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EBERLEIN INTEGRAL AND POLYNOMIAL INTERPOLATION

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1. Introduction. An important and pertinent problem in connection with $N + 1$-point Lagrangian Interpolation for a continuous real valued function defined on, say, the interval $[-1, 1]$ is the choice of a set of points where we require the interpolating polynomial to coincide with the function. The set of zeros of Tchebychev polynomial $T_{N+1}(x)$ is the classical solution of the problem [cf. [1] p. 262 appendix II]. However, this is inapplicable in interpolation for a function of 2 or more variables, because there is no adequate generalization of these polynomials to cover the case of more than one variable.

In the present paper a method not depending on orthogonal polynomial is outlined for choosing the best points for Lagrangian Interpolation for functions of one variable.

If we make the interpolating polynomial agree with the function at the zeros of a Tchebychev polynomial of appropriate degree, we obtain the best approximation in terms of the uniform norm. We shall work with the "least-squares norm" i.e., we seek to minimize the average of

$$\frac{1}{2}\int_{-1}^{1}[x(t) - p_N(t)]^2 \, dt$$

over a class of functions which for the moment we shall denote by C. Obviously we need a probability measure on C. If we denote this measure by u, then we seek to minimize

$$\frac{1}{2}\int_C \left\{\int_{-1}^{1}[x(t) - p_N(t)]^2 \, dt\right\} du \, .$$

We take C as the set of all functions

$$x(t) = \sum_{i=0}^{\infty} x_i t^i \, [-1 \leqq t \leqq 1]$$

where

$$\|x\|_1 = \sum_{i=0}^{\infty} |x_i| \leqq 1 \, .$$

The set C, then, is clearly the unit ball S_∞ of the sequence space l^1. We observe that given any $x \in l^1$, an appropriate scalar multiple $\alpha . x$ is in S_∞. Also l^1 includes all polynomials of all degrees. EBERLEIN [2] defined an integral on W^*-continuous real functions on S_∞, which we shall call Eberlein integral. The completely additive measure induced on S_∞ by this integral which we denote by $\mathrm{d}_E x$, is the measure which we use in the following work.

I wish to thank Dr. V. L. N. SARMA, who has suggested the investigation and helped me in various stages of the work.

II. Formulation of error. Let $J = \{a_0, a_1, \ldots, a_N\} \subset [-1, 1]$. Consider the polynomial

$$(2.0) \qquad p_N(t) = \sum_{j=0}^{N} s_j(t)\, x(a_j)$$

where

$$(2.1) \qquad s_j(t) = \frac{(t - a_0)(t - a_1) \ldots (t - a_{j-1})(t - a_{j+1}) \ldots (t - a_N)}{(a_j - a_0) \ldots (a_j - a_{j-1})(a_j - a_{j+1}) \ldots (a_j - a_N)} .$$

"Mean square error" in representing $x(t)$ by $p_N(t)$ is

$$(2.2) \qquad I[x(t) - p_N(t)]^2 = \frac{1}{2}\int_{-1}^{1} [x(t) - \sum_{j=0}^{n} s_j(t)\, x(a_j)]^2 \,\mathrm{d}t .$$

If we regard the root mean square error as a random variable over S_∞, a global measure for the error associated with our choice of interpolation points is given by the variance of this random variable. Accordingly we write

$$(2.3) \qquad \sigma^2(J) = \int_{S\infty} [I\{x(t) - p_N(t)\}^2] \,\mathrm{d}_E x .$$

The intergrand in R.H.S. of (2.3) is

$$(2.4) \qquad I(x^2) - 2\sum_{j=0}^{N} x(a_j)\, I(s_j . x) + \sum_{i=0}^{N} \sum_{j=0}^{N} x(a_i)\, x(a_j)\, I(s_i s_j) .$$

Now

$$(2.5) \qquad I(x^2) = \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x_m x_n \int_{-1}^{1} t^{m+n} \,\mathrm{d}t = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e_{m+n+1}}{m+n+1} x_m x_n$$

where $e_n = 0$ (n is even) or $e_n = 1$ (n is odd). As is easily verified that

$$s_j(t) = \frac{1}{\pi_j} \sum_{i=0}^{N} (-1)^i \sigma_j^i t^{N-i} \quad (0 \leqq j \leqq N)$$

where

$$(2.5\mathrm{A}) \quad \pi_j = (a_j - a_0)(a_j - a_1) \ldots (a_j - a_{j-1})(a_j - a_{j+1}) \ldots (a_j - a_N) .$$

and

(2.5B) $$\sigma_j^i = \sum_{m_1 > m_2 > \ldots > m_i} a_{m_1} \cdot a_{m_2} \ldots a_{m_i}$$

where m_t takes values $0, 1, 2, \ldots, j-1, j+1, \ldots, N$ and $t = 1, 2, \ldots, i$, and

$$\sigma_j^o = 1 \quad \text{for} \quad j = 0, 1, \ldots, N .$$

We have

$$s_j \cdot x = \frac{1}{\pi_j} \sum_{i=0}^{N} \sum_{n=0}^{\infty} (-1)^i \sigma_j^i x_n t^{N-i+n}$$

and

$$I(s_j \cdot x) = \frac{1}{\pi_j} \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{+N} (-1)^i \frac{e_{N-i+n+1}}{N-i+n+1} \sigma_j^i \right\} x_n .$$

Then

$$x(a_j)\, I(s_j \cdot x) = \frac{1}{\pi_j} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{N} (-1)^i \frac{e_{N-i+n+1}}{N-i+n+1} \sigma_j^i a_j^m \right\} x_m x_n$$

and

(2.6) $$-2 \sum_{j=0}^{N} x(a_j)\, I(s_j \cdot x) =$$

$$= -2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{N} \sum_{j=0}^{N} (-1)^i \frac{e_{N-i+n+1}}{N-i+n+1} \sigma_j^i a_j^m \right\} x_m x_n .$$

Again

(2.6A) $$I(s_i \cdot s_j) = \frac{1}{\pi_i \pi_j} \sum_{p=0}^{N} \sum_{q=0}^{N} (-1)^{q+p} \sigma_i^p \sigma_j^q \frac{e_{2N-p-q+1}}{2N-p-q+1}$$

and

(2.7) $$\sum_{i=0}^{N} \sum_{j=0}^{N} x(a_j)\, x(a_i)\, I(s_i \cdot s_j) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{ \sum_{i=0}^{N} \sum_{j=0}^{N} I(s_i \cdot s_j)\, a_i^m a_j^n \}\, x_m x_n .$$

Thus $\sigma^2(J)$ is integral of sum of the right sides of (2.5), (2.6) and (2.7) over S_∞.

We now compute three integrals separately:

(2.8) $$\int_{S_\infty} I(x^2)\, d_E x = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{e_{m+n+1}}{m+n+1} \right) \int_{S_\infty} x_m x_n\, d_E x =$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{e_{m+n+1}}{m+n+1} \right) \frac{\delta_{mn}}{3^{n+1}} \quad \text{[cf. [2]]} =$$

$$= \frac{1}{\sqrt{3}} \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3}} \right)^{2n+1} \Big/ (2n+1) = \frac{1}{2\sqrt{3}} \log_e \left(\frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \right) = \frac{1}{2\sqrt{3}} \log_e (2 + \sqrt{3})$$

(2.9) $$\int_{S_\infty} [-2\sum_{j=0}^{N} x(a_j)\, I(s_j x)]\, d_E x =$$

$$= -2\sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{n=0}^{\infty}(-1)^i \frac{e_{N-i+1}}{N-i+n+1}\,\sigma_j^i a_j^n \frac{1}{3^{n+1}}\cdot\frac{1}{\pi_j} =$$

$$= -2\sum_{j=0}^{N}\frac{1}{\pi_j}\sum_{i=0}^{N}(-1)^i \sigma_j^i \sum_{n=0}^{\infty}\frac{e_{N-i+1}}{N-i+n+1}\,\frac{a_j^n}{3^{n+1}} = -2\sum_{i=0}^{N}\sum_{j=0}^{N}(-1)^i \frac{\sigma_j^i}{\pi_j} A_{ij}$$

where we have used the abbreviation

(2.9A) $$A_{ij} = \sum_{n=0}^{\infty}\frac{e_{N-i+1}}{N-i+n+1}\,\frac{a_j^n}{3^{n+1}}$$

and

(2.10) $$\int_{S_\infty} [\sum_{i=0}^{N}\sum_{j=0}^{N} x(a_i)\, x(a_j)\, I(s_i s_j)]\, d_E x = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{m=0}^{\infty} I(s_i s_j)\frac{(a_i a_j)^m}{3^{m+1}}$$

$$= \sum_{i=0}^{N}\sum_{j=0}^{N}\frac{I(s_i s_j)}{3 - a_i a_j}.$$

Thus finally we have $\sigma^2(J) =$ the sum of right sides of (2.8), (2.9) and (2.10) i.e.

(2.11) $$\sigma^2(J) = \frac{1}{2\sqrt{3}}\log_e(2+\sqrt{3}) - 2\sum_{i=0}^{N}\sum_{j=0}^{N}(-1)^i\frac{\sigma_j^i}{\pi_j}A_{ij} + \sum_{i=0}^{N}\sum_{j=0}^{N}\frac{I(s_i s_j)}{3 - a_i a_j}.$$

The best choice, in the "least-squares norm", of the points $a_0, a_1, \ldots, a_N$ is that which makes $\sigma^2(J)$ a minimum as a function of $a_0, \ldots, a_N$.

III. Outline of method for the choice of the almost best symmetrical points with respect to origin for Lagrangian interpolation in some special cases. First, take $N = 1$ then (2.11) reduced to

(3.1) $$\sigma^2(J) = \frac{1}{2\sqrt{3}}\log_e(2+\sqrt{3}) - 2\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{(-1)^i\sigma_j^i}{\pi_j}A_{ij} + \sum_{i=0}^{1}\sum_{j=0}^{1}\frac{I(s_i s_j)}{3 - a_i a_j}$$

take $J = \{-a, a\}$ where $a \in (0, 1]$.

Substituting the values of σ_j^i, π_j, A_{ij} and $I(s_i s_j)$ [which are obtained by using (2. 5B) (2.5A), (2.6A) and (2.9A)] in right side of (3.1) we get

(3.2) $$\sigma^2(J) = \frac{1}{2\sqrt{3}}\log_e(2+\sqrt{3}) + \frac{2}{a^2} + \frac{10}{3(9-a^4)} - \frac{3+a^2}{a^3}\log_e\frac{3+a}{3-a}.$$

For convenience we denote right side of (3.2) by $f(a)$. Now the problem is to choose the point "a" for which $f(a)$ is a minimum. To solve this problem we find $f'(a)$ which comes out to be

$$\frac{40a^3}{3(9-a^4)^2} + \frac{9+a^2}{a^4}\log_e\frac{3+a}{3-a} - \frac{1}{a^3}\left(\frac{2a^2+54}{9-a^2}\right). \tag{3.3}$$

(3.3) can be written as

$$\frac{40a^3}{3^5}\left\{1+\frac{2a^4}{9}+\frac{3a^8}{9^2}+\ldots\right\}+\frac{2(9+a^2)}{a^4}\left\{\frac{a}{3}+\frac{1}{3}\left(\frac{a}{3}\right)^3+\frac{1}{5}\left(\frac{a}{3}\right)^5+\ldots\right\}- \tag{3.4}$$

$$-\frac{2a^2+54}{9a^3}\left\{1+\frac{a^2}{9}+\frac{a^4}{9^2}+\frac{a^6}{9^3}+\ldots\right\}.$$

By truncating the term whose degree in a is greater than or equal to 6 from (3.4) we get the polynomial

$$\frac{1}{3^3}\left(\frac{136}{7\,.\,3^6}a^5 - \frac{3992}{35\,.\,3^3}a^3 + \frac{8}{5}a\right). \tag{3.5}$$

(3.5) has a real root in close neighborhood of point 0·6 which is approximately equal to 0·61 ... Keeping this fact in mind, we started trial approach to find zero of $f'(a)$ and we obtained the point 0·607 ... as a better value for which $f(a)$ is approximately a minimum. We find $f'(0{\cdot}607) = {\cdot}000013$ and $f(0{\cdot}607) = {\cdot}004288$.

Secondly, take $N = 2$. In this case let $J = \{-a, 0, a\}$ for $0 < a \leqq 1$, then after substitution and simplification in the same parallel way as in a previous case (2.11) reduces to

$$\sigma^2(J) = -\ .28649122 - \frac{9+3a^2}{a^5}\log_e\frac{3+a}{3-a} + \tag{3.6}$$

$$+\frac{1}{(9-a^4)}\left\{\frac{54}{a^4}+\frac{20}{a^2}-\frac{13}{5}-\frac{20}{9}a^2-\frac{1}{3}a^4\right\}$$

we denote right side of (3.6) by $F(a)$. The problem of minimization of $\sigma^2(J)$ in this case reduces to the problem of minimization of $F(a)$. We get, therefore,

$$F'(a) = \frac{45+9a^2}{a^6}\log_e\frac{3+a}{3-a} - \frac{18a^2+54}{a^5(9-a^2)} + \tag{3.7}$$

$$+\frac{1}{(9-a^4)^2}\left\{\frac{432}{a}-\frac{360}{a^3}+80a-\frac{1944}{a^5}-\frac{112}{5}a^3-\frac{40}{9}a^5\right\}.$$

The polynomial that we obtain after cutting out the terms having degree 5 or greater than 5 in a, is

$$\frac{1}{3^5}\left(\frac{56}{11 \,.\, 3^4} a^4 - \frac{12258}{35 \,.\, 3^4} a^2 + \frac{104}{35}\right) \tag{3.8}$$

(3.8) has a real root near 0·8. By the same trial approach as in a previous case 0·799 is obtained as a better value at which $F(a)$ attains its approximate minimum, which is equal to ·000420.

The Method generalizes easily to functions of several variables; but useful results cannot be obtained without the aid of computer.

References

[1] *Z. Kopal:* Numerical Analysis, Second edition.

[2] *W. F. Eberlein:* An integral over function space, Can. Jour. Math., *14* (1962), 379—384.

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