.1.1)\qquad \mathscr{K}_a^U = \{K \mid K \subset U \text{ compact}\}\,.$$

Then $\mathscr{K}_a^U$ satisfy 1.3.3. Following 1.1.4–7 we introduce the closure of uniform convergence on the cofiltrations 2.1.1, in all A_U.

2.1.2. Definition. This closure will be called closure of uniform convergence on compact sets and denoted by $l(t)$. The closure t in P will be called normal, similarly as in 1.1.3, if all the maps

$$(2.1.3)\qquad p_U^t : (S_U, \tau_U) \to (A_U, l(t))$$

are homeomorphisms. We study again the existence of a normal closure. For this sake we use the results developed in Chap. 1.

Following 1.2.8–18 we may construct for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_U^a$, (i.e. $K \subset U$ compact) the filter base $\mathscr{F}\ \text{gr}_K\, a$. Then 1.2.19 gives

2.1.4. Proposition. *All the maps p_U^t are continuous iff for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact there is $\mathscr{F}$ $\mathrm{gr}_K\, a \leqq \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$.*

Following 1.2.20–22 we may construct the closure k from the bases $\mathscr{F}$ $\mathrm{gr}_K\, a$. With respect to the special choice of cofilters $\mathscr{K}_a^U$ we will denote it in this chapter by the letter l. The closure l depends on the collection μ from which it was constructed. Therefore, when necessary, we will denote it by l_μ. The closure t in P is called seminormal, if $l(t) = l$ in every A_U.

The propositions 1.2.25–27 have the form

2.1.5. Proposition. A. *All the maps p_U^l are continuous.*

B. *All the maps p_U^t are continuous iff $l \leqq l(t)$ in all A_U.*

2.1.6. Proposition. *Let for the closure t in P all the $(p_U^t)^{-1}$ be continuous. Then in every A_U $l(t) \leqq l$.*

The conditions 1.2.29–32 have the form

2.1.7. Theorem. *Every normal closure is seminormal. Thus a necessary condition for the existence of a normal closure is the existence of a seminormal closure t in P.*

2.1.8. Theorem. *A necessary condition for the existence of a normal closure in P is the continuity of all $(p_U^l)^{-1}$.*

2.1.9. Theorem. A. *Let all the $(p_U^l)^{-1}$ be continuous. Then there exists a normal closure t in P iff there exists a seminormal closure t_1 in P. If this condition is satisfied, then t_1 is normal.*

B. *If there exists a seminormal closure t_1 in P, then there exists a normal closure in P iff all the $(p_U^l)^{-1}$ are continuous. If this condition is satisfied, then t_1 is normal.*

2.1.10. Theorem. *The necessary and sufficient condition for the existence of a normal closure in P is the simultaneous validity of both necessary conditions 2.1.7, 8.*

So that we could decide whether there exists a normal closure in P, we must study the simultaneous validity of 2.1.7, 8. In the following paragraph we study the second condition.

2. Continuity of $(p_U^l)^{-1}$

Theorem 1.3.1 has the form

2.2.1. Theorem. *All the the $(p_U^l)^{-1}$ are continuous iff the following condition holds: "If $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, then there exist compact sets $K_1, \ldots, K_n \subset U$ and $F_i \in \mathscr{F}$ $\mathrm{gr}_{K_i}\, a$, $i = 1, \ldots, n$ such that*

$$(2.2.2) \qquad \bigcap_{i=1}^{n} \xi_{UK_i}^{-1}(F_i) \subset p_U(W)\,."$$

Following 1.3.5, 6 we can construct for our collection $\mu = \{\tau_U\}$ its modification μ^0 which is again a collection, $\mu \leqq \mu^0$. By 1.3.8 we get

2.2.3. Theorem. *Let $\mathcal{S}$ be a presheaf over X, μ and ν its closure collections, $\mu \leqq \nu$. Let all the $(p_U^{l_\nu})^{-1}$ be continuous. Then we have $\mu^0 \leqq \nu$.*

By 1.3.11 we get

2.2.4. Theorem. (a) *$l_{\mu^0} = l_\mu$ in all A_U,*

(b) *the all $p_U^{l_0}$ are continuous. (We have put $l_{\mu^0} = l_0$.)*

Because 2.2.3, 4 hold for μ^0, we will call it cofilter modification of the collection μ with respect to the cofiltration $\varkappa$ from 2.1.1. Then from 1.3.11 we get.

2.2.5. Corollary. *The collection μ^0 is the finest of the all collections ν which are coarser than μ and for which the all $(p_U^{l_\nu})^{-1}$ are continuous. These three properties are equivalent:*

(a) *The all $(p_U^{l_\mu})^{-1}$ are continuous,*

(b) *$\mu = \mu^0$,*

(c) *the collection μ satisfies the condition of Theorem 2.2.1.*

2.2.6. Proposition. *Let $\mathcal{S}$ be a full presheaf* (see [8], 1.1.46) *over a locally compact space X with the projective closure collection* (see [8], 1.1.4). *Then the all $(p_U^l)^{-1}$ are continuous.*

Proof. Let $U \in \mathcal{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$. By [8], 1.1.44 we have $\varrho_{UV}^{-1}(W^V) \subset W$ for some $V \in \mathcal{B}(U)$, and some $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$, where $\overline{V} \subset U$ is compact. Further we have $\xi_{VV}^{-1}\xi_{VV}(W') \subset W^V$ for some $W' \in \Delta(\varrho_{UV}(a); \tau_V)$. For $V' \in \mathrm{B}(\overline{V}; U)$ let us set $W^{V'} = \varrho_{V'V}^{-1}(W')$. Hence $F = S(R(U, a, \overline{V}, W^{V'})) \in \mathcal{F}\ \mathrm{gr}_{\overline{V}}\, a$ and $\tilde{\xi}_{UV}^{-1}(F) \subset \tilde{\xi}_{UV}^{-1}(F \cap \cap\, \psi^{-1}(V)) \subset \tilde{\xi}_{UV}^{-1}\xi_{VV}(W') \subset \tilde{\rho}_{UV}^{-1}\tilde{\xi}_{VV}^{-1}\xi_{VV}(W') \subset \tilde{\rho}_{UV}^{-1}p_V\xi_{VV}^{-1}\xi_{VV}(w') \subset \tilde{\rho}_{UV}^{-1}p_V(W^V) = p_U\varrho_{UV}^{-1}(W^V) \subset p_U(W)$, which finishes the proof.

From 1.3.14 we get also one sufficient condition for the equality $\mu = \mu^0$:

2.2.7. Proposition. *Let all the (S_U, τ_U) be compact topological spaces, all the (A_U, l) Hausdorff topological spaces. Then we have: $\mu = \mu^0$.*

3. Seminormal closure

Here we study when the necessary condition 2.1.7 for the existence of a normal closure holds, i.e. when there exists a seminormal closure in P. By 1.4.1, 2 every generating closure is seminormal and therefore we try to find a generating closure in P. 1.4.4 gives

2.3.1. Proposition. *For the existence of a generating closure in P the following condition is necessary: "If $U \in \mathscr{B}(X)$, $a \in S_U$ and if $K \subset L \subset U$ are compact, then $\mathscr{F}\ \mathrm{gr}_K\ a \sim \mathscr{F}\ \mathrm{gr}_L\ a \cap \psi^{-1}(K)$."*

So that we could construct the generating closure t^+ from 1.4.14, the conditions 1.4.5, 7 must be satisfied. Because our cofilters are of the form 2.1.1, the condition 1.4.5a is satisfied and 1.4.5b requires local compactness of X. 1.4.7 gives

2.3.2. Assumption. Let the following hold up to the end of the paragraph: "Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset L \subset U$ compact. Then $\mathscr{F}\ \mathrm{gr}_K\ a \sim \mathscr{F}\ \mathrm{gr}_L\ a \cap \psi^{-1}(K)$."

Thus if X is locally compact and 2.2.2 holds, we may construct following 1.4.8 – 15 the closure t^+ which is by 1.4.17 the generating one. Thus the condition in 2.3.2, which is by 2.3.1 necessary for the existence of a generating closure, is also sufficient in the case of a locally compact X. Thus we have (by 1.4.18)

2.3.3. Proposition. *Let X be locally compact. Then the condition* 2.3.2 *is necessary and sufficient for the existence of a generating closure. If it is fulfilled, t^+ is the generating closure. By* 1.4.19 *we see that t^+ is seminormal.*

4. The importance of assumption 2.3.2.

The special form of cofilters 2.1.1, which we are studying here, allows us without difficulties to show some further relations, which we did not study in the previous chapter.

2.4.1. Proposition. *Let X be locally compact and let* 2.3.2 *hold. Let $\alpha \in P$, $U, U' \in \mathscr{B}(X)$, let $\overline{U} \subset U'$ be compact, $a' \in S_{U'}$, $y = \psi(\alpha) \in U$, $a = \varrho_{U'U}(a')$ a generating element for α* (see 0.18). *Then the filter bases*

$$(2.4.2)\quad \tilde{\mathscr{B}}(\alpha; U, a) = \{\xi_{UV}(W) \mid V \in \mathrm{B}(y; U),\ W \in \Delta(a; \tau_U)\},$$

$$(2.4.3)\quad \mathscr{D}(\alpha; U, a) = \{F \cap \psi^{-1}(\overline{M}) \mid F \in \mathscr{F}\ \mathrm{gr}_U\ a,\ M \in \mathrm{B}(y; U),\ \overline{M} \subset U\ compact\},$$

$$(2.4.4)\quad \mathscr{B}'(\alpha; U, a) = \{F \cap \psi^{-1}(M) \mid F \in \mathscr{F}\ \mathrm{gr}_U\ a,\ M \in \mathrm{B}(y; U),\ \overline{M} \subset U\ compact\},$$

are equivalent with the filter base $L(\alpha; U', a', \overline{U})$ of t^+-neighborhoods of α (see 1.4.11 – 14).

Proof. From the local compactness of X follows immediately the equivalence of $\mathscr{D}(\alpha; U, a)$ and $\mathscr{B}'(\alpha; U, a)$. Because 2.3.2 is satisfied, we get easily that $\mathscr{D}(\alpha; U, a)$ and $L(\alpha; U', a', \overline{U})$ are equivalent. We are going to prove the equivalence of $\mathscr{B}'(\alpha; U, a)$ and $\tilde{\mathscr{B}}(\alpha; U, a)$.

1. Let $F \cap \psi^{-1}(M) \in \mathscr{B}'(\alpha; U, a)$. Then $F \in \mathscr{F}\ \mathrm{gr}_{\bar{U}}\ a$ and $\bar{M} \subset U$. With respect to the form of the sets $\bar{F} \in \mathscr{F}\ \mathrm{gr}_{\bar{M}}\ a$ from 1.2.13, there exists (by 2.3.2) $W \in \Delta(a; \tau_U)$ such that $\xi_{U\bar{M}}(W) \subset F \cap \psi^{-1}(\bar{M})$. Hence $\xi_{UM}(W) \subset F \cap \psi^{-1}(M)$ and $\xi_{UM}(W) \in \tilde{\mathscr{B}}(\alpha; U, a)$.

2. Let $\xi_{UV}(W) \in \tilde{\mathscr{B}}(\alpha; U, a)$. For $\tilde{V} \in \mathrm{B}(\bar{U}; U')$ let us set $W^{\tilde{V}} = \varrho_{\tilde{V}V}^{-1}(W)$. Then $F = S(R(U', a', \bar{U}, W^{\tilde{V}})) \in \mathscr{F}\ \mathrm{gr}_{\bar{U}}\ a'$ and $F \cap \psi^{-1}(V) \subset \xi_{UV}(W)$. Moreover $F \cap \psi^{-1}(V) \in \mathscr{B}'(\alpha; U, a)$.

2.4.2. Corollary. *Let X be locally compact and* 2.3.2 *hold. Then the filter base of t^+-neighborhoods of α is formed by any base from* 2.4.1. *For our purposes we can choose that, which is most convenient for us.*

2.4.3. Corollary. *Let X be locally compact and* 2.3.2 *hold. Assertion*: "*Let $\alpha \in P$, $U, V \in \mathscr{B}(X)$, let $a \in S_U$, $b \in S_V$ be generating elements for α and let $U' \in \mathscr{B}(U)$, $V' \in \mathscr{B}(V)$, $y = \psi(\alpha) \in U' \cap V'$ where $\bar{U}' \subset U$, $\bar{V}' \subset V$ are compact, $W \in \Delta(\varrho_{UU'}(a); \tau_{U'})$ and $U'' \in \mathrm{B}(y; U')$.*" *Then there exists $W' \in \Delta(\varrho_{VV'}(b); \tau_{V'})$ and $V'' \in \mathrm{B}(y; U'')$ such that $\xi_{V'V''}(W') \subset \xi_{U'U''}(W)$.*

Proof. Under this assumption we have $\xi_{U'U''}(W) \in \tilde{\mathscr{B}}(\alpha; U', a')$, where $a' = \varrho_{UU'}(a)$. By 2.4.1 this base is equivalent to the base $\tilde{\mathscr{B}}(\alpha; V', b')$ where $b' = \varrho_{VV'}(b)$. And this is our corollary.

2.4.4. Proposition. *Let X be locally compact and let the assertion from* 2.4.3 *hold. Then* 2.3.2 *holds.*

Proof. Let $U' \in \mathscr{B}(X)$, $a' \in S_{U'}$, $K \subset L \subset U'$ compact. There exists $U \in \mathrm{B}(L; U')$ such that $\bar{U} \subset U'$ is compact. Let $a = \varrho_{U'U}(a')$, $F = S(R(U, a, L, W^V)) \in \mathscr{F}\ \mathrm{gr}_L\ a$. For all $V \in \mathrm{B}(L; U)$ let us set $\tilde{W}^V = W^V$. If $V \in \mathrm{B}(K; U) - \mathrm{B}(L; U)$, we can find for the neighborhood W^U and for $\alpha \in \mathrm{gr}_K\ a$ (by the assertion 2.4.3) the sets $W_\alpha \in \Delta(\varrho_{UV}(a); \tau_V)$ and $V_\alpha \in \mathrm{B}(\psi(\alpha); V)$, such that $\xi_{VV_\alpha}(W_\alpha) \subset \xi_{UV_\alpha}(W^U)$. Then there is $K \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$ where $\alpha_1, \ldots, \alpha_n \in \mathrm{gr}_K\ a$. Let us set $\tilde{W}^V = \bigcap_{i=1}^{n} \varrho_{UV_i}^{-1}(W_{\alpha_i})$, where we have put $V_i = V_{\alpha_i}$, $i = 1, \ldots, n$. Then we have $\xi_{VK}(\tilde{W}^V) \subset \xi_{UK}(W^U) = \xi_{UK}(W^U) \cap \psi^{-1}(K) \subset F \cap \psi^{-1}(K)$. For $F' = S(R(U, a, K, \tilde{W}^V)) \in \mathscr{F}\ \mathrm{gr}_K\ a$ we have $F' \subset F \cap \psi^{-1}(K)$.

2.4.5. Corollary. *Let X be locally compact. Then the assertion* 2.4.3 *and the assumption* 2.3.2 *are equivalent.*

5. Generating closure for X not locally compact

If X is locally compact and the assertion 2.4.3 holds, we can construct the generating closure t^+, since 2.3.2 holds. The property included in 2.4.3 can be extended to the spaces which are not locally compact.

For $\alpha \in P$, $U \in \mathscr{B}(X)$, $a \in S_U$ generating for α let

$$\mathscr{H}(\alpha; U, a) = \{\xi_{UM}(W) \mid W \in \Delta(a; \tau_U),\ M \in \mathrm{B}(\psi(\alpha); U)\}\,. \tag{2.5.1}$$

2.5.2. Definition. The presheaf $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ is called topologically trivial (briefly t.t.), if all the bases $\mathscr{H}(\alpha; U, a)$ are equivalent for all $\alpha \in P$.

2.5.3. Remark. The topological triviality is the strengthening of the property from the assertion 2.4.3. If the presheaf is t.t., the bases $\mathscr{H}(\alpha; U, a)$ can be denoted by $\mathscr{H}(\alpha)$ and we may always use the most convenient one.

2.5.4. Notation. For the 4-tuple (K, M, U, a) from 1.2.11 let

$$\mathscr{G}(M; U, a) = \{\xi_{UK}(W) \mid W \in \Delta(a; \tau_U)\}\,. \tag{2.5.5}$$

2.5.6. Proposition. *Let the presheaf be t.t. If $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact, then $\mathscr{G} = \mathscr{G}(M; U, a) \sim \mathscr{F}\, \mathrm{gr}_K\, a$. Thus the presheaf is simple with respect to the compact cofiltration* (see 1.5.2).

Proof. Clearly $\mathscr{G} \leqq \mathscr{F}\, \mathrm{gr}_K\, a$. Now, let $G = \xi_{UK}(W) \in \mathscr{G}$. If $V \in \mathrm{B}(K; U)$ we can for every $x \in K$ find (as a consequence of t.t.) $V_x \in \mathrm{B}(x; V)$ and $W_x \in \Delta(\varrho_{UV}(a); \tau_V)$ such that $\xi_{VV_x}(W_x) \subset \xi_{VV_x}(W)$. Therefore $K \subset V_{x_1} \cup \ldots \cup V_{x_n}$. For $W^V = \bigcap_{i=1}^{n} W_{x_i}$ we have $\xi_{VK}(W^V) \subset \xi_{UK}(W)$. Let us form W^V for every $V \in \mathrm{B}(K; U)$ in this way. Then $S = S(R(U, a, K, W^V)) \in \mathscr{F}\, \mathrm{gr}_K\, a$ and $S \subset \xi_{UK}(W)$.

2.5.7. Corollary. *If (K, M, U, a) is from* 1.2.11 *and if the presheaf is t.t., then all the bases* 2.5.5 *are equivalent.*

Thus we may denote them only by $\mathscr{G}_M$. Where we need the bases $\mathscr{F}\, \mathrm{gr}_K\, a$, we may use the equivalent bases $\mathscr{G}\, \mathrm{gr}_K\, a$ if it is convenient for us. So, for example, the filter base $\Delta(a(x); l)$ is of the form

$$\{\tilde{\xi}_{UK}^{-1}(\xi_{UK}(W)) \mid K \subset U \text{ compact},\ W \in \Delta(a; \tau_U)\}\,, \tag{2.5.8}$$

instead of more complicated form described in 1.2.20. For example, Theorem 2.2.1 has the more clear form

2.5.9. Theorem. *Let the presheaf be t.t.. The all $(p_U^l)^{-1}$ are continuous iff the following holds: "If $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, then there exists $K \subset U$ compact and $W' \in \Delta(a; \tau_U)$ such that $\tilde{\xi}_{UK}^{-1}\xi_{UK}(W') \subset p_U(W)$."*

The advantage of t.t. presheaves consist in the fact that the relations are more easy to survey and can be more easily verified. Moreover we can construct for them the generating closure without the assumption of local compactness of X.

For t.t. presheaves the bases $\mathscr{H}(\alpha)$ from 2.5.1 generate a closure in P.

2.5.10. Notation. The closure generated in P by the bases $\mathscr{H}(\alpha)$ will be denoted by t_1^+.

When X was locally compact and 2.3.2 held, we constructed the closure t^+ following 1.4.14. On the other hand, the closure t_1^+ was constructed for t.t. presheaves. If moreover the presheaf is t.t., X is locally compact and 2.3.2 holds, then the bases $\mathscr{H}(\alpha)$ from 2.5.1 and the bases $\overline{\mathscr{B}}(\alpha; U, a)$ from 2.4.2 coincide. Hence (by 2.4.1 and 2.5.10) $t^+ = t_1^+$. Therefore we may the closure from 2.5.10 denote by t^+ and we will really do so. Condition 2.5.10 is only the extension of the definition of the closure t^+ from 1.4.14 to presheaves over the spaces, which are not locally compact. So that we could realise this extension, the presheaf must be t.t.

2.5.11. Theorem. *Let the presheaf be t.t. Then the closure t^+ from* 2.5.10 *is a generating one.*

Proof. We use 2.5.6. Let $G = \xi_{UK}(W) \in \mathscr{G} \operatorname{gr}_K a$. By 2.5.10, 11 we have $\xi_{UU}(W) \in \in \Delta(\operatorname{gr}_K a; t^+)$. Here we have $G = \xi_{UU}(W) \cap \psi^{-1}(K)$. Thus $\Delta(\operatorname{gr}_K a; t^+) \cap \psi^{-1}(K) \leqq \leqq \mathscr{G} \operatorname{gr}_K a$. Conversely, let $O \in \Delta(\operatorname{gr}_K a; t^+)$. By 2.5.10 we can assume that $O = = U\{\xi_{UM_\alpha}(W_\alpha) \mid \alpha \in \operatorname{gr}_K a\}$, where $M_\alpha \in \mathrm{B}(\psi(\alpha); U)$ and $W_\alpha \in \Delta(a; \tau_U)$ for every $\alpha \in \operatorname{gr}_K a$. The compactness on K implies that only finite number of the sets M_α covers K. For convenience we may assume that they are only two. Let $M_{\alpha_i} = M_i$, $W_{\alpha_i} = W_i$, $X_i = \psi(\alpha_i)$, $i = 1, 2$. Then $G = \xi_{UK}(W_1 \cap W_2) \subset (\xi_{UM_1}(W_1 \cap W_2) \cup \cup\, \xi_{UM_\alpha}(W_1 \cap W_2)) \cap \psi^{-1}(K) \subset O \cap \psi^{-1}(K)$ and $G \in \mathscr{G} \operatorname{gr}_K a$.

2.5.12. Remark. The topological triviality of $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ can be localised as follows: $\mathscr{S}$ is locally t.t. (briefly l.t.t.) if the following holds: "If $\alpha \in P$, $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact, $y = \psi(\alpha) \in K$, $a(y) = \alpha$, then there exists $U' \in \mathrm{B}(K; U)$ such that for every $V, V' \in \mathrm{B}(y; U')$ the bases $\mathscr{H}(\alpha; V, \varrho_{UV}(a))$, $\mathscr{H}(\alpha; V', \varrho_{UV'}(a))$ from 2.5.1 are equivalent. If $\mathscr{S}$ is l.t.t., we can define as in 2.5.10 the closure t_1^+ and easily (similarly as in the case when $\mathscr{S}$ is t.t.) prove, that 2.5.6, 11 hold.

We can easily see that if X is locally compact and 2.3.2 holds, then $\mathscr{S}$ is l.t.t. (see 2.4.1, 2). Thus 2.5.10 for the l.t.t. presheaves is an extension of 1.4.15.

6. Natural closure

For every $U \in \mathscr{B}(X)$ there is a natural map $\delta_U : S_U \times U \to P$ defined as follows: $a \in S_U$, $y \in U \Rightarrow \delta_U(a, y) = a(y)$ (see 0.11). Then we have the family of maps

$$(2.6.1) \qquad \Delta = \{\delta_U \mid U \in \mathscr{B}(X)\}$$

of the sets $S_U \times U$ into P. We can define a natural closure in P with respect to the set Δ, namely the finest of all closures t, for which all the maps $\delta_U \in \Delta$, $\delta_U : (S_U \times U,$

$\tau_U \times u) \to (P, t)$ are continuous (u is the closure in U induced by the closure from X), i.e. $\varinjlim_{\Delta} \tau_U \times u$.

2.6.2. Definition. The closure $\varinjlim_{\Delta} \tau_U \times u$ will be denoted by t^* and called natural closure.

Let us notice how the sets $\Delta(\alpha; t^*)$ look like.

2.6.3. Agreement. Let $\alpha \in P$, $U \in \mathscr{B}(X)$, $y = \psi(\alpha) \in U$ and let $a \in S_U$ be a generating element for α. Every map φ_{Uy} of the set $\mathrm{B}(y; U)$ into itself such that for $V \in \mathrm{B}(y; U)$ we have $\varphi_{Uy}(V) \subset V$ will be called choice. (In 1.2.8 something else was called choice, but there is not danger of confusion, because the both choices are clearly different.)

Let $\alpha \in P$, $U \in \mathscr{B}(X)$, $y = \psi(\alpha) \in U$, $a \in S_U$ a generating element for α. If $R(U, a, y, W^V)$ is a chain from 1.2.10 and φ_{Uy} a choice from 2.6.3, let us form

$$(2.6.4) \qquad S'(R(U, a, y, W^V), \varphi_{Uy}) = \bigcup_{V \in \mathrm{B}(y;U)} \xi_{V_{\varphi}(V)}(W^V)$$

(we put $\varphi_{U_y} = \varphi$)

$$(2.6.5) \qquad \mathscr{B}(\alpha; U, a) = \{S'(R(U, a, y, W^V), \varphi_{Uy}) \mid R(U, a, y, W^V) \text{ is a chain from } U \text{ to } y,\ \varphi_{Uy} \text{ is a choice}\}.$$

2.6.5. Proposition. *Let $\alpha \in P$, $U_1, U_2 \in \mathscr{B}(X)$, $a_1 \in S_{U_1}$, $a_2 \in S_{U_2}$, $y = \varphi(\alpha) \in U_1 \cap \cap U_2$, $a_1(y) = a_2(y) = \alpha$. Then the sets $\mathscr{B}_1 = \mathscr{B}(\alpha; U_1, a_1)$ and $\mathscr{B}_2 = \mathscr{B}(\alpha; U_2, a_2)$ are equivalent filter-bases in P round α.*

Proof. Let $S_1 = S'(R(U_1, a_1, y, W_1^V), \varphi_{U_1y}) \in \mathscr{B}_1$. Then we have $\varrho_{U_1U_3}(a_1) = \varrho_{U_2U_3}(a_2)$ for some $U_3 \in \mathrm{B}(y; U_1 \cap U_2)$. For $V \in \mathrm{B}(y; U_2)$ let us set $W_2^V = \overset{-1}{\varrho}_{VV \cap U_3}(W_1^{V \cap U_3})$ and $\varphi_{U_2y}(V) = \varphi_{U_1y}(V \cap U_3)$. Hence $S'(R(U_2, a_2, y, W_2^V), \varphi_{U_2y}) = S_2 \in \mathscr{B}_2$ and $S_2 \subset S_1$.

2.6.7. Corrolary. *Instead of $\mathscr{B}(\alpha; U, a)$ we may write simply $\mathscr{B}(\alpha)$ and use always the more convenient base $\mathscr{B}(\alpha; U, a)$.*

2.6.8. Corollary. *The bases $\mathscr{B}(\alpha)$ form a closure in P. From the form of these bases we can easily see, that they are filter-bases of t^*-neighborhoods of the elements α.*

2.6.9. Proposition. *Let X be locally compact and let 2.3.2 hold. If t^* is the closure from 1.4.14, then $t^+ = t^*$.*

Proof. By 2.4.1, 2 we get from 2.6.4 that $t^+ \leqq t^*$, because $\tilde{\mathscr{B}}(\alpha; U, a)$ from 2.4.2 is finer than $\mathscr{B}(\alpha)$ from 2.6.5. Now, we show (using 2.4.1), that $\mathscr{B}(\alpha)$ is finer than $\mathscr{B}'(\alpha; U, a)$ from 2.4.4. Let $F \cap \psi^{-1}(V) \in \mathscr{B}'(\alpha; U, a)$. For every $V' \in \mathrm{B}(\psi(\alpha); V)$

we can find $\varphi(V') \in \mathrm{B}(\psi(\alpha); V)$ such that $\overline{\varphi(V')} \subset V'$ is compact. By 2.3.2 we have $\xi_{V'\overline{\varphi(V')}}(W^{V'}) \subset F \cap \psi^{-1}(\overline{\varphi(V')})$ for some $W^{V'} \in \Delta(\varrho_{UV'}(a); \tau_{V'})$. Thus we get $\xi_{V'\varphi(V')}(W^{V'}) \subset F \cap \psi^{-1}(\varphi(V))$. Hence $S' = S'(R(U, a, \psi(\alpha), W^{V'}), \varphi) \subset F \cap \psi^{-1}(V)$ and $S' \in \Delta(\alpha; t^*)$, which proves the proposition.

2.6.10. Proposition. *Let the presheaf be t.t.* (see 2.5.2), *let t^+ be the closure from* 2.5.10. *Then $t^+ = t^*$.*

Proof. It follows immediately from the t.t. of $\mathscr{S}$ that the bases $\mathscr{H}(\alpha)$ from 2.5.1 and $\Delta(\alpha; t^*)$ from 2.6.5 are equivalent. Further $\mathscr{H}(\alpha)$ is a base of the filter of t^+-neighborhoods of α.

2.6.9, 10 can be generalised as follows: *Let $\mathscr{S}$ be l.t.t.* (see 2.5.12), *t^+ the closure from* 2.5.10 (see 2.5.12). *Then $t^* = t^+$.*

The proof follows easily from 2.5.12 and 2.4.1.

2.6.11. Corollary. We can see the importance of the natural closure t^*. The closures t^+ from 1.7.14 and 2.5.10 coincide with t^*.

2.6.12. Proposition. *All the maps $p_U^{t^*}$ are continuous.*

Proof. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact, $O \in \Delta(\mathrm{gr}_K\, a, t^*) \cap \psi^{-1}(K)$. We can assume that $O = \bigcup \{O_\alpha \cap \psi^{-1}(K) \mid \alpha \in \mathrm{gr}_K\, a\}$, where for every $\alpha \in \mathrm{gr}_K\, a$ there is $O_\alpha \in \Delta(\alpha; t^*)$. By 2.6.4 for every $\alpha \in \mathrm{gr}_K\, a$ there exist $V_\alpha \in \mathrm{B}(\psi(\alpha); U)$ and $W_\alpha \in \Delta(a; \tau_U)$ such that $\xi_{UV_\alpha}(W) \subset O_\alpha$. Then we have $K \subset V_{\alpha_1} \cap \ldots \cap V_{\alpha_n}$. For $W = \bigcap_{i=1}^{n} W_{\alpha_i}$ we get $\xi_{UK}(W) \subset O$, which finishes the proof.

2.6.13. Proposition. *Let $\mathscr{S}$ be a presheaf over a regular space X. Let t be a closure in P such that for every $U \in \mathscr{B}(X)$ and every compact $K \subset U$ we have $\mathscr{F}\, \mathrm{gr}_K\, a \leqq \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$. Then $\Delta(\mathrm{gr}_K\, a; t^*) \cap \psi^{-1}(K) \leqq \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$.*

Proof. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact, $O \cap \psi^{-1}(K) \in \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$. It suffices to prove: If $\alpha \in \mathrm{gr}_K\, a$, then for some $S \in \Delta(\alpha; t^*)$ we have $S \cap \psi^{-1}(K) \subset O \cap \psi^{-1}(K)$. Let $\alpha \in \mathrm{gr}_K\, a$ and $F = S(R(U, a, K, W^V)) \in \mathscr{F}\, \mathrm{gr}_K\, a$ such that $F \subset O \cap \psi^{-1}(K)$. For $V \in \mathrm{B}(K; U)$ let us set $\tilde{W}^V = W^V$ and $\varphi(V) = V$. For $V \in \mathrm{B}(\psi(\alpha); U) - \mathrm{B}(K; U)$ we can find $\varphi(V) \in \mathrm{B}(\psi(\alpha); U)$ such that $\overline{\varphi(V)} \subset V$. Then $\overline{\varphi(V)} \cap K = L$ is a compact subset in K. Beeause $\mathscr{F}\, \mathrm{gr}_L\, a \leq \Delta(\mathrm{gr}_L\, a; t)$, we have $F_1 \subset O \cap \psi^{-1}(L)$ for some $F_1 = S(R(U, a, L, \tilde{W}^{V_1})) \in \mathscr{F}\, \mathrm{gr}_L\, a$, and thus $\xi_{VL}(\tilde{W}^V) \subset F_1$. Hence $S'(R(U, a, \psi(\alpha), \tilde{W}^V), \varphi) = S \in \Delta(\alpha; t^*)$ and $S \cap \psi^{-1}(K) \subset O \cap \psi^{-1}(K)$, which finishes the proof.

2.6.14. Theorem. *Let $\mathscr{S}$ be a presheaf over a regular space X. If there exists a generating closure in P, then t^* is a generating one.*

Proof. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact. By 2.1.4, 2.6.12, we have $\mathscr{F}\,\mathrm{gr}_K\, a \leqq \Delta(\mathrm{gr}_K\, a; t^*) \cap \psi^{-1}(K)$. If t is a generating closure, then we get by 2.6.13 $\Delta(\mathrm{gr}_K\, a; t^*) \cap \psi^{-1}(K) \leqq \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K) \leqq \mathscr{F}\,\mathrm{gr}_K\, a$, which finishes the proof.

2.6.15. Theorem. *Let $\mathscr{S}$ be a presheaf over a regular space X. If there exists a seminormal closure in P, then t^* is seminormal.*

Proof. By 2.6.12 and 2.1.4 we have $l \leqq l(t^*)$ in all A_U. If t is seminormal, then by 2.1.4, 5 $\mathscr{F}\,\mathrm{gr}_K\, a \leqq \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$, where $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact. By 2.6.13 we have $\Delta(\mathrm{gr}_K\, a; t^*) \cap \psi^{-1}(K) \leqq \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$, hence in every A_U we have $l(t^*) \leqq l(t) = l$. Thus $l(t^*) = l$.

2.6.16. Theorem. *Let $\mathscr{S}$ be a presheaf over a regular space X. If there exists a normal closure in P, then t^* is normal.*

Proof. Let t be a normal closure. By 1.2.29 in every A_U we have $l(t) = l$. By 2.6.15 in every A_U we have also $l(t^*) = l$ and thus t^* is normal.

7. Sufficient conditions for the normality of t^*

2.7.1. Theorem. *Let $\mathscr{S}$ be a full presheaf over a locally compact X* (see [8], 1.1.46), *with a projective closure collection $\mu = \{\tau_U\}$* (see [8], 1.1.4) *for which* 2.3.2 *holds. Then t^* is normal.*

Proof. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$. Then for some $V \in \mathscr{B}(U)$ (where $\overline{V} \subset U$ is compact) and for some $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$ we get $\varrho_{UV}^{-1}(W^V) \subset W$ (see 1.1.45 in [8]). Further we have $\xi_{VV}^{-1}\xi_{VV}(W') \subset W^V$ for some $W' \in \Delta(\varrho_{UV}(a); \tau_V)$. Let us choose $U' \in \mathrm{B}(\overline{V}; U)$ such that $\overline{U}' \subset U$ is compact. We set $\tilde{W} = \varrho_{U'V}^{-1}(W')$. By 2.4.1, 2 and 2.6.9 we have $O = \xi_{U'U'}(\tilde{W}) \in \Delta(\mathrm{gr}_{\overline{V}}\, a; t^*)$. Then $\tilde{\xi}_{UV}^{-1}(O \cap \psi^{-1}(\overline{V})) \subset \tilde{\xi}_{UV}^{-1}(O \cap \cap \psi^{-1}(V)) \subset \tilde{\xi}_{UV}^{-1}(\xi_{VV}(W')) \subset \tilde{\varrho}_{UV}^{-1}\xi_{VV}^{-1}\xi_{VV}(W') \subset \tilde{\varrho}_{UV}^{-1}(W^V) \subset p_U(W)$.

Now we will construct one closure t^1 in P, which will be used in the next. From now we assume X is locally connected.

Let $\alpha \in P$, $U \in \mathscr{B}(X)$, $a \in S_U$, $y = \psi(\alpha) \in U$, $a(y) = \alpha$. Let us set

(2.7.2) $\mathscr{M}(\alpha; U, a) = \{S^*(R(U', \varrho_{UU'}(a), y, W^V)) = \bigcup\{\xi_{VV}(W^V) \mid V \in \mathrm{B}(y; U'),$ connected$\} \mid U' \in \mathrm{B}(y; U)$, $R(U', \varrho_{UU'}(a), y, W^V)$ is a chain from U' to y round $\varrho_{UU'}(a)\}$.

2.7.3. Proposition. *Let $U, V \in \mathscr{B}(X)$, $y = \psi(\alpha) \in U \cap V$, $a \in S_U$, $b \in S_V$, $a(y) = b(y) = \alpha$. Then $\mathscr{M}(\alpha; U, a) \sim \mathscr{M}(\alpha; V, b)$. Thus we may denote these bases briefly by $\mathscr{M}(\alpha)$. These bases form a closure in P which is coarses than t^*. We will denote it by t^1.*

Proof. Let $M = S^*(R(U', \varrho_{UU'}(a), y, W^V)) \in \mathscr{M}(\alpha; U, a)$. We can find $\tilde{V} \in$ $\in \mathrm{B}(y; U' \cap V)$ such that $\varrho_{U\tilde{V}}(a) = \varrho_{V\tilde{V}}(b)$. For connected $V' \in \mathrm{B}(y; \tilde{V})$ let us set $W_1^{V'} = W^{V'}$. Then $M_1 = S^*(R(\tilde{V}, \varrho_{V\tilde{V}}(b), y, W_1^{V'})) = \bigcup \{\xi_{V'V'}(W_1^{V'}) \mid V' \in \mathrm{B}(y; \tilde{V}), V'$ connected$\} \in \mathscr{M}(\alpha; V, b)$ and we have $M_1 \subset M$, which finishes the proof of the equivalence. The rest follows by comparing 2.7.2 and 2.6.3–5.

2.7.4. Agreements and notations. For $U \in \mathscr{B}(X)$ and $K \subset U$ compact let us denote by Π_K^0 the set of all finite open covers of K by open connected sets in U. If $\mathscr{V} \in \Pi_K^0$, then every map $\sigma_{\mathscr{V}} : \mathscr{V} \to \prod_{V \in \mathscr{V}} \Delta(\varrho_{UV}(a); \tau_V)$ of $\mathscr{V}$ into the cartesian product of the sets $\Delta(\varrho_{UV}(a); \tau_V)$, such that for every $V \in \mathscr{V}$ we have $\sigma_{\mathscr{V}}(V) \in \Delta(\varrho_{UV}(a); \tau_V)$ will be called choice. If $\sigma_{\mathscr{V}}$ is a choice, then

$$B(K; \mathscr{V}, W_{\mathscr{V}}^V) = \{W_{\mathscr{V}}^V \mid V \in \mathscr{V},\ W_{\mathscr{V}}^V = \sigma_{\mathscr{V}}(V)\}\,, \tag{2.7.5}$$

will be called envelope. Further, let us set

$$S(B(K; \mathscr{V}, W_{\mathscr{V}}^V)) = \bigcup\{\xi_{VV}(W_{\mathscr{V}}^V) \mid V \in \mathscr{V}\}\,, \tag{2.7.6}$$

$$\mathscr{B}\,\mathrm{gr}_K\, a = \{\bigcup_{\mathscr{V} \in \Pi_K{}^0} S(B(K; \mathscr{V}, W_{\mathscr{V}}^V)) \mid B(K; \mathscr{V}, W_{\mathscr{V}}^V) \text{ is some envelope, } \mathscr{V} \in \Pi_K^0\}\,. \tag{2.7.7}$$

It is clear that $\mathscr{B}\,\mathrm{gr}_K\, a$ is a filter base round $\mathrm{gr}_K\, a$ in P.

2.7.8. Lemma. *Let* $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ *compact. Then* $\Delta(\mathrm{gr}_K\, a; t^1) \cap$ $\cap\, \psi^{-1}(K) \leqq \mathscr{B}\,\mathrm{gr}_K\, a \cap \psi^{-1}(K)$.

Proof. Let $B = \bigcup\{S(B(K; \mathscr{V}, W_{\mathscr{V}}^V) \mid \mathscr{V} \in \Pi_K^0\} \in \mathscr{B}\,\mathrm{gr}_K\, a$. Let us choose a cover $\mathscr{V} = \{U_x \mid x \in K\}$ of K such that for every $x \in K$ there is $U_x \in \mathrm{B}(x; U)$. Let $x \in K$, $V_x \in \mathrm{B}(x, U_x)$ connected. Then there exists $\mathscr{V} \in \Pi_K^0$ such that $V_x \in \mathscr{V}$. Let us set $\tilde{W}^x = W_{\mathscr{V}}^{V_x}$. For every $x \in K$ and every connected $V_x \in \mathrm{B}(x; U_x)$ let us form this $\tilde{W}^x \in \Delta(\varrho_{UV_x}(a); \tau_{V_x})$. Then we have $O = \bigcup_{x \in K} \bigcup_{V_x} \{\xi_{V_xV_x}(\tilde{W}^x) \mid x \in V_x \subset U_x, V_x$ connected$\} \in \Delta(\mathrm{gr}_K\, a; t^1)$ and we have $O \subset B$.

2.7.9. Theorem. *Let* $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ *be a presheaf with the unique continuation over a locally compact space* X *with a projective closure collection* $\mu =$ $= \{\tau_U\}$ (see 0.32 and 1.1.4 in [8]). *Then* t^* *is normal.*

Proof. We will prove that the natural maps $(p_U^{t^1})^{-1} : (A_U, l(t^1)) \to (S_U, \tau_U)$ are continuous. Because (by 2.7.3) there is $t^* \leqq t^1$, we have $l(t^*) \leqq l(t^1)$ and thus all the maps $(p_U^{t^*})^{-1}$ are continuous. By 2.6.12 t^* is normal.

Let $U \in \mathscr{B}(X)$ be connected, $a \in S_U$, $W \in \Delta(a; \tau_U)$. By [8], 1.2.1–4 there exists a compact set $M(W) = K \subset U$ such that if $V \in B(K; U)$, then for some $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$ we have

(2.7.10) $$\varrho_{UV}^{-1}(W^V) \subset W.$$

For $\mathscr{V} \in \Pi_K^0$ let us set $V_{\mathscr{V}} = \bigcup\{V | V \in \mathscr{V}\}$. Then $\varrho_{UV_{\mathscr{V}}}^{-1}(W^{V_{\mathscr{V}}}) \subset W$ for some $W^{V_{\mathscr{V}}} \in \Delta(\varrho_{UV_{\mathscr{V}}}(a); \tau_{V_{\mathscr{V}}})$. Further, for every $V \in \mathscr{V}$ there exists $W_{\mathscr{V}}^V \in \Delta(\varrho_{UV}(a); \tau_V)$ such that $W^{V_{\mathscr{V}}} \supset \bigcap_{V \in \mathscr{V}} \varrho_{V_{\mathscr{V}}V}^{-1}(W_{\mathscr{V}}^V)$ (see 0.32 and 1.1.4 in [8]). For these $W_{\mathscr{V}}^V$, $V \in \mathscr{V}$ let us take the envelope $\mathscr{B}_{\mathscr{V}} = B(K; \mathscr{V}, W_{\mathscr{V}}^V)$ (see 2.7.4–5). If we form such $\mathscr{B}_{\mathscr{V}}$ for every $\mathscr{V} \in \Pi_K^0$, then we have

(2.7.11) $$B = \bigcup_{\mathscr{V} \in \Pi_K^0} S(\mathscr{B}_{\mathscr{V}}) = \bigcup_{\mathscr{V} \in \Pi^0_K} \bigcup_{V \in \mathscr{V}} \xi_{VV}(W_{\mathscr{V}}^V) \in \mathscr{B} \operatorname{gr}_K a$$

and (by 2.7.8) we have $I = \xi_{UK}^{-1}(B \cap \psi^{-1}(K)) \in \Delta(a(x); l(t^1))$. Let $b(x) \in I$. Then for every $y \in K$ there exists (by 2.7.11) $\mathscr{V}_y \in \Pi_K^0$ and $V_y \in \mathscr{V}_y$ such that $\xi_{Uy}(b) = b(y) \in \xi_{V_yV_y}(W^{V_y})$. Let $\{Vy_1, \ldots, Vy_n\} = \mathscr{V} \in \Pi_K^0$, $\mathscr{V} \subset \{V_y \mid y \in K\}$. For $i = 1, \ldots, n$ let us write $V_{y_i} = V_i$, $W^{V_i} = W^i$. Then for $i = 1, \ldots, n$ we have $\varrho_{UV_i}(b) \in W^i$ (V_i is connected). Putting $V_{\mathscr{V}} = \bigcup_{i=1}^n V_i$, we get $\varrho_{UV_{\mathscr{V}}}(b) \in \bigcap_{i=1}^n \varrho_{V_{\mathscr{V}}V_i}^{-1}(W^i) \subset W^{V_{\mathscr{V}}}$. Thus $b \in \varrho_{UV_{\mathscr{V}}}^{-1}(W^{V_{\mathscr{V}}}) \subset W$, which implies $\xi_{UK}^{-1}(B \cap \psi^{-1}(K)) \subset W$. This proves the continuity of the all maps $(p_U^{t_1})^{-1}$ for connected U. Let $U \in \mathscr{B}(X)$ be arbitrary. For every component V of U let us notice the following commutative diagram:

(2.7.12) $$\begin{array}{ccc} (S_U, \tau_U) & \xleftarrow{(p_U{}^{t^1})^{-1}} & (A_U, l(t^1)) \\ \downarrow{\scriptstyle \varrho_{UV}} & & \downarrow{\scriptstyle \tilde{\varrho}_{UV}} \\ (S_V, \tau_V) & \xleftarrow{(p_V{}^{t^1})^{-1}} & (A_V, l(t^1)) \end{array}$$

The projectivity of $\mu = \{\tau_U\}$ implies that $(p_U^{t^1})^{-1}$ is continuous iff $\varrho_{UV} \circ (p_U^{t^1})^{-1}$ is continuous for every component V of U. By 2.7.12 this map coincides with the map $(p_V^{t^1})^{-1} \circ \tilde{\varrho}_{UV}$. By the first part of the proof, the map $(p_V^{t^1})^{-1}$ is continuous, and the continuity of $\tilde{\varrho}_{UV}$ is obvious (see 1.1.12–14). Theorem is proved.

CHAPTER 3

POINTWISE CONVERGENCE

The results from Chap. 1 will be used here in the case, when for every $U \in \mathscr{B}(X)$, $a \in S_U$, $\mathscr{K}_a^U$ is cofilter of all finite sets in U.

1. Existence of the normal closure

For every $U \in \mathscr{B}(X)$, $a \in S_U$ let

(3.1.1) $$\mathscr{K}_a^U = \{K \mid K \subset U \text{ is finite}\} .$$

Then $\mathscr{K}_a^U$ satisfies 1.3.3. Following 1.1.4–7 we introduce in all A_U the closure of uniform couvergence on the cofilters 3.1.1.

3.1.2. Definition. This closure will be called closure of pointwise convergence and denoted by $b(t)$. The closure t in P will be called normal (in accordance with 1.1.3) if the all natural maps

(3.1.3) $$p_U^t : (S_U, \tau_U) \to (A_U, b(t))$$

are homeomorsphisms.

We study again the existence of the normal closure. For this purpose we use the tools from chap. 1. Similarly as in 1.2.8–18 we form for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$ the bases $\mathscr{F}$ $\text{gr}_K\, a$. For $K = \{x\}$ we write here $\mathscr{F}_x(a)$ instead of $\mathscr{F}$ $\text{gr}_x\, a$. Following 1.2.20–22 we construct in every A_U the closure k from the bases $\mathscr{F}_x(a)$ The closure k will be denoted in this chapter by b in accordance to the special choice of cofiltration.*) The natural map $(S_U, \tau_U) \to (A_U, b)$ will be denoted by p_U^b. The closure t in P will be called seminormal, if $b(t) = b$ in every A_U. The reader can easily rephrase the propositions 1.2.19, 251–27 and therefore we will not present them here. From 1.2.29, 30, 32 we get

3.1.4. Theorem. *Every normal closure is seminormal. Thus a necessary condition for the existence of a normal closure in P is the existence of a seminormal closure.*

3.1.5. Theorem. *A necessary condition for the existence of a normal closure in P is the continuity of all* $(p_U^b)^{-1}$.

3.1.6. Theorem. *The necessary and sufficient condition for the existence of a normal closure in P is the simultaneous validity of both necessary conditions* 3.1.4, 5.

Thus we must study these conditions. First of all we take the first one.

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*) Using bases $\mathscr{F}_x(a)$, ($\mathscr{F}$ $\text{gr}_K\, a$), we get by 1.2.20–22 a filter-base $\{\bigcup_{x \in K} \tilde{\xi}_{Ux}^{-1}(F_x);\ K \subset U$ is finite, $F_x \in \mathscr{F}_x(a)$ for $x \in K\}$, resp. $\{\tilde{\xi}_{UK}^{-1}(F);\ K \subset U$ finite, $F \in \mathscr{F}$ $\text{gr}_K\, a\}$. These bases generate a closure b', resp. b. We have only $\mathscr{F}$ $\text{gr}_K\, a \cap \psi^{-1}(x) \leqq \mathscr{F}_x(a)$, thus $b \leqq b'$. The equality is not generally true. If t is normal, then $b' \leqq b(t)$ and $b \leqq b' \leqq b(t) \leqq b$, thus $b' = b$. To be precise, in 3.2.1 we do not prove the existence of a seminormal closure t for which $b(t) = b$, but the existence of a pseudonormal one from 4.1.10 (see the next chapter), for which $b(t) = b'$. 3.1.1 is a condition for the continuity of $(p_U^{b'})^{-1} : (A_U, b') \to (S_U, \tau_U)$. Theorem 3.3.3 is correct if the modification μ° concerns the bases $\mathscr{F}_x(a)$ and the map $p_U^{b\mu}$ concerns the closure b'_U. 3.4.6, 3.4.10, 3.4.11 are true for both b, b'. These questions are studied in the next chapter.

2. Seminormal closure

Here we study the existence of a seminormal closure in P. The conditions 1.4.1, 2 say, that every generating closure is seminormal. The special form of cofilters $\mathscr{K}_a^U$ allows us without using further results from Chap. 1 to prove directly

3.2.1. Theorem. *The natural closure t^* is a generating and therefore a seminormal one.*

Proof. Let $\alpha \in P$, $U \in \mathscr{B}(X)$, $y = \psi(\alpha) \in U$, $a \in S_U$ a generating element for α, $F = S(R(U, a, y, W^V)) = \bigcup\{\xi_{Vy}(W^V) \mid V \in \mathrm{B}(y; U)\} \in \mathscr{F}_y(a)$, let φ_{Uy} be any choice from 2.6.3. Then (we put $\varphi_{Uy} = \varphi$)

$$(3.2.2) \qquad S' = S'(R \mid U, a, y, W^V), \varphi_{Uy}) = \bigcup\{\xi_{\varphi(V)}(W^V) \mid V \in \mathrm{B}(y; U\} \in \Delta(\alpha; t^*)\,,$$

hence $S' \cap \psi^{-1}(y) \subset F$. Thus $\Delta(\alpha; t^*) \cap \psi^{-1}(y) \leqq \mathscr{F}_y(a)$. Conversely, let S' be an element from $\Delta(\alpha; t^*)$ of the form 3.2.2. Then $S' \cap \psi^{-1}(y) \in \Delta(\alpha; t^*) \cap \psi^{-1}(y)$. Let us set $F = S(R(U, a, y, W^V))$. Then $F \in \mathscr{F}_y(a)$ and $F = S' \cap \psi^{-1}(y)$ which finishes the proof.

Now, 3.1.5 implies

3.2.3. Theorem. *The necessary and sufficient condition for the existence of a normal closure is the continuity of all $(p_U^b)^{-1}$. If this condition holds, then t^* is normal.*

3.2.4. Remark. In every stalk $\psi^{-1}(x)$ let us take the closure $u_x = \mathrm{ind}_{\psi^{-1}(x)}\, t^*$. Let $\tilde{t}$ be the closure in P, which is the topological sum of the closures u_x. Under the closure $\tilde{t}$ any two distinct stalks are separated, for every stalk is a clopen (closed and open) set in $(P, \tilde{t})$. Thus there is not $A_U \subset \Gamma(U, \tilde{t})$ (see 0.34). It can be easily seen that $\tilde{t}$ is the finest closure in P, for which the all p_U^t are continuous.

3.2.5. Notation. Let t_s be the sheaf topology in P (see 0.34). Let us denote by $\hat{t}$ the topological sum of the closures $\tilde{t}$ and t_s (see 0.36). Then it is clear that for all $x \in X$

$$(3.2.6) \qquad \mathrm{ind}_{\psi^{-1}(x)}\, \hat{t} = \mathrm{ind}_{\psi^{-1}(x)}\, t^*\,.$$

It follows from 1.1.9 that $b(t^*) = b(\hat{t})$ in all A_U. And 3.2.3 implies

3.2.7. Theorem. *If there exists a normal closure in P, then $\hat{t}$ is normal.*

3.2.8. Proposition. *The finest closure in the set of all closures t in P, for which every p_U^t is continuous and simultaneously $A_U \subset \Gamma(U, t)$ for all $U \in \mathscr{B}(X)$, is the closure $\hat{t}$.*

3. Continuity of $(p_U^b)^{-1}$

2.3.1 gets the form

3.3.1. Theorem. *All the* $(p_U^b)^{-1}$ *are continuous iff the following condition holds: "If* $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, *then there exist* $x_1, \ldots, x_n \in U$ *and* $F_i \in \mathscr{F}_{x_i}(a)$ *such that*

$$\bigcap_{i=1}^{n} \xi_{Ux_i}^{-1}(F_i) \subset p_U(W) \,.\text{"} \tag{3.3.2}$$

Following 1.3.5–6 we will form to our collection μ its modification μ^0.

3.3.3. Theorem. *The collection* μ^0 *is the finest one in the set of all collections* ν *which are coarser than* μ *and for which the all* $(p_U^{b_\nu})^{-1}$ *are continuous. The following four properties are equivalent:*

(a) *The all* $(p_U^{b_\mu})^{-1}$ *are continuous,*
(b) $\mu = \mu^0$,
(c) *for the collection* μ *the condition from* 3.3.1 *holds,*
(d) *there exists a normal closure in P.*

Proof. a, b, c follow from the assertion of 1.3.13, (d) follows from 3.2.3.

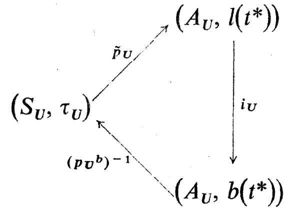
3.3.4. Definition. The collection μ^0 (i.e. the modification of μ with respect to the cofilters 3.1.1) will be called pointwise modification of μ.

3.3.5. Proposition. *Let all the* (S_U, τ_U) *be compact topological spaces, all the* (A_U, b) *Hausdorff topological spaces. Then* $\mu = \mu^0$.

3.3.6. Remark. For the presheaf which is simple with respect to the cofilters 3.1.1 (see 1.5.2) Theorem 3.3.1 has the simple form. From the special form of $\mathscr{K}_a^U$ we can see that the simplicity with respect to this cofilters could be localised for example as follows: "If $U \in \mathscr{B}(X)$, $a \in S_U$, $y \in U$, then there exists $V \in \mathrm{B}(y; U)$ such that the filter bases $\{\xi_{Vy}(W) \mid W \in \Delta(\varrho_{UV}(a); \tau_V)\}$ and $\mathscr{F}_y(a)$ are equivalent." The form of 3.3.1 is sufficiently clear in this case.

3.3.7. Remark. For the natural closure t^* and $l(t^*)$ from 2.1.2 we mention the commutative diagram

(3.3.8)



By 2.6.12 the all $\tilde{p}_U$ are continuous. Because the cofilter of all compact sets contains the cofilter of all finite sets, every i_U is continuous. If $\mu = \mu^0$ (μ^0 from 3.3.4), by 3.2.3 every $(p_U^b)^{-1}$ is continuous. Hence the all three spaces in 3.3.8 are homeomorphic, thus $l(t^*) = b(t^*)$ in all A_U and all the maps $p_U : (S_U, \tau_U) \to (A_U, l(t^*))$ are homeomorphisms. Similarly, if q is any method of constructing of a closure $q(t)$ in A_U from t, such that $l(t^*) \leqq q(t^*)$ in every A_U and if all the natural maps $(A_U, q(t^*)) \to (S_U, \tau_U)$ are continuous, then $q(t^*) = l(t^*)$ in all A_U and all the maps $p_U : (S_U, \tau_U) \to (A_U, l(t^*))$ are homeomorphisms.

4. Examples

Theorem 3.3.1 solves the problem of the existence of a normal closure. In 3.4.1 we show, that sometimes it is possible to verify the continuity of all $(p_U^b)^{-1}$ without the verification of 3.3.1. The expamles 3.4.11, 12 refer to 1.1.9, Example 3.4.13 refers to 1.4.3.

3.4.1. Example. Let the presheaf $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ have the property

(3.4.2) If $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, $\mathscr{V} \in \Pi_U^0$ (see 0.27) then there exists $V \in \mathscr{V}$ such that $\varrho_{UV}(W) \in \Delta(\varrho_{UV}(a), \tau_V)$.

Then we have: If $U \in \mathscr{B}(X)$, $\overline{U}$ compact, $a \in S_U$, $W \in \Delta(a; \tau_U)$, then there exists a point $y \in \overline{U}$ which has the filter base $\Delta(y)$ of open neighborhoods such that $V \in \Delta(y) \Rightarrow \varrho_{UV \cap U}(W) \in \Delta(\varrho_{UV \cap U}(a); \tau_{V \cap U})$.

Proof. Easy, the negation leads to contradiction with 3.4.2.

3.4.3. Corollary. *If* 3.4.2 *holds and if we have moreover*

(3.4.4) "*For* $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$ *there exists* $V \in \mathscr{B}(U)$, $\overline{V} \subset U$ *such that* $\varrho_{UV}(W) \in \Delta(\varrho_{UV}(a); \tau_V)$,"

then even exists $y \in U$ *satisfying the assertion of* 3.4.1.

For example, it is in the case, when X is regular and 3.4.2 *holds for any* $\mathscr{V} \in \Pi_U$.

3.4.5. Definition. Let $U \in \mathscr{B}(X)$, $a \in S_U$. The base $\Delta(a; \tau_U)$ will be called saturated if the following holds: "If $W \in \Delta(a; \tau_U)$, then there exists $W' \in \Delta(a; \tau_U)$ such that $\xi_{Ux}^{-1}\xi_{Ux}(W') \subset W$ for all $x \in U$."

In the presheaf with the unique continuation (see 0.32), then $\Delta(a; \tau_U)$ is saturated for any connected $U \in \mathscr{B}(X)$ and any $a \in S_U$.

3.4.6. Proposition. *Let* $\mathscr{S}$ *satisfy* 3.4.2, 4 *and let* $\Delta(a; \tau_U)$ *be saturated for any* $U \in \mathscr{B}(X)$, $a \in S_U$. *Then the all* $(p_U^b)^{-1}$ *are continuous.*

Proof. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$. For W we can find $W' \in \Delta(a; \tau_U)$ such that W and W' satisfy 3.4.5. For W' we can find $y \in U$ and $\Delta(y)$ which are mentioned in 3.4.1. If $V \in \mathrm{B}(y; U)$, let us set $W^V = \varrho_{VV'}^{-1} \varrho_{VV'}(W')$, where V' is an element from $\Delta(y) \cap \mathrm{B}(y; V)$. Then for $F = S(R(U, a, y, W^V)) \in \mathscr{F}_y(a)$ we have $F \subset \xi_{Uy}(W')$. Hence $\xi_{Uy}^{-1}(F) \subset \xi_{Uy}^{-1U}\xi_{Uy}(W') \subset p_U(W)$, which finishes the proof.

3.4.7. Remark. It can happen that 3.4.2, 4 hold, but $\mathscr{S}$ has not any saturated base $\Delta(a; \tau_U)$. Even then every $(p_U^b)^{-1}$ may be continuous. The condition 3.3.1 can be satisfied for the points $x_1, \ldots, x_n \in U$ which are all different from y (y from 3.4.1.).

3.4.8. Example. Let the presheaf $\mathscr{S}$ over a complete metric space X have this property:

(3.4.9) If $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, then there exists $U' \in \mathscr{B}(U)$ such that for every $V \in \mathscr{B}(U')$ there exists $W' \in \Delta(\varrho_{UV}(a); \tau_V)$ and $V' \in \mathscr{B}(V)$ such that $\varrho_{VV'}(W') \subset \varrho_{UV'}(W)$.

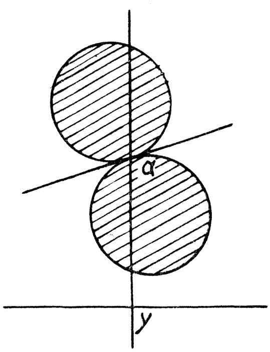
Then we have: If $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, then there exists $y \in U$ which has a base $\Delta(y)$ of open neighborhoods such that if $V \in \Delta(y)$, then there exists $W' \in \Delta(\varrho_{UV}(a); \tau_V)$ and $V'' \in \mathrm{B}(y; V)$ such that $\varrho_{VV''}(W') \subset \varrho_{UV''}(W)$.

Proof. Let $U \in \mathscr{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$. By the assumption we can choose $U' \in \mathscr{B}(U)$ as in 3.4.9. By induction we can construct (according to 3.4.9) the sequence $\{K_n\}$ of open balls round x_n with radius r_n and a sequence $\{V_n\}$ of open sets such that

(a) $U' = V_0 \supset \bar{K}_1 \supset K_1 \supset V_1 \supset \bar{K}_2 \supset K_2 \supset V_2 \supset \ldots$, $r_n \to 0$,

(b) if $n = 1, 2, \ldots$, then $\varrho_{K_nV_n}(W^n) \subset \varrho_{UV_n}(W)$ for some $W^n \in \Delta(\varrho_{UK_n}(a); \tau_{K_n})$.

Thus there exists the unique $y \in \bigcap_{n=1}^{\infty} \bar{K}_n$. It can be easily seen that y is the desired point



3.4.10. Remark. In the same way as in 3.4.6 we can find out, that if $\mathscr{S}$ satisfies 3.4.8 and has the saturated bases $\Delta(a; \tau_U)$, then the all $(p_U^b)^{-1}$ are continuous.

3.4.11. Example. Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; R\}$ (R is the real line) be a presheaf where S_U is the set of all continuous functions on $U \in \mathscr{B}(R)$ and τ_U is the topology of uniform convergence. In the covering space of $\mathscr{S}$ we take the discrete topology d. Then $b(d)$ in A_U is not discrete. The $(p_U^b)^{-1}$ are not continuous, nor p_U^b are continuous.

3.4.12. Example. Let $\mathscr{S}$ be the presheaf of all constant real functions over R. Then $P = R \times R$. If $\alpha \in P$, $y \in \psi(\alpha)$, then $\psi^{-1}(y)$ is the vertical line through α. Let t_2 be the Euclidean topology, let t_1 be the topology, a neighborhood of which is on Fig. 1.

Then $t_1 \leqq t_2$, $t_1 \neq t_2$ but both they induce the same Euclidean topology in the stalks.

3.4.13. Example. Let $\mathscr{S}$ be the same as in 3.4.12. Let us provide the sets S_U with the closure τ_U of uniform convergence. We introduce the topology t_1 in P as follows: If $x \neq 0$, $\alpha \in \psi^{-1}(x)$ let $\Delta(\alpha; t_1)$ be an ordinary Euclidean filter. For $\alpha \in \psi^{-1}(0)$ let $\Delta(\alpha; t_1) = \psi^{-1}(0)$. Let t_2 be the Euclidean topology. Then there is not $t_1 \sim t_2$, but nevertheless $b(t_1) = b(t_2)$ (see 1.1.9).

CHAPTER 4

ANOTHER METHOD FOR THE STUDY OF THE EXISTENCE OF A NORMAL CLOSURE

We get another method for the study of the existence of a normal closure. We use it for the study of uniform convergence. This method we compare with the method developed in Chapter 1.

1. Introduction of notions and the existence of a normal closure

4.1.1. Assumption. Let the assumption 1.3.3 for the cofiltration $\varkappa = \{\mathscr{K}^U_a \mid U \in \mathscr{B}(X),\ a \in S_U\}$ of the presheaf $\mathscr{S} = \{(S_U, \tau_U);\ \varrho_{UV};\ X\}$ hold, i.e. if $U, V \in \mathscr{B}(X)$, $V \subset U$, $a \in S_U$, then $\mathscr{K}^V_{\varrho_{UV}(a)} = \{K \mid K \in \mathscr{K}^U_a,\ K \subset V\}$.

Similarly as in 1.1.7 let us denote by $k(t)$ the closure of uniform convergence on $\varkappa$. Let p^t_U be the natural map $p^t_U : (S_U, \tau_U) \to (A_U, k(t))$.

Let the all p^t_U be continuous, $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}^U_a$. By 1.2.19 we have $\mathscr{F}\ \mathrm{gr}_K\, a \leqq \Delta(\mathrm{gr}_K\, a;\ t) \cap \psi^{-1}(K)$. Let

$$\mathscr{N}(K) = \{L \mid L \in \mathscr{K}^U_a,\ L \subset K\}\,. \tag{4.1.2}$$

The continuity of all p^U_a and 1.2.19 imply that for every $L \in \mathscr{N}(K)$

$$\mathscr{F}\ \mathrm{gr}_L\, a \leqq \Delta(\mathrm{gr}_L\, a;\ t) \cap \psi^{-1}(L) \leqq \Delta(\mathrm{gr}_K\, a;\ t) \cap \psi^{-1}(K)\,. \tag{4.1.3}$$

We can see the following: If we denote by

$$\mathscr{H}\ \mathrm{gr}_K\, a = \varliminf\ \{\mathscr{F}\ \mathrm{gr}_L\, a \mid L \in \mathscr{N}(K)\} \tag{4.1.4}$$

the finest base of the all filter-bases in $\psi^{-1}(K)$, which are coarser than every $\mathscr{F}\ \mathrm{gr}_L\, a$, $L \in \mathscr{N}(K)$, then we have obviously

$$\mathscr{H}\ \mathrm{gr}_K\, a \leqq \Delta(\mathrm{gr}_K\, a;\ t) \cap \psi^{-1}(K)\,. \tag{4.1.5}$$

Thus the continuity of all p_U^t implies 4.1.5 for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$. Obviously for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$

$$(4.1.6) \qquad \mathscr{F} \operatorname{gr}_K a \leqq \mathscr{H} \operatorname{gr}_K a .$$

By 1.1.19 we get from 4.1.6, that if for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$ 4.1.5 holds, then the all p_U^t are continuous. We have proved

4.1.7. Proposition. *All the maps p_U^t are continuous iff for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$ 4.1.5 holds.*

For $U \in \mathscr{B}(X)$, $a \in S_U$, $K \in \mathscr{K}_a^U$ let us set

$$(4.1.8) \qquad \mathscr{J}(a(x)) = \{\xi_{UK}^{-1}(H) \mid H \in \mathscr{H} \operatorname{gr}_K a\} ,$$

which is a filter base in A_U round $a(x)$.

4.1.9. Notation. The closures in the sets A_U formed by the bases $\mathscr{J}(a(x))$ will be denoted by m. The natural map $p_U : (S_U, \tau_U) \to (A_U, m)$ will be denoted by p_U^m. Obviously we have $k \leqq m$ in every A_U (see 1.2.22).

4.1.10. Definition. A closure t in P will be called pseudonormal, if $k(t) = m$ in all A_U.

In the quite obvious way (similarly as in 1.2.25–32) we could prove

4.1.11. Proposition. A. *All the maps p_U^m are continuous.*

B. *All the p_U^t are continuous iff $m \leqq k(t)$ in every A_U.*

C. *Let all the $(p_U^t)^{-1}$ be continuous. Then $k(t) \leqq m$ in every A_U.*

D. *If the closure t is normal, then it is pseudonormal.*

4.1.12. Theorem. *A necessary condition for the existence of a normal closure in P is the existence of a pseudonormal closure.*

4.1.13. Theorem. *A necessary condition for the existence of a normal closure in P is the continuity of all $(p_U^m)^{-1}$.*

4.1.14. Theorem. *The necessary and sufficient condition for the existence of a normal closure in P is the simultaneous validity of both necessary conditions* 4.1.12, 13.

4.1.15. Remark. Comparing 1.2.20–22 and 4.1.6–10 we can see, that $k \leqq m$ in every A_U. If t is normal then by 1.2.26 and 4.1.11 D $k(t) \leqq k \leqq m \leqq k(t)$ in every A_U and thus $k = m = k(t)$.

Now, we could deal with the continuity of $(p_U^m)^{-1}$ and in the same way as in Chap. 1, §3 construct for a collection $\mu = \{\tau_U\}$ its modification μ^1, for which we could get a similar proposition as in 1.3.11. Similarly as in Chap. 1, §4 we could construct under the assumption 1.4.5 a pseudonormal closure t in P. We will not do so, because the method has already been described in Chap. 1.

2. Pseudonormal closure in the case of compact cofiltration

We will notice which results we get in the case of compact cofiltration from 2.1.1 if we use the results from the previous paragraph. If we study the existence of a normal closure, we study following 4.1.14 the conditions 4.1.12, 13. Now, we are going to treat the first one.

4.2.1. Definition. A closure t in P will be called m-generating, if for every $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact, we have $\mathscr{H}\ \mathrm{gr}_K\, a \sim \Delta(\mathrm{gr}_K\, a; t) \cap \psi^{-1}(K)$.

Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a presheaf over a locally compact X. If $y \in U$, $V \in \mathrm{B}(y; U)$ then there exists $K \subset V$ compact, such that $y \in \mathrm{int}\, K$. As in 1.4.9 we denote

$$G(K, y) = \{L \mid y \in \mathrm{int}\, L \subset L \subset K,\ L \text{ is compact}\}\,. \tag{4.2.2}$$

Let $\alpha \in P$, $U \in \mathscr{B}(X)$, $K \subset U$ compact, $y = \psi(\alpha) \in \mathrm{int}\, K$, $a \in S_U$ a generating element for α (see 0.18). Let us set

$$T(\alpha; U, a, K) = \{H \mid H \in \mathscr{H}\ \mathrm{gr}_L\, a,\ L \in G(K, y)\}\,. \tag{4.2.3}$$

Let $U, V \in \mathscr{B}(X)$, $K \subset U$, $L \subset V$ compact, $y = \psi(\alpha) \in \mathrm{int}\, K \cap \mathrm{int}\, L$, $a \in S_U$, $b \in S_V$, $a(y) = b(y) = \alpha$ (see 0.18). Then we get easily from 1.2.17, 18 that $T(\alpha; U, a, K)$, $T(\alpha; V, b, L)$ are equivalent filter bases round α in P. Thus we write briefly $T(\alpha)$.

4.2.4. Notation. The closure in P generated by the bases $T(\alpha)$ will be denoted by $\bar{t}$.
The definition of the bases $\mathscr{H}\ \mathrm{gr}_K\, a$ in 4.1.4 implies easily that if $M, N \subset U$ are compact, $M \subset N$, $H \in \mathscr{H}\ \mathrm{gr}_K\, a$, then $H \cap \psi^{-1}(M) \in \mathscr{H}\ \mathrm{gr}_M\, a$. Conversely, for every $H_1 \in \mathscr{H}\ \mathrm{gr}_M\, a$ there exists $H \in \mathscr{H}\ \mathrm{gr}_N\, a$ such that $H \cap \psi^{-1}(M) \subset H_1$.

4.2.5. Proposition. *Let $U \in \mathscr{B}(X)$, $a \in S_U$, $K \subset U$ compact. Then $\mathscr{H}\ \mathrm{gr}_K\, a \sim \Delta(\mathrm{gr}_K\, a, \bar{t}) \cap \psi^{-1}(K)$. Thus $\bar{t}$ is a m-generating closure.*

Proof. Let $H \in \mathscr{H}\ \mathrm{gr}_K\, a$. There exists $L \subset U$ compact, such that $K \subset \mathrm{int}\, L$. Then by the definition of $\bar{t}$, every $H_1 \in \mathscr{H}\ \mathrm{gr}_L\, a$ is a $\bar{t}$-neighborhood of $\mathrm{gr}_K\, a$. We can find $H_1 \in \mathscr{H}\ \mathrm{gr}_L\, a$ such that $H_1 \cap \psi^{-1}(K) \subset H$. Because $H_1 \in \Delta(\mathrm{gr}_K\, a; \bar{t})$, we get $\Delta(\mathrm{gr}_K\, a; \bar{t}) \cap \psi^{-1}(K) \leqq \mathscr{H}\ \mathrm{gr}_K\, a$. Conversely, let $O \in \Delta(\mathrm{gr}_K\, a; \bar{t})$. By 4.2.1, 2 we can ssume, that $O = \bigcup\{O_\alpha \mid \alpha \in \mathrm{gr}_K\, a\}$, where $O_\alpha \in \mathscr{H}\ \mathrm{gr}_{L_\alpha}\, a$, $\psi(\alpha) \in \mathrm{int}\, L_\alpha$ and that

$L_\alpha \subset U$ is compact for every $\alpha \in \mathrm{gr}_K\, a$. It follows easily from 1.2.14–18 that for $M, N \subset U, M, N$ compact, $M \subset N$ we have $\mathscr{F}\, \mathrm{gr}_N\, a \cap \psi^{-1}(M) \leqq \mathscr{F}\, \mathrm{gr}_M\, a$. Let $L \subset K$ be compact. Let us choose a finite subcover $\{L_{\alpha_1}, \ldots, L_{\alpha_n}\}$ from the cover $\{L_\alpha \mid \alpha \in \mathrm{gr}_K\, a\}$. For every $i = 1, \ldots, n$ let $X_i = L \cap L_i$. Because of $X_i \subset L_i$, there exist $F_i \in \mathscr{F}\, \mathrm{gr}_{X_i}\, a$ such that $F_i \subset O_{\alpha_i}$, $i = 1, \ldots, n$. Then we can find $F_L \in \mathscr{F}\, \mathrm{gr}_L\, a$ such that $F_L \cap \psi^{-1}(X_i) \subset F_i$, $i = 1, \ldots, n$. For every compact set $L \subset K$ let us form such a set F_L. Then $\bigcup\{F_L \mid L \subset K \text{ compact}\} = H \in \mathscr{H}\, \mathrm{gr}_K\, a$ (by 4.1.4) and $H \subset O$. Thus $\mathscr{H}\, \mathrm{gr}_K\, a \leqq \Delta(\mathrm{gr}_K\, a; \bar{t}) \cap \psi^{-1}(K)$.

4.2.6. Corollary. *Let $\mathscr{S}$ be a presheaf over a locally compact space X. Then the necessary and sufficient condition for the existence of a normal closure in P is the continuity of all $(p_U^m)^{-1}$. If this condition holds, then $\bar{t}$ is normal and thus the natural closure t^* from* 2.6.2 *is normal.*

Proof. Follows from 4.1.14, 2.6.16.

4.2.7. Remark. Let $\mathscr{S}$ be a presheaf over a locally compact space X. The necessary and sufficient condition for the existence of a normal closure in case of compact cofiltration is the simultaneous validity of 4.1.12, 13 or 2.1.7, 8. The conditions 2.1.8 and 4.1.13 are similar. The first one requires the continuity of all $(p_U^l)^{-1} : (A_U, l) \to$ $\to (S_U, \tau_U)$, the second one the continuity of all $(p_U^m)^{-1} : (A_U, m) \to (S_U, \tau_U)$. Because by 4.1.9 there is $l \leqq m$, 4.1.13 requires more than 2.1.8 and thus 2.1.8 holds more often than 4.1.13. It can be expected than conversely 4.1.12 is weaker than 2.1.7. Indeed, we see that for the compact cofiltration and a locally compact X we can always construct a pseudonormal (even a m-generating) closure, whereas the seminormal closure t^+ from 1.4.15 we have constructed under the assumption 2.3.2. (More generally in the case when $\mathscr{S}$ is l.t.t. (see 2.5.12).)

3. Uniform convergence

We study the case where for $U \in \mathscr{B}(X)$, $a \in S_U$

$$\mathscr{K}_a^U = \{U\}\,. \tag{4.3.1}$$

Then the cofilters $\mathscr{K}_a^U$ satisfy 1.2.1. Following 1.1.4–7 we may introduce in every A_U the closure of uniform convergence on the cofilters 4.3.1.

4.3.2. Definition. This closure will be called closure of uniform convergence and denoted by $s(t)$. The closure t in P will be called normal if the all natural maps

$$p_U^t : (S_U, \tau_U) \to (A_U, s(t)) \tag{4.3.3}$$

are homeomorphisms.

Following 1.2.8 – 18 we may form the bases $\mathcal{F}\ \mathrm{gr}_K\ a$ for every $U \in \mathcal{B}(X)$, $a \in S_U$, $K \in \mathcal{K}^U_a$. According to 4.3.1 we denote them by $\mathcal{F}\ \mathrm{gr}_U\ a$. Following 1.2.2 – 22 (resp. 4.1.8, 9) we can form the closure k (resp. m) in every A_U. We will denote it here by s (resp. σ). The natural map $p_U : (S_U, \tau_U) \to (A_U, s)$, $((S_U, \tau_U) \to (A_U, \sigma))$ will be denoted by p^s_U, (p^δ_U). The closure t in P will be called seminormal (pseudonormal) if in all A_U we have $s(t) = s$, $(s(t) = \sigma)$. For the maps p^t_U, p^s_U, and p^δ_U Theorems 1.2.29 – 32 (resp. 4.1.12 – 14) hold. The form of these theorems in this case is clear and therefore we will not present them here.

With respect to 4.3.1 Theorem 1.3.1 is of the form

4.3.4. Theorem. *All the $(p^s_U)^{-1}$ are continuous iff the following condition holds: "If $U \in \mathcal{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, then for some $W' \in \Delta(a; \tau_U)$ there is*

$$\xi^{-1}_{UU}\xi_{UU}(W') \subset p_U(W)\,.\text{"} \tag{4.3.5}$$

Thus by 1.1.46 from the all $(p^s_U)^{-1}$ are continuous iff the presheaf is full. (See [8], 1.1.46).

Because the cofilters 4.1.1 do not satisfy 1.3.3, we can not further use the results from chapter 1.

4.3.6. Theorem. *All the $(p^m_U)^{-1}$ are continuous iff the following condition holds: "If $U \in \mathcal{B}(X)$, $a \in S_U$, $W \in \Delta(a; \tau_U)$, then for every $V \in \mathcal{B}(U)$ there exists $W^V \in \Delta(\varrho_{UV}(a); \tau_V)$ such that*

$$\xi^{-1}_{UU}\Big(\bigcup_{V \in \mathcal{B}(U)} \xi_{VV}(W^V)\Big) \subset p_U(W)\,.\text{"} \tag{4.3.7}$$

Proof. Easily follows (with respect to 4.3.1) from 4.1.4 – 9.

4.3.8. Remark. 4.3.6 is a much stronger condition for the existence of a normal closure than 4.3.4.

4. Pseudonormal closure

Here we study the condition 4.1.12 in the case of open cofiltration from 4.3.1. The pseudonormal closure will be constructed in a certain quite special case.

4.4.1. Definition. Let $U, V \in \mathcal{B}(X)$, $a \in S_U$, $b \in S_V$. The element b will be called continuation of a if $U \subset V$ and $\varrho_{VU}(b) = a$.

Using the maximality principle we get easily

4.4.2. Lemma. *Let $\mathcal{S}$ be a projective presheaf* (see 0.31), *$U \in \mathcal{B}(X)$ connected, $a \in S_U$. Then there exists a connected $V \in \mathcal{B}(X)$ and $b \in S_V$ such that b is a continuation of a. Moreover, if $V' \in \mathcal{B}(X)$ is connected and $b' \in S_{V'}$ is a continuation of b then $V = V'$, $b = b'$. Thus b is a maximal continuation of a on the connected set.*

Let $\mathscr{S}$ be a projective presheaf (see 0.31) over a locally connected space X, $\alpha \in P$. Then there exists a connected $U \in \mathscr{B}(X)$ and $a \in S_U$ such that a is a generating element for α. Every such a can be continued following 4.4.2 to a maximal one. For every α let $\mathscr{N}_\alpha$ be the set of all its maximal elements.

4.4.3. Assumption. Let $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ be a projective presheaf (see 0.35), X locally connected. Then for every $\alpha \in P$ the set $\mathscr{N}_\alpha$ is a single point.

If 4.4.3 holds, then every $\alpha \in P$ has the unique maximal generating element (on the connected set). No two maximal elements $b \in S_U$, $c \in S_{U'}$ form anywhere the same germ.

4.4.4. Definition. Let 4.4.3 hold, let $\alpha \in P$, $b \in \mathscr{N}_\alpha$, $b \in S_V$. Then the base $\mathscr{H}$ $\text{gr}_V\, b$ from 4.1.4 is a filter base in P round α. The closure generated in P by these bases will be denoted by t_0. All germs β of a maximal generating element have the same set $\Delta(\beta; t_0)$. The space (P, t_0) is not Hausdorff space except some very special cases.

4.4.5. Proposition. *Let* $\mathscr{S} = \{(S_U, \tau_U); \varrho_{UV}; X\}$ *be projective, X locally connected, let* 4.4.3 *hold. Then the closure* t_0 *is a m-generating closure. Therefore it is pseudo-normal. Thus the normal closure in P exists iff* 4.1.13 *holds. If it is satisfied, then* t_0 *is normal.*

Added in proof: After the paper was accepted for publication, the author was told by professor Frolík about the paper [10], where related problems are studied. That paper concerns the uniform fields of algebraic structures and the question of representation is studied here in a more natural and useful way than we have done in the first chapter of [9]. The case which is studied in the first chapter of [9] is more special, the tools are quite different, and perhaps not so natural.

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