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# POLYNOMIALS IN TOPOLOGICAL ALGEBRAS

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Several members of Marczewski's seminar in Wroclaw and Grätzer's seminar in Winnipeg were interested in the solution of the following problem. Let $\mathfrak{A} = (A; \mathbf{F})$ be an algebra (e.g. in the sense of [4]). $p_n(\mathfrak{A})$ denotes the number of all essentially $n$-ary polynomials over $\mathfrak{A}$. The question is what can be said about the sequence $p_0(\mathfrak{A}), p_1(\mathfrak{A}), \ldots$ (let us emphazise that, for card $A > 1$, identity is considered as an essentially unary polynomial – in contradistinction to [3]). Papers [6], [7] dealt with this problem under the condition that $\mathfrak{A}$ was an ordered algebra (i.e. $A$ is ordered and all polynomials are isotone in every variable). In this note some observations connected with the upper problem, as far as topological algebras are concerned, will be described. A topology or a topological space is meant in the sense of [2]. Thus $(P, \mathscr{U})$ being a topological space, $\mathscr{U}$ is the set of all open sets in it. This set $\mathscr{U}$ will be called a topology on the set $P$. Let $\mathfrak{A} = (A; \mathbf{F})$ be an algebra and $\mathscr{U}$ a topology on $A$. Let all functions $f \in \mathbf{F}$ be continuous in the topology $\mathscr{U}$. Then the pair $(\mathfrak{A}, \mathscr{U})$ is called a topological algebra. Principal question is:

Let $(\mathfrak{A}, \mathscr{U})$ be a topological algebra. What can be said about the sequence $p_0(\mathfrak{A})$, $p_1(\mathfrak{A}), \ldots$, when $\mathscr{U}$ has certain topological property (i.e. $(A, \mathscr{U})$ is compact or connected etc.)?

Having in mind difficulties of the general question for universal algebras one can hardly settle the upper problem in this generality. In present investigations we shall deal with two related topics.

a) Following URBANIK [8], $\mathscr{S}(\mathfrak{A}) = \{n : p_n(\mathfrak{A}) \neq 0\}$. What can be said about $\mathscr{S}(\mathfrak{A})$ in the case of a topological algebra?

b) How far algebras realizing one of the sequence studied in [3] (see below) can be chosen among topological algebras with "rich" topological properties?

The result on b) will be applied to a) in the same way as in [7].

First of all, let us exhibit some topological property which has quite restrictive influence on the structure of $\mathscr{S}(\mathfrak{A})$. A topology $\mathscr{U}$ will be called a $d$-topology if it is the right or left topology of some bidirected set (see [2]), i.e. if the following is true:

1. $\mathcal{U}$ is a $T_0$-topology.

2. Every point has the smallest neighborhood.

3. Let $x$, $y$ be two points, $O_x(O_y)$ the smallest neighborhood of $x$ (of $y$ respectively). There exist points $v$, $z$ such that the following is valid for their smallest neighborhoods $O_v$, $O_z$:

$$O_v \subset O_x \cap O_y\,, \quad O_x \cup O_y \subset O_z\,.$$

The result from [6], 3.5 can now be formulated in the following proposition.

**Proposition 1.** *Let* $(\mathfrak{A} = (A; \mathsf{F}), \mathcal{U})$ *be a topological algebra, where* $\mathcal{U}$ *is a d-topology. Then* $\mathcal{S}(\mathfrak{A})$ *is one of the following sets*

$$\{0, 1, \ldots, n\},\ \{1, 2, \ldots, n\},\ \{1, 3, 4, 5, \ldots\},\ \{0, 1, 2, 3, \ldots\},\ \{1, 2, 3, 4, \ldots\}\,.$$

Proof. Define the following partial order $\leqq$ in $A$.

$$x \leqq y \equiv O_x \subset O_y\,.$$

Therefore, $\mathcal{U}$ is the left topology for $(A, \leqq)$. Let $A^n = \underbrace{A \ldots A}_{n\text{-times}}$ be the cardinal power of $(A, \leqq)$ (see [1]), i.e.

$$\langle a_1, \ldots, a_n \rangle \lesssim \langle b_1, \ldots, b_n \rangle \quad \text{iff} \quad a_i \leqq b_i \quad \text{for} \quad i = 1, \ldots, n\,.$$

Denote the product topology for $A^n$ (induced in $A^n$ by $\mathcal{U}$) as $\mathcal{W}$. The topology $\mathcal{W}$ is a $T_0$-topology and $O_{a_1} \times \ldots \times O_{a_n}$ is the smallest neighbourhood of $\langle a_1, \ldots, a_n \rangle$ in $\mathcal{W}$. Let $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_n \rangle$ be two points in $A^n$. Then

$$O_{a_1} \times \ldots \times O_{a_n} \subset O_{b_1} \times \ldots \times O_{b_n} \equiv O_{a_i} \subset O_{b_i} \equiv a_i \leqq b_i \equiv$$
$$\equiv \langle a_1, \ldots, a_n \rangle \lesssim \langle b_1, \ldots, b_n \rangle\,.$$

So $\mathcal{W}$ is the left topology for $\lesssim$. By [2] chapter I, §4, problem 3, $f$ being some $n$-ary polynomial over $\mathfrak{A}$ and hence $f$ being a continuous mapping of $(A^n, \mathcal{W})$ to $(A, \mathcal{U})$, $f$ is isotone mapping of $(A^n, \lesssim)$ in $(A, \leqq)$, in other words, $\mathfrak{A}$ is an ordered algebra with the order $\leqq$. As $\mathcal{U}$ is a $d$-topology, $\leqq$ is bidirected.

By [6], 3.5, $\mathcal{S}(\mathfrak{A})$ is one of the sets mentioned above.

Now, let us examine what can be said about connection of $\mathcal{S}(\mathfrak{A})$ and compact connected topologies. The final result will be that there is no essential restriction on the type of $\mathcal{S}(\mathfrak{A})$ for compact connected topological algebras. We shall get more detailed results concerning the sequence $p_0(\mathfrak{A})$, $p_1(\mathfrak{A})$, … for such algebras.

In [3] the following theorem can be found.

**Theorem.** *Let* $p_0, p_1, \ldots, p_n, \ldots$ *be a sequence of cardinal numbers, satisfying one of the following conditions*:

(i) $p_0 > 0$.

(ii) $p_0 = 0$, $p_n > 0$ *for all* $n \geqq 1$.

(iii) $p_0 = 0$, $2n$ *divides* $p_{2n}$ *and* $p_{2n-1} > 0$ *for all* $n > 0$.

(iv) $p_0 = 0$, $p_1 > 0$ *and* $n$ *diwides* $p_n$ *for all* $n > 0$.

*Then there exists an algebra* $\mathfrak{A}$ *such that* $p_1(\mathfrak{A}) = p_1 + 1$, $p_n(\mathfrak{A}) = p_n$ *otherwise.*

If we confine ourselves to Hausdorff topologies, the algebras constructed in [3], can be topologized only by topologies, which are very near to discrete topologies. E.g. for the algebra $\mathfrak{A}$ belonging to the case (i) and containing at least one nonidentical, nonconstant polynomial the following is true: Let $\beta, \gamma$ be the smallest ordinals with $\bar{\beta} = \sup_{n \geqq 2} (p_n + 1)$, $\bar{\gamma} = \sup_{n \geqq 1} (p_n + 1)$. $(\mathfrak{A}, \mathcal{U})$ is a topological algebra ($\mathcal{U}$ being a Hausdorff topology) if and only if

1. The points of $\bigcup_{i<\beta} A_i$ are isolated.
2. All the sets $A_i$, $i < \gamma$, are simultaneously closed and open in $\mathcal{U}$.

We see, such a topology $\mathcal{U}$ is neither compact nor connected.

On the other hand, we shall prove the following theorem.

**Theorem 2.** *Let* $A$ *be a compact interval of the real line different from one-point set. Let* $p_0, p_1, \ldots$ *be a sequence of cardinal numbers less or equal to* $2^{\aleph_0}$ *and fulfilling one of the conditions* (i), (ii), (iv). *Then there exists a topological algebra* $\mathfrak{A}$ *defined on* $A$ *such that*

$$p_1(\mathfrak{A}) = p_1 + 1\,, \quad p_n(\mathfrak{A}) = p_n \quad \text{for} \quad n \neq 1\,.$$

Remark. Restriction $p_n \leqq 2^{\aleph_0}$ is natural as, $A$ being a compact interval, there are only $2^{\aleph_0}$ continuous mappings of $A^n$ in $A$.

Proof of Theorem. First one auxiliary construction. Let $x_1, \ldots, x_n \in [\frac{1}{2}, 1]$ $c \in (0, \frac{1}{2})$. Put $d_c(x_1, \ldots, x_n) = c \min [\{|x_i - x_j| : i, j = 1, \ldots, n, i \neq j\} \cup \{x_i - \frac{1}{2} : : i = 1, \ldots, n\}]$. $d_c$ is continuous, $d_c(x_1, \ldots, x_n) \in [0, \frac{1}{4})$, $d_c(x_1, \ldots, x_n) = 0$ if and only if $x_i = x_j$ for some $i, j$, $i \neq j$ or $x_i = \frac{1}{2}$ for some $i$.

Now, we shall construct algebras realizing the cases (i), (ii) (iv) mentioned above.

Case (i). Put $A = [0, 1]$. For $c \in [0, \frac{1}{2})$ put $g_c(x) = c$ (so $g_c$ is a constant function).

For $c \in (0, \frac{1}{2})$ $f_c^n(x_1, \ldots, x_n)$ will be defined as follows.

$$f_c^n(x_1, \ldots, x_n) = d_c(x_1, \ldots, x_n) \quad \text{for} \quad x_1, \ldots, x_n \in [\tfrac{1}{2}, 1]\,.$$
$$f_c^n(x_1, \ldots, x_n) = 0 \quad \text{otherwise}\,.$$

$f_c^n$ are continuous, symmetrical, essentially $n$-ary and mutually different. One easily sees that

$$g_c(f_d^n(x_1, \ldots, x_n)) = g_c(g_d(x)) = c\,,$$
$$f_c^n(f_d^m(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) = f_c^n(g_d(x_1), x_2, \ldots, x_n) = 0\,.$$

By definition, after any identification of variables in $f_c^n(x_1, \ldots, x_n)$ we get the function $g_0$.

Let $\mathscr{F}$ be some subset of $\{g_c : c \in [0, \frac{1}{2})\} \cup \{f_c^n : c \in (0, \frac{1}{2}), n = 1, 2, \ldots\}$ containing $g_0$. Then $\mathscr{F}$ is the set of all nontrivial polynomials over $(A; \mathscr{F})$.

Case (ii).

Put $A = [-1, 1]$. Let us define $g^n(x_1, \ldots, x_n), f_c^n(x_1, \ldots, x_n)$ for $c \in (0, \frac{1}{2})$ as follows.

$$g^n(x_1, \ldots, x_n) = 0 \quad \text{for} \quad x_1, \ldots, x_n \in [0, 1] .$$
$$g^n(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} \quad \text{otherwise} .$$
$$f_c^n(x_1, \ldots, x_n) = d_c(x_1, \ldots, x_n) \quad \text{for} \quad x_1, \ldots, x_n \in [\tfrac{1}{2}, 1] ,$$
$$f_c^n(x_1, \ldots, x_n) = 0 \quad \text{for} \quad x_1, \ldots, x_n \in [0, 1] , \quad \{x_1, \ldots, x_n\} \cap [0, \tfrac{1}{2}) \neq \emptyset ,$$
$$f_c^n(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} \quad \text{otherwise} .$$

The functions $g^n, f_c^n$ are clearly continuous, symmetrical, essentially $n$-ary and mutually different. One can easily verify that $g^n(x_1, x_1, x_3, \ldots, x_n) = f_c^n(x_1, x_1, x_3, \ldots, x_n) = g^{n-1}(x_1, x_3, \ldots, x_n)$. $g^n(g^m(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) = g^n(f_c^m(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) = f_c^n(g^m(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) = f_c^n(f_d^m(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) = g^{n+m-1}(x_1, \ldots, x_{n+m-1})$.

Hence, if $\mathscr{F}$ is some subset of $\{f_c^n : c \in (0, \frac{1}{2}), n = 1, 2, \ldots\}$ then $\mathscr{F} \cup \{g^n : n = 1, 2, \ldots\}$ is the set of all nontrivial polynomials over $(A; \mathscr{F} \cup \{g^n : n = 1, 2, \ldots\})$.

Case (iv). Put $A = [-1, 1]$. Define $g(x), f_{c,i}^n(x_1, \ldots, x_n)$ as follows.

$$g(x) = 0 \quad \text{for} \quad x \in [0, 1] ,$$
$$g(x) = x \quad \text{otherwise} .$$

For $i = 1, 2, \ldots, n$, $c \in (0, \frac{1}{2})$

$$f_{c,i}^n(x_1, \ldots, x_n) = d_c(x_1, \ldots, x_n) \quad \text{for} \quad x_1, \ldots, x_n \in [\tfrac{1}{2}, 1] ,$$
$$f_{c,i}^n(x_1, \ldots, x_n) = 0 \quad \text{for} \quad x_i \in [0, 1] , \quad \{x_1, \ldots, x_n\} \cap [-1, \tfrac{1}{2}) \neq \emptyset ,$$
$$f_{c,i}^n(x_1, \ldots, x_n) = x_i \quad \text{otherwise} .$$

All the functions just defined are continuous, mutually different, $f_{c,i}^n$ is essentially $n$-ary and symmetrical in $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. If the variables in $f_{c,i}^n$ are permuted and $x_i$ is replaced by $x_j$ in this permutation we get $f_{c,j}^n$ from $f_{c,i}^n$. Further

$$g(g(x)) = g(x) .$$
$$g(f_{c,i}^n(x_1, \ldots, x_n)) = g(x_i) .$$
$$f_{c,1}^n(g(x_1), x_2, \ldots, x_n) = g(x_1) = f_{c,1}^n(x_1, g(x_2), x_3, \ldots, x_n) .$$
$$f_{c,1}^n(f_{d,i}^m(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) = g(x_i) .$$
$$f_{c,1}^n(x_1, f_{d,i}^m(x_2, \ldots, x_{m+1}), x_{m+2}, \ldots, x_{m+n-1}) = g(x_1) .$$
$$f_{c,1}^n(x_1, x_1, x_3, \ldots, x_n) = f_{c,1}^n(x_1, x_2, x_2, x_4, \ldots, x_n) = g(x_1) .$$

Hence, $\mathscr{F}$ being a nonempty subset of $\{g\} \cup \{f^n_{c,1} : c \in (0, \frac{1}{2}),\ n = 1, 2, \ldots\}$ then $\{g\} \cup \{f^n_{c,i} : f^n_{c,i} \in \mathscr{F},\ i = 1, \ldots, n\}$ is the set of all nontrivial polynomials over $(A; \mathscr{F})$.

**Corollary.** *Let $M$ be a subset of the set of all nonnegative integers containing* 1. *There exists a topological algebra $\mathfrak{A}$ on a compact interval with $\mathscr{S}(\mathfrak{A}) = M$.*

The case (iii) cannot be, in general, involved in Theorem 2 as the folllowing proposition is valid.

**Proposition 3.** *Let $(\mathfrak{A}, \mathscr{U})$ be a Hausdorff compact connected algebra with $p_{2n}(\mathfrak{A}) = 0$ for all $n$. Then $p_{2n+1}(\mathfrak{A}) \neq 1$ for $n = 1, 2, 3, \ldots$*

Proof. Let $p_k(\mathfrak{A}) = 1$ for certain odd $k > 1$. Let $f(x_1, \ldots, x_k)$ be the essentially $k$-ary polynomial over $\mathfrak{A}$. As $p_k(\mathfrak{A}) = 1$, $f(x_1, \ldots, x_k)$ is symmetrical in its variables. Put

$$C = \{f(a_1, \ldots, a_k) : a_1, \ldots, a_k \in \mathfrak{A}\}\,.$$

$C$ is compact, connected and closed in respect to $f$. Let $\bar{f}$ be the restriction of $f$ to $C$. Put $\mathfrak{C} = (C; \bar{f})$. There is no constant polynomial in $\mathfrak{C}$. Namely, suppose that $\bar{G}(x_1, \ldots, x_n)$ is a polynomial symbol over $\mathfrak{C}$ giving a constant polynomial. Put $f$ instead $\bar{f}$ in $\bar{G}(x_1, \ldots, x_n)$. We get a polynomial symbol $G(x_1, \ldots, x_n)$ over $\mathfrak{A}$. Then $G(f(x_1, \ldots, x_n), \ldots, f(x_1, \ldots, x_n))$ is a polynomial symbol over $\mathfrak{A}$ yielding a constant polynomial. But $p_0(\mathfrak{A}) = 0$, a contradiction.

Let $\bar{g}(x_1, \ldots, x_n)$ be an essentially $n$-ary polynomial over $\mathfrak{C}$ induced by a polynomial symbol $\bar{G}(x_1, \ldots, x_n)$. Let $G(x_1, \ldots, x_n)$ be the polynomial symbol over $\mathfrak{A}$ gained from $\bar{G}(x_1, \ldots, x_n)$ by replacing $\bar{f}$ by $f$. $g(x_1, \ldots, x_n)$ be the corresponding polynomial over $\mathfrak{A}$. This polynomial is clearly essentially $n$-ary ($\bar{g}$ is the restriction of $g$ to $C$). As there is no polynomial of even arity in $\mathfrak{A}$, $n$ is odd.

Let $\bar{h}(x)$ be an unary polynomial over $\mathfrak{C}$. Let $\bar{H}(x)$ be a polynomial symbol yielding $\bar{h}(x)$. $\bar{H}(x)$ yields $H(x)$ by replacing $\bar{f}$ by $f$ and $H(x)$ defines $h(x)$ over $\mathfrak{A}$. By $p_k(\mathfrak{A}) = 1$, $p_0(\mathfrak{A}) = 0$ it is $h(f(y_1, \ldots, y_k)) = f(y_1, \ldots, y_k)$. This implies $\bar{h}(c) = c$ for $c \in C$.

We see that $\mathfrak{C}$ is a compact connected idempotent algebra with $p_{2n}(\mathfrak{C}) = 0$ for all $n$. Further, from symmetry of $f$, we get the symmetry of $\bar{f}$ and therefore $\bar{f}$ is essentialy $k$-ary. By [8] Theorem 1 and Theorem 2 $\mathscr{S}(\mathfrak{C}) = \{1, 3, 5, \ldots\}$ and $\mathfrak{C}$ is a reduct of at least two-point Boolean group $(C, +)$ with $x + y + z$ as the fundamental operation.

As $x + y + z$ is continuous in $C$, $x + y = x + y + 0$ is continuous, too. It is $x = -x$ therefore we can consider $(C, +)$ as a compact connected commutative group. We have $2x = 0$ four all $x \in C$. Hence each character of this group is the zero mapping. But this contradicts the Duality theorem for compact groups ([5], Theorem 39), by which $(C, +)$ is isomorphic to the group of characters of its group of characters.

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