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SOMEWHAT CONTINUOUS FUNCTIONS*)

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1. INTRODUCTION

The ideas of feebly continuous functions and feebly open functions were first introduced by ZDENĚK FROLÍK in [2]. In [2] he proves that the almost continuous, feebly open image of a Baire space (a space of second category in itself) is a Baire space (resp. a space of second category in itself) and observes that almost continuous can be replaced by one-to-one and feebly continuous. In this paper we study the concepts of somewhat continuous functions and somewhat open functions which are Zdeněk Frolík's functions except that we have dropped the requirement that the functions be onto. These ideas are also closely related to the idea of weakly equivalent topologies which was first introduced in [5].

In Section 2, we study properties of somewhat continuous functions. We also give a characterization for somewhat continuous functions which parallels a characterization for feebly continuous functions given by Zdeněk Frolík in [2].

In Section 3, we define somewhat open functions and get results which parallel the results for somewhat continuous functions.

In Section 4, we show that there are other properties that carry over under our weaker functions and give counterexamples for others.

In Section 5, conditions are found which make somewhat continuous and somewhat open equivalent to continuous and open.

The reader is referred to [1], [2], and [4] for definitions not stated in this paper. Throughout this paper theorems, which only require a straightforward approach using standard methods, will be stated only and the proofs left as easy exercises for the reader.

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2. SOMEWHAT CONTINUOUS FUNCTIONS

Definition 1. Let $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ be topological spaces. A function $f : (X, \mathscr{S}) \to (Y, \mathscr{T})$ is said to be *somewhat continuous* provided that if $U \in \mathscr{T}$ and $f^{-1}(U) \neq \emptyset$, then there is a $V \in \mathscr{S}$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$.

It is clear that every continuous function and every quasi-continuous function is somewhat continuous.

Theorem 1. *If* $f : (X, \mathscr{S}) \to (Y, \mathscr{T})$ *and* $g : (Y, \mathscr{T}) \to (Z, \mathscr{U})$ *are somewhat continuous functions and* $f(X)$ *is dense in* Y, *then* $gf : (X, \mathscr{S}) \to (Z, \mathscr{U})$ *is somewhat continuous.*

Theorem 2. *If* $f : (X, \mathscr{S}) \to (Y, \mathscr{T})$ *is somewhat continuous and* $g : (Y, \mathscr{T}) \to (Z, \mathscr{U})$ *is continuous, then* $gf : (X, \mathscr{S}) \to (Z, \mathscr{U})$ *is somewhat continuous.*

The following example shows that not only is the composite of two somewhat continuous functions not necessarily somewhat continuous but that $f : (X, \mathscr{S}) \to (Y, \mathscr{T})$ can be continuous and $g : (Y, \mathscr{T}) \to (Z, \mathscr{U})$ somewhat continuous and yet $gf : (X, \mathscr{S}) \to (Z, \mathscr{U})$ need not be somewhat continuous.

Example 1. Let $X = \{a, b, c\}$, $\mathscr{S} = \{\emptyset, X\}$, $\mathscr{T} = \{\emptyset, X, \{b\}, \{a, c\}\}$, and $\mathscr{U} = \{\emptyset, X, \{a, b\}\}$. Define $f : (X, \mathscr{S}) \to (X, \mathscr{T})$ by $f(a) = a$, $f(b) = c$, and $f(c) = c$. Let g be the identity function from $(X, \mathscr{T})$ onto $(X, \mathscr{U})$. Then f is continuous, g is somewhat continuous, and yet gf is not somewhat continuous.

In the previous example g is an example of a somewhat continuous function which is neither continuous nor quasi-continuous.

Theorem 3. *If* $f : (X, \mathscr{S}) \to (Y, \mathscr{T})$ *is a function, then the following three conditions are equivalent:*

(1) *f is somewhat continuous,*

(2) *if C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper closed subset D of X such that $D \supset f^{-1}(C)$, and*

(3) *if M is a dense subset of X, then $f(M)$ is a dense subset of $f(X)$.*

Theorem 4. *If* $f : (X, \mathscr{S}) \to (Y, \mathscr{T})$ *is somewhat continuous and* A *is a dense subset of* X *and* $\mathscr{S}_A$ *is the induced topology for* A, *then* $f|_A : (A, \mathscr{S}_A) \to (Y, \mathscr{T})$ *is somewhat continuous.*

The following example shows that just requiring the subset to be either open or closed is not sufficient to insure that the restriction is somewhat continuous.

Example 2. Let $X = \{a, b, c\}$, $\mathscr{S} = \{\emptyset, X, \{a, b\}, \{c\}\}$, and $\mathscr{T} = \{\emptyset, X, \{b, c\}\}$. Let $A = \{a, b\}$. Then A is both an open and closed subset of $(X, \mathscr{S})$. The identity function from $(X, \mathscr{S})$ onto $(X, \mathscr{T})$ is somewhat continuous but its restriction to A is not somewhat continuous.

Theorem 5. *If $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ are topological spaces and A is an open subset of X and $f:(A, \mathscr{S}_A) \to (Y, \mathscr{T})$ is a somewhat continuous function such that $f(A)$ is dense in Y, then any extension F of f mapping $(X, \mathscr{S})$ into $(Y, \mathscr{T})$ is somewhat continuous.*

The next two examples show that neither A being open in X nor $f(A)$ being dense in Y can be omitted in the previous theorem.

Example 3. Let $X = \{a, b\}$, $\mathscr{S} = \{\emptyset, X, \{a\}\}$, and $\mathscr{T} = \{\emptyset, X, \{b\}\}$. Let $A = \{a\}$. Then $\mathscr{S}_A = \{\emptyset, A\}$. Define $f:(A, \mathscr{S}_A) \to (X, \mathscr{T})$ by $f(a) = a$. Then f is continuous and A is an open subset of X. The extension F of f defined by $F(a) = a$ and $F(b) = b$ is a function from $(X, \mathscr{S})$ onto $(X, \mathscr{T})$ which is not somewhat continuous.

Example 4. Let $X = \{a, b\}$, $\mathscr{S} = \{\emptyset, X\}$, and $\mathscr{T} = \{\emptyset, X, \{a\}\}$. Let $A = \{a\}$. Define $f:(A, \mathscr{S}_A) \to (X, \mathscr{T})$ by $f(a) = a$. Then $f(A)$ is dense in $(X, \mathscr{T})$ and f is continuous. The extension F of f defined by $F(a) = a$ and $F(b) = b$ is a function from $(X, \mathscr{S})$ onto $(X, \mathscr{T})$ which is not somewhat continuous.

Theorem 6. *If $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ are spaces and $X = A \cup B$ where A and B are open subsets of X and $f:(X, \mathscr{S}) \to (Y, \mathscr{T})$ is a function such that $f|_A$ and $f|_B$ are somewhat continuous, then f is somewhat continuous.*

In the previous theorem it is not enough to assume that either $A \cap B$ is closed or that $A \cap B$ is open as the next example shows.

Example 5. Let $X = \{a, b, c\}$, $\mathscr{S} = \{\emptyset, X, \{a\}, \{b, c\}\}$, and $\mathscr{T} = \{\emptyset, X, \{b\}\}$. Let $A = \{a, b\}$ and $B = \{a, c\}$. Then $A \cap B$ is both an open and closed subset of $(X, \mathscr{S})$. The identity function $f:(X, \mathscr{S}) \to (X, \mathscr{T})$ has the property that $f|_A$ and $f|_B$ are continuous and yet f is not somewhat continuous.

Definition 2. If X is a set and $\mathscr{T}$ and $\mathscr{T}'$ are topologies for X then $\mathscr{T}$ is said to be *weakly equivalent* to $\mathscr{T}'$ provided if $U \in \mathscr{T}$ and $U \neq \emptyset$, then there is a $V \in \mathscr{T}'$ such that $V \neq \emptyset$ and $V \subset U$ and if $U \in \mathscr{T}'$ and $U \neq \emptyset$, then there is a $V \in \mathscr{T}$ such that $V \neq \emptyset$ and $V \subset U$.

It is useful to observe that two topologies $\mathscr{T}$ and $\mathscr{T}'$ for X are weakly equivalent if and only if the identity function from $(X, \mathscr{T})$ onto $(X, \mathscr{T}')$ is somewhat continuous in both directions.

Theorem 7. *If $f:(X, \mathscr{S}) \to (Y, \mathscr{T})$ is somewhat continuous and $\mathscr{S}'$ is a topology for X which is weakly equivalent to $\mathscr{S}$, then $f:(X, \mathscr{S}') \to (Y, \mathscr{T})$ is somewhat continuous.*

Theorem 8. *If $f:(X,\mathscr{S})\to(Y,\mathscr{T})$ is a somewhat continuous function from X onto Y and $\mathscr{S}'$ and $\mathscr{T}'$ are topologies for X and Y respectively such that $\mathscr{S}'$ is weakly equivalent to $\mathscr{S}$ and $\mathscr{T}'$ is weakly equivalent to $\mathscr{T}$, then $f:(X,\mathscr{S}')\to(Y,\mathscr{T}')$ is somewhat continuous.*

3. SOMEWHAT OPEN FUNCTIONS

Definition 3. Let $(X,\mathscr{S})$ and $(Y,\mathscr{T})$ be topological spaces. A function $f:(X,\mathscr{S})\to$ $\to(Y,\mathscr{T})$ is said to be *somewhat open* provided that if $U\in\mathscr{S}$ and $U\neq\emptyset$, then there is a $V\in\mathscr{T}$ such that $V\neq\emptyset$ and $V\subset f(U)$.

Example 6. Let $X=\{a,b,c\}$, $\mathscr{S}=\{\emptyset,X,\{a,b\}\}$, and $\mathscr{T}=\{\emptyset,X,\{a\}\}$. Then the identity function $f:(X,\mathscr{S})\to(X,\mathscr{T})$ is somewhat open but not open.

It is clear that every open function is somewhat open.

Theorem 9. *If $f:(X,\mathscr{S})\to(Y,\mathscr{T})$ and $g:(Y,\mathscr{T})\to(Z,\mathscr{U})$ are somewhat open functions, then $gf:(X,\mathscr{S})\to(Z,\mathscr{U})$ is somewhat open.*

Theorem 10. *If $f:(X,\mathscr{S})\to(Y,\mathscr{T})$ is a function then the following two conditions are equivalent:*

(1) *f is somewhat open, and*

(2) *if M is a dense subset of Y, then $f^{-1}(M)$ is a dense subset of X.*

Theorem 11. *If $f:(X,\mathscr{S})\to(Y,\mathscr{T})$ is one-to-one and onto then the following two conditions are equivalent:*

(1) *f is somewhat open, and*

(2) *if C is a closed subset of X such that $f(C)\neq Y$, then there is a closed subset D of Y such that $D\neq Y$ and $D\supset f(C)$.*

The following two examples show that neither one-to-one nor onto can be omitted in Theorem 11 for either direction.

Example 7. Let $X=\{a\}$, $Y=\{a,b\}$, $\mathscr{S}=\{\emptyset,X\}$, $\mathscr{T}=\{\emptyset,Y,\{a\}\}$, and $\mathscr{U}=$ $=\{\emptyset,Y,\{b\}\}$. Define $f:X\to Y$ by $f(a)=a$. Then f is one-to-one, $f:(X,\mathscr{S})\to$ $\to(Y,\mathscr{T})$ is somewhat open but does not fulfill (2) of Theorem 11, and $f:(X,\mathscr{S})\to$ $\to(Y,\mathscr{U})$ fulfills (2) of Theorem 11 but is not somewhat open.

Example 8. Let $X=\{a,b,c\}$, $Y=\{a,b\}$, $\mathscr{S}=\{\emptyset,X,\{a,b\}\}$, $\mathscr{T}=\{\emptyset,X,\{c\}\}$, and $\mathscr{U}=\{\emptyset,Y\}$. Define $f:X\to Y$ by $f(a)=a$, $f(b)=b$, and $f(c)=a$. Then f is onto, $f:(X,\mathscr{S})\to(Y,\mathscr{U})$ is somewhat open but does not fulfill (2) of Theorem 11, and $f:(X,\mathscr{T})\to(Y,\mathscr{U})$ fulfills (2) of Theorem 11 but is not somewhat open.

Theorem 12. *If* $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ *is somewhat open and* A *is an open subset of* X, *then* $f|_A : (A, \mathcal{S}_A) \to (Y, \mathcal{T})$ *is somewhat open.*

Theorem 13. *If* $(X, \mathcal{S})$ *and* $(Y, \mathcal{T})$ *are spaces and* A *is a dense subset of* X *and* $f : (A, \mathcal{S}_A) \to (Y, \mathcal{T})$ *is somewhat open, then any extension* F *of* f *mapping* $(X, \mathcal{S})$ *into* $(Y, \mathcal{T})$ *is somewhat open.*

Theorem 14. *If* $(X, \mathcal{S})$ *and* $(Y, \mathcal{T})$ *are spaces and* $X = A \cup B$ *where* A *and* B *are subsets of* X *and* $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ *is a function such that* $f|_A$ *and* $f|_B$ *are somewhat open, then* f *is somewhat open.*

It is useful to observe that two topologies $\mathcal{T}$ and $\mathcal{T}'$ for X are weakly equivalent if and only if the identity function from $(X, \mathcal{T})$ onto $(X, \mathcal{T}')$ is somewhat open in both directions.

Theorem 15. *If* $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ *is a somewhat open function and* $\mathcal{S}'$ *and* $\mathcal{T}'$ *are topologies for* X *and* Y *respectively such that* $\mathcal{S}'$ *is weakly equivalent to* $\mathcal{S}$ *and* $\mathcal{T}'$ *is weakly equivalent to* $\mathcal{T}$, *then* $f : (X, \mathcal{S}') \to (Y, \mathcal{T}')$ *is somewhat open.*

4. TOPOLOGICAL PROPERTIES THAT CARRY OVER UNDER OUR WEAKER FUNCTIONS

Theorem 16. *If* f *is a somewhat continuous function from* X *onto* Y *and* X *is separable, then* Y *is separable.*

Definition 4. A function $f : X \to Y$ is said to be a *somewhat homeomorphism* provided f is one-to-one, onto, somewhat continuous, and somewhat open.

It is clear that if f is a somewhat homeomorphism, then f^{-1} is a somewhat homeomorphism.

Theorem 17. *Let* $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ *be a somewhat homeomorphism. If* A *is a nowhere dense subset of* X, *then* $f(A)$ *is a nowhere dense subset of* Y.

The following corollary follows immediately from Theorem 17.

Corollary 17.1. *Let* $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ *be a somewhat homeomorphism. If* $(X, \mathcal{S})$ *is of first category (second category), then* $(Y, \mathcal{T})$ *is of first category (resp. second category).*

Theorem 18. *If* $(X, \mathcal{T})$ *is a topological space and* $\mathcal{T}'$ *is a topology for* X *which is weakly equivalent to* $\mathcal{T}$, *then* $(X, \mathcal{T})$ *is a Baire space (first category) (second category) if and only if* $(X, \mathcal{T}')$ *is a Baire space (resp. first category) (resp. second category).*

Proof. From previous comments the identity function from $(X, \mathcal{T})$ onto $(X, \mathcal{T}')$ is a somewhat homeomorphism and thus the result follows from [2, Coro. 1, p. 383] and Corollary 17.1.

LEVINE defines in [3] the notion of a D-space. A topological space $(X, \mathcal{T})$ is said to be a D-space provided every nonempty open subset of X is dense in X.

Theorem 19. *If $f:(X, \mathcal{S}) \to (Y, \mathcal{T})$ is a somewhat continuous function from X onto Y and X is a D-space, then Y is a D-space.*

Theorem 20. *Suppose $f: X \to Y$ is a continuous, somewhat open function from X onto Y. If X is locally compact on a dense subset of X (i.e. there is a dense subset D of X such that if $p \in D$, then there is an open subset U of X containing p such that $\overline{U}$ is compact) and Y is Hausdorff, then Y is locally compact on a dense subset of Y.*

Proof. Let U be a nonempty open subset of Y. If we can find a point of U at which Y is locally compact we will have proved the theorem. Since f is continuous and onto, $f^{-1}(U)$ is a nonempty open subset of X. Since X is locally compact on a dense subset of X, there is an element $x \in f^{-1}(U)$ such that X is locally compact at x. Thus we can find an open subset W of X containing x and contained in $f^{-1}(U)$ such that $\overline{W}$ is compact. Since W is a nonempty open subset of X and f is somewhat open, there is a nonempty open subset V of Y such that $V \subset f(\overline{W})$. Let $z \in V$. Then $z \in U$. Since Y is Hausdorff and $f(\overline{W})$ is compact, $f(\overline{W})$ is closed. Thus $\overline{V} \subset f(\overline{W})$. Hence $\overline{V}$ is compact and thus Y is locally compact at z which is a point of U.

The following example shows that the continuous, somewhat open image of a locally compact space need not be locally compact everywhere.

Example 9. Let X = Reals − {nonzero integers}, let $\mathcal{S}$ be the induced topology for X gotten from the usual topology for the reals, and let $\mathcal{T}$ be the topology for X which has as a base

$$(\mathcal{S} - \{V \mid 0 \in V, V \in \mathcal{S}\}) \cup \{[(-\infty, -r) \cup (-(1/r), 1/r) \cup (r, \infty)] \cap X \mid r > 0\}.$$

Then $(X, \mathcal{S})$ is locally compact and $(X, \mathcal{T})$ is Hausdorff. The identity function from $(X, \mathcal{S})$ onto $(X, \mathcal{T})$ is continuous and somewhat open. Yet $(X, \mathcal{T})$ is not locally compact at 0.

The following example shows that there is no hope for any separation axioms carrying over under somewhat homeomorphisms.

Example 10. Let X be the reals and let $\mathcal{S}$ be the usual topology for X. Let $\mathcal{T}$ be the topology for X gotten by throwing out all the elements of $\mathcal{S}$ which contain either 0 or 1 except X. Then the identity function from $(X, \mathcal{S})$ onto $(X, \mathcal{T})$ is a somewhat homeomorphism. However $(X, \mathcal{S})$ is a metric space and $(X, \mathcal{T})$ is not even a T_0-space.

5. WHEN SOMEWHAT CONTINUOUS (SOMEWHAT OPEN) IS EQUIVALENT TO CONTINUOUS (RESP. OPEN)

Theorem 21. *Let $(X, \mathcal{S}, \cdot)$ and $(Y, \mathcal{T}, \circ)$ be topological groups. If $f : (X, \mathcal{S}, \cdot) \to$ $\to (Y, \mathcal{T}, \circ)$ is a somewhat continuous homomorphism, then f is continuous.*

Proof. Let e be the identity element of $(X, \cdot)$ and let i be the identity element of $(Y, \circ)$. Let U be an element of $\mathcal{T}$ which contains i. Then there is a $W \in \mathcal{T}$ such that $i \in W$ and $W \circ W^{-1} \subset U$. Since f is a homomorphism, $f(e) = i$ and thus $f^{-1}(W) \neq \emptyset$. Since f is somewhat continuous there is a $V \in \mathcal{S}$ such that $V \neq \emptyset$ and $V \subset f^{-1}(W)$. Let $p \in V$. Then clearly $e \in V \cdot p^{-1}$ and $V \cdot p^{-1} \in \mathcal{S}$. Since $f(V) \subset W$ and f is a homomorphism,

$$f(V \cdot p^{-1}) \subset f(V) \circ f(p^{-1}) = f(V) \circ [f(p)]^{-1} \subset W \circ W^{-1} \subset U .$$

Hence by [4, Th. B, Part a, p. 63], f is continuous.

Theorem 22. *Let $(X, \mathcal{S}, \cdot)$ and $(Y, \mathcal{T}, \circ)$ be topological groups. If $f : (X, \mathcal{S}, \cdot) \to$ $\to (Y, \mathcal{T}, \circ)$ is a somewhat open homomorphism, then f is open.*

Proof. This follows from the proof of [4, Th. 13, p. 65] and observing that in this proof, Pontrjagin first proves a condition equivalent to somewhat open and using this proves the function is open.

The following example shows that f being a homomorphism can not be omitted in Theorem 21 and Theorem 22.

Example 11. Let X be the reals, $\mathcal{T}$ the usual topology for X, and $\cdot$ the usual addition for X. Then $(X, \mathcal{T}, \cdot)$ is a topological group. Then the function f defined by

$$f(x) = \begin{cases} x & \text{if} \quad x \neq 0 \quad \text{and} \quad x \neq 1 \\ 0 & \text{if} \quad x = 1 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is a one-to-one, onto, somewhat continuous, somewhat open function from $(X, \mathcal{T}, \cdot)$ onto $(X, \mathcal{T}, \cdot)$ which is neither continuous nor open.

Theorem 23. *Suppose $(X, \mathcal{T}, \cdot)$ and $(X, \mathcal{T}', \cdot)$ are topological groups. If $\mathcal{T}$ is weakly equivalent to $\mathcal{T}'$, then $\mathcal{T} = \mathcal{T}'$.*

Proof. This result follows from Theorem 21 applied on the identity function in both directions.

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