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FUNCTION SPACES FOR SOMEWHAT CONTINUOUS FUNCTIONS*)

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1. INTRODUCTION

NAIMPALLY in [3] and [4] investigates function spaces considering functions other than continuous functions. In particular he shows that the class of connectivity maps is closed in the topology of uniform convergence and that the class of almost continuous functions is closed in the graph topology. In this paper I shall show that the class of somewhat continuous functions is closed in the topology of uniform convergence but not closed in the graph topology. In fact it will be shown that in certain instances the class of somewhat continuous functions is dense in the set of all functions under the graph topology.

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2. UNIFORM CONVERGENCE TOPOLOGY

Definition 1. Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be topological spaces. A function $f : (X, \mathscr{S}) \to \to (Y, \mathscr{T})$ is said to be *somewhat continuous* provided if $U \in \mathscr{T}$ and $f^{-1}(U) \neq \emptyset$, then there is a $V \in \mathscr{S}$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$.

Let X be a topological space and let $(Y, \mathscr{V})$ be a uniform space. Then a basis for the uniformity of uniform convergence for Y^X is the collection $\{W(V) \mid V \in \mathscr{V}\}$ where

$$W(V) = \{(f, g) \in Y^X \times Y^X \mid (f(x), g(x)) \in V \text{ for all } x \in X\}$$

[See 2, p. 226].

If X and Y are topological spaces, then the set of all somewhat continuous functions from X into Y will be denoted by $S(X, Y)$.

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Theorem 1. *If X is a topological space and $(Y, \mathscr{V})$ is a uniform space, then $S(X, Y)$ is closed in Y^X under the topology of uniform convergence.*

Proof. Let f be an element of Y^X which is a limit point of $S(X, Y)$. Let M be a dense subset of X, let O be a non-empty open subset of $f(X)$, and let z be an element of O. There is an element $Q \in \mathscr{T}_{\mathscr{V}}$ (the topology for Y generated by $\mathscr{V}$) such that $O = Q \cap f(X)$. Let y be an element of X such that $f(y) = z$. Since $Q \in \mathscr{T}_{\mathscr{V}}$, there is a $V \in \mathscr{V}$ such that $V[z] \subset Q$. There is an open symmetric member U of $\mathscr{V}$ such that $U \circ U \circ U \subset V$. Since f is a limit point of $S(X, Y)$, there is a $g \in S(X, Y)$ such that $g(x) \in U[f(x)]$ for all $x \in X$. Thus $g(y) \in U[f(y)]$. Since g is somewhat continuous, by [1, Th. 3], $g(M)$ is a dense subset of $g(X)$. Since $g(y)$ is contained in the $\mathscr{T}_{\mathscr{V}}$ – interior of $U[g(y)]$, there is an element $m \in M$ such that $g(m) \in U[g(y)]$. Since $m \in X$, $g(m) \in U[f(m)]$. Since U is symmetric,

$$f(m) \in U[g(m)] \subset U \circ U[g(y)] \subset U \circ U \circ U[f(y)] \subset V[f(y)] = V[z] \subset Q .$$

Hence, there is an element m of M such that $f(m) \subset O$. Therefore $f(M)$ is a dense subset of $f(X)$. By [1, Th. 3], f is somewhat continuous. Thus $S(X, Y)$ is a closed subset of Y^X under the topology of uniform convergence.

3. GRAPH TOPOLOGY

Let X and Y be topologicial spaces. For each $f \in Y^X$, let $G(f)$ denote the graph of f in $X \times Y$. For each open subset U of $X \times Y$ under the usual product topology, let $F_U = \{f \in Y^X \mid G(f) \subset U\}$. The set $\{F_U \mid U$ open in $X \times Y\}$ is a basis for the graph topology Γ for Y^X [See 4, p. 268].

Example 1. Let X and Y be the closed unit interval with the usual topology. Then $S(X, Y)$ is not a closed subset of (Y^X, Γ).

Proof. Define $f : X \to Y$ by $f(x) = 0$ if $x \neq 1$ and $f(1) = 1$. Then $f^{-1}((1/2, 1]) \neq \emptyset$ and does not contain any non-empty open subset of X. Thus f is not somewhat continuous. Let U be an open subset of $X \times Y$ containing $G(f)$. Then U contains the point (1,1). There is a disc D containing (1,1) such that $D \cap (X \times Y) \subset U$. Let r be the radius of D and let s be the maximum of 1/2 and $1 - (r/2)$. Define $g : X \to Y$ by $g(x) = 0$ if $0 \leqq x < s$ and $g(x) = 1$ if $s \leqq x \leqq 1$. If V is an open subset of Y, then $g^{-1}(V)$ is either empty, contains $[0, s)$, or contains $[s, 1]$. Thus, since $0 < s < 1$, g is somewhat continuous. Hence f is a limit point of $S(X, Y)$ which is not in $S(X, Y)$. Hence $S(X, Y)$ is not a closed subset of (Y^X, Γ).

Definition 2. Let X be a topological space and let $\mathscr{U}$ be a cover of X. Then $\mathscr{U}$ is said to be *interior preserving* provided that if $U \in \mathscr{U}$, then the interior of $U - \bigcup\{V \in \mathscr{U} \mid V \neq U\}$ is not empty.

Definition 3. A topological space X is said to be an *$\mathscr{I}$-space* provided if $\mathscr{U}$ is an open cover of X, then $\mathscr{U}$ has an interior preserving refinement.

It should be noted in Definition 3 that there is no requirement that the refinement consist of open subsets of X. The following theorem and example show that there is such a thing as an $\mathscr{I}$-space and that not all topological spaces are $\mathscr{I}$-spaces.

Theorem 2. *Each paracompact space is an $\mathscr{I}$-space.*

Proof. Let $\mathscr{U}$ be an open cover of a paracompact space X. Let a family $\{G_\alpha \mid \alpha \in A\}$ be a locally finite open refinement of $\mathscr{U}$. Suppose A is well-ordered by $<$ and each G_α is non-void. Let β be the first element of A under $<$. Put $G'_\beta = G_\beta$ and, as X is regular, choose a closed subset F_β of G_β such that the interior of F_β is non-void. Suppose $\gamma \in A$ and G'_α and F_α have been defined for all $\alpha < \gamma$. Put $G'_\gamma = G_\gamma - \bigcup\{F_\alpha \mid \alpha < \gamma\}$. The set G'_γ is open. In case $G'_\gamma = \emptyset$ put $F_\gamma = \emptyset$, and clearly $G_\gamma \subset \bigcup\{G_\alpha \mid \alpha < \gamma\}$. If $G'_\gamma \neq \emptyset$ choose a closed subset F_γ of G'_γ such that the interior of F_γ is non-void. Now, let $B = \{\alpha \in A \mid G'_\alpha \neq \emptyset\}$. The subfamily $\{G_\alpha \mid \alpha \in B\}$ also covers X, the family $\{F_\alpha \mid \alpha \in B\}$ is disjoint. Finally put, for $\alpha \in B$, $H_\alpha = G_\alpha - \bigcup\{F_\gamma \mid \gamma \in B, \gamma \neq \alpha\}$. Each H_α is open. We shall show that $\{H_\alpha \mid \alpha \in B\}$ is the required cover. If a point x of X belongs to some, and only one, F_α then $x \in H_\alpha$, if not then it must belong to some G_α and $x \in H_\alpha$. For any fixed γ, if $\alpha \neq \gamma$ then $H_\alpha \cap F_\gamma = \emptyset$, hence $H_\gamma - \bigcup\{H_\alpha \mid \alpha \in B, \alpha \neq \gamma\} \supset F_\gamma$. As the interior of F_γ is non-void, the family $\{H_\alpha \mid \alpha \in B\}$ is an interior preserving cover.

Example 2. Let $X = \{a, b, c\}$ and let $\mathscr{T} = \{\emptyset, X\{a, b\}, \{b, c\}, \{b\}\}$. Then $(X, \mathscr{T})$ is not an $\mathscr{I}$-space. Actually let $U_1 = \{a, b\}$ and $U_2 = \{a, c\}$. Let $\mathscr{U} = \{U_1, U_2\}$. Then $\mathscr{U}$ is an open cover of X which has no interior preserving refinement.

Theorem 3. *If X is an $\mathscr{I}$-space, then $S(X, Y)$ is dense in (Y^X, Γ).*

Proof. Let X be an $\mathscr{I}$-space and Y a topological space. Let $f \in Y^X$ and let F_U be an element of Γ which contains f. For each $x \in X$, since $(x, f(x)) \in U$, there exist sets U_x open in X and V_x open in Y such that $x \in U_x$, $f(x) \in V_x$, and $U_x \times V_x \subset U$. Then $\{U_x \mid x \in X\}$ is an open cover of X. Since X is an $\mathscr{I}$-space, there is an interior preserving refinement $\mathscr{V}$ of $\{U_x \mid x \in X\}$. For each element $V \in \mathscr{V}$, there is an element $y \in X$ such that $V \subset U_y$. Pick one such y and call it w_V. Let $<$ be a well-ordering of $\mathscr{V}$. Define a function $g : X \to Y$ as follows:

Let V be the first element of $\mathscr{V}$ under $<$. For each $x \in V$ define $g(x) = f(w_V)$.

Suppose g has been defined through a certain point of $\mathscr{V}$ under $<$ and that M is the first element remaining of $\mathscr{V}$ under $<$ at which g has not yet been defined. For each $x \in M$ at which g has not yet been defined, define $g(x) = f(w_M)$.

Let $x \in X$. Let N be the first element of $\mathscr{V}$ under $<$ which contains x. Then $g(x) = f(w_N)$. Now $x \in N \subset U_{w_N}$ and $g(x) = f(w_N) \in V_{w_N}$. Thus $(x, g(x)) \in U_{w_N} \times V_{w_N} \subset U$. Hence $G(g) \subset U$ and thus $g \in F_U$.

It remains to be shown that $g \in S(X, Y)$. Let A be an open subset of Y such that $g^{-1}(A) \neq \emptyset$. Let $p \in X$ such that $g(p) \in A$. Let D be the first element of $\mathscr{V}$ under $<$ which contains p. Thus $g(p) = f(w_D)$. Let B be the subset of D consisting of all elements of X not contained in any member of $\mathscr{V}$ preceding D under $<$. Then for each $x \in B$, $g(x) = f(w_D) = g(p)$. Therefore $B \subset g^{-1}(A)$. But since $\mathscr{V}$ is interior preserving, the interior of B is not empty. So the interior of B is a non-empty open subset of X contained in $g^{-1}(A)$. Hence, g is somewhat continuous and thus $S(X, Y)$ is dense in (Y^X, Γ).

Example 3. Let $(X, \mathscr{T})$ be the space from Example 2, let $\mathscr{S}$ be the discrete topology on X. The identity function $f : (X, \mathscr{T}) \to (X, \mathscr{S})$ is not somewhat continuous. The set $U = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$ is open in $(X, \mathscr{T}) \times (X, \mathscr{S})$ and contains $G(f)$. If $g : (X, \mathscr{T}) \to (X, \mathscr{S})$, $G(g) \subset U$, then $g(a) = a$, $g(c) = c$ and if still $g \neq f$, $g(b) \neq b$. Consider $g(b) = a$ only. Then $g^{-1}(\{c\})$ contains no open non-void set, hence g is not somewhat continuous. Thus Theorem 3 need not hold without the assumption on X of being an $\mathscr{I}$-space.

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