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LATTICE FORMULATION OF GENERAL ALGEBRAIC DEPENDENCE

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1. INTRODUCTION

Presently, the treatment of dependence in the lattice theory follows much the same line (e.g. G. BIRKHOFF [1], M. L. DUBREIL-JACOTIN, L. LESIEUR and R. CROISOT [10], H. HERMES [11], D. E. RUTHERFORD [16] or G. SZÁSZ [18]):

A finite subset I of atoms of a lattice L is said to be (linearly) independent if, for any $a \in I$, $a \nleq \bigvee_{x \in I \setminus (a)} x$. *If L is an (upper) semimodular lattice of finite length, then all maximal independent subsets of atoms of L have the same cardinality called the rank of the lattice L.*

Hand in hand, the concept of linear dependence, as well as the restriction to the atoms and to a particular type of lattices in this formulation impose limitations for certain algebraic applications. In some problems of universal algebra, the appropriate concept of dependence is that of direct dependence which may be defined as follows:[1])

A set I of elements of a (finitary) universal algebra $\mathscr{A}$ is said to be directly independent if, for every $x \in I$, the subalgebras generated, respectively by $\{x\}$ and by $I \setminus \{x\}$ intersect trivially (i.e. in the subalgebra of all constants of $\mathscr{A}$).

Such a concept has proved to be of value in abelian groups and, more generally, in modules and rings (e.g. L. FUCHS [9], the author [3], [5], [6] and [7]).

Observe that direct dependence is, in fact, defined on the set of all monogenic (i.e. one-generator) subalgebras of $\mathscr{A}$ and can be therefore naturally extended to the lattice of all subalgebras of $\mathscr{A}$. The next obvious step is to modify the definition of direct

[1]) To be distinguished from some other concepts of dependence in algebras (cf. e.g. E. MARCZEWSKI [12] and J. SCHMIDT [17]). As one would expect, in algebras of some type the notions coincide; for example, Marczewski's dependence coincide with direct dependence in v- and v^*-algebras of [13] and [14], respectively. Incidentally, this may explain the fact that the authors have succeeded in establishing an invariant rank in these cases.

dependence for arbitrary lattices (Definition 4) and study this concept in the framework of the lattice theory; this is the modus operandi of the present paper.

We shall introduce certain properties of lattices to be able to prove the invariance of the rank of a lattice, as well as of some other cardinal numbers (useful in universal algebra applications). In effect, we shall show that, with respect to direct dependence, a subset of a lattice is under certain conditions a GA-dependence structure of [2] (see also [4]). We can see in [8] that these conditions are very natural. For, in [8], every (abstract) GA-dependence structure is shown to be of such a lattice direct dependence type.

As an application of our results to an arbitrary (upper) semimodular lattice, we get the following extension of the result mentioned at the very beginning of our introduction:

All maximal independent subsets of atoms of a semimodular lattice have the same cardinality.

Also, the relation between direct dependence and J. VON NEUMANNS's dependence of [15] is described.

2. PRELIMINARIES

Unless stated otherwise, L is always a lattice with the least and the greatest element, denoted by 0 and 1, respectively. The symbols $\wedge$ and $\vee$ are used to denote the meet- and join-operation in L; in particular, for a subset $X \subseteq L$, $\bigwedge(X)$ and $\bigvee(X)$ denote $\bigwedge_{x \in X} x$ and $\bigvee_{x \in X} x$ (provided that they exist), respectively.

Definition 1. *Two non-zero elements a, b of L are said to be essentially equal (in symbol, $a \overset{e}{\sim} b$) if, for every $x \in L$,*

$$x \wedge a \neq 0 \quad \textit{if and only if} \quad x \wedge b \neq 0 .$$

If, in particular, $a \leqq b$, we shall say that a is essential in b.

One can see easily that a meet of two elements essential in $b \in L$ is essential in b, as well, and that for every triple $a \leqq b \leqq c$ of non-zero elements of L, a is essential in c if and only if a is essential in b and b is essential in c.

Definition 2. *A non-zero element $u \in L$ is said to be uniform if every $x \in L$ such that $0 \neq x \leqq u$ is essential in u. Denote the set of all uniform elements of L by U_L.*

Notice that $0 \neq v \leqq u$ and $u \in U_L$ imply that $v \in U_L$. We shall need also the following concept of a tidy lattice or, more generally, of a tidy subset of a lattice.

Definition 3. *A subset S of a lattice L is said to be tidy if, for every non-zero $a \in S$, there is a uniform element $u \in S$ such that $u \leqq a$.*

Since every atom of L is trivially a uniform element of L, we can conclude that every subset S of L having the property that each of its elements contains an atom of L which belongs to S, is tidy. In particular, every lattice satisfying minimum condition, or having finite length is tidy. More generally, an atomistic lattice, i.e. a lattice L such that, for every $x \in L$ there is an atom $a \in L$ with $a \leqq x$, is tidy. Observe that, if L is tidy and the join $\bigvee(U_L)$ exists, then $\bigvee(U_L)$ is essential in $1 \in L$.

3. THE CONCEPT OF DIRECT DEPENDENCE

Although we assume, as before, L to be an arbitrary lattice with 0 and 1, the reader may find it helpful to specialize the formulations to the case when L is a complete algebraic lattice, i.e. a lattice of all subalgebras of a universal algebra (see [2]); in view of [8], this is a very natural test case. In our presentation, we include several brief remarks to this effect.

The letter S denotes again a (fixed) subset of non-zero elements of L; the set of all non-zero elements of L will be denoted by L^*.

Definition 4. *A subset I of L is said to be d-independent (directly independent) if $I \subseteq L^*$ and if, for every $a \in I$ and every finite subset F of $I \setminus \{a\}$,*

$$a \wedge \bigvee(F) = 0 .$$

A subset of L^* which is not d-independent will be called d-dependent.

Given a subset S of L^*, denote the set of all d-independent subsets contained in S ($\emptyset$ including) by $\mathscr{I}_{S,L}$ and the set of all maximal (by inclusion) elements of $\mathscr{I}_{S,L}$ by $\mathscr{M}_{S,L}$; furthermore, denote by $\mathscr{I}^{\circ}_{S,L}$ the set of all independent subsets of uniform elements of S and by $\mathscr{M}^{\circ}_{S,L}$ the set of all maximal elements of $\mathscr{I}^{\circ}_{S,L}$.

Trivially, every subset of a d-independent subset is d-independent. Equally easily, we can see that a subset I belongs to $\mathscr{I}_{S,L}$ if and only if every finite subset of I belongs to $\mathscr{I}_{S,L}$.[2]) Thus, applying Zorn's lemma we have the first part of the following

Proposition 1. *For every $I \in \mathscr{I}_{S,L}$, or $I^{\circ} \in \mathscr{I}^{\circ}_{S,L}$, there exists $M \in \mathscr{M}_{S,L}$, or $M^{\circ} \in \mathscr{M}^{\circ}_{S,L}$ containing I, or I°, respectively. In particular,*

$$\mathscr{M}_{S,L} \neq \emptyset \quad \text{and} \quad \mathscr{M}^{\circ}_{S,L} \neq \emptyset .$$

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[2]) I.e. $(S, \mathscr{I}_{S,L})$ is an A-dependence structure in the sense of [4].

Evidently, always $\mathscr{I}^{\circ}_{S,L} \subseteq \mathscr{I}_{S,L}$. *If S is a tidy subset, then also*

$$\mathscr{M}^{\circ}_{S,L} \subseteq \mathscr{M}_{S,L}\,.$$

Proof. In order to complete the proof, assume that the last inclusion does not hold, i.e. that there exists

$$M^{\circ} \in \mathscr{M}^{\circ}_{S,L} \setminus \mathscr{M}_{S,L}\,.$$

In view of the first part of our proposition, there is a set $M \in \mathscr{M}_{S,L}$ properly containing M°. Take $a \in M \setminus M^{\circ}$. Since S is tidy, there exists $u \in U_L$ such that $u \leqq a$. Now, obviously, $M^{\circ} \cup \{u\} \notin \mathscr{I}_{S,L}$ and thus, there is a finite subset F of M° and an element $v \in F \cup \{u\}$ satisfying

$$v \wedge \bigvee((F \cup \{u\}) \setminus \{v\}) \neq 0\,.$$

Hence, if $v \neq u$, i.e. if $v \in F$, then

$$v \wedge \bigvee((F \cup \{a\}) \setminus \{v\}) \neq 0$$

and if $v = u$, then

$$a \wedge \bigvee(F) \geqq u \wedge \bigvee(F) \neq 0\,;$$

therefore, $F \cup \{a\} \notin \mathscr{I}_{S,L}$ – a contradiction.

Remark. Let us point out that in an atomistic complete algebraic lattice L, a subset I of L^* is d-independent if and only if

$$a \wedge \bigvee(I \setminus \{a\}) = 0 \quad \text{for every} \quad a \in I\,.$$

Definition 5. *A subset S of* L^* *is said to be balanced if, for every* $u_j \in S \cap U_L$ *and every finite* $I^{\circ} \in \mathscr{I}^{\circ}_{S,L}$ *such that* $u_j \wedge \bigvee(I^{\circ}) = 0$ $(j = 1, 2)$,

$$u_1 \wedge (u_2 \vee \bigvee(I^{\circ})) \neq 0 \quad \text{if and only if} \quad u_2 \wedge (u_1 \vee \bigvee(I^{\circ})) \neq 0\,.$$

If U_L *is balanced, we call simply the lattice L balanced.*

Remark. Notice that every modular lattice is balanced. For, in such a lattice,

$$a \vee [x_1 \wedge (a \vee x_2)] = (a \vee x_1) \wedge (a \vee x_2) = a \vee [x_2 \wedge (a \vee x_1)]$$

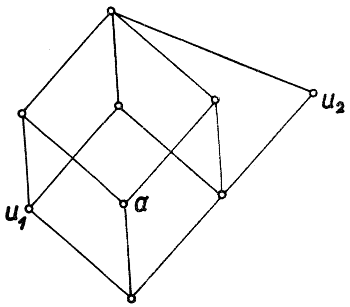
holds in general for any three elements; thus, if

$$a \wedge x_1 = a \wedge x_2 = 0\,,$$

then

$$x_1 \wedge (a \vee x_2) = 0 \quad \text{is equivalent to} \quad x_2 \wedge (a \vee x_1) = 0\,.$$

On the other hand, not every (upper) semimodular lattice is balanced; this can be illustrated simply by the following diagram:



However, one can see easily that the set of all atoms of a semimodular lattice is balanced (cf. G. SZÁSZ [18], p. 150).

Proposition 2. *Let $S \subseteq L^*$ be a balanced subset. Then, for every finite $I \subseteq L$ and $u \notin I$,*

$$I \cup \{u\} \in \mathscr{I}^{\circ}_{S,L} \quad \textit{if and only if} \quad I \in \mathscr{I}^{\circ}_{S,L} \,\&\, u \in S \cap U_L \,\&\, u \wedge \bigvee(I) = 0 \,.$$

Proof. Only sufficiency requires a proof. Assume that $I \cup \{u\} \notin \mathscr{I}^{\circ}_{S,L}$; this means that there is $u' \in I$ such that

$$u' \wedge (u \vee \bigvee(I \setminus \{u'\})) \neq 0 \,.$$

Hence, since S is balanced,

$$u \wedge (u' \vee \bigvee(I \setminus \{u'\})) = u \wedge \bigvee(I) \neq 0 \,,$$

in contradiction to our assumption.

Remark. Evidently, in an atomistic complete algebraic lattice L, the condition in Definition 5 that I be finite can be waived. The same remark applies to several propositions of this section; as a matter of fact, some formulations can be simplified as a consequence. Also, for the same reason, some results can be extended; a typical instance is the following corollary of Proposition 2:

$$M^{\circ} \in \mathscr{M}^{\circ}_{S,L} \quad \text{if and only if} \quad u \wedge \bigvee(M^{\circ}) \neq 0 \quad \text{for every} \quad u \in S \cap U_L \,.$$

Lemma. *Let $S \subseteq L^*$ be a balanced subset. Let ψ be a one-to-one mapping of a subset I of $S \cap U_L$ into $S \cap U_L$ such that*

$$\psi(x) \sim x \quad \textit{for all} \quad x \in I \,.$$

Then,

$$I \in \mathscr{I}^{\circ}_{S,L} \quad \text{if and only if} \quad \psi(I) \in \mathscr{I}^{\circ}_{S,L} .$$

As a consequence, in the case when $I \in \mathscr{I}^{\circ}_{S,L}$ *is finite,*

$$u \wedge \bigvee(I) \neq 0 \quad \text{if and only if} \quad u \wedge \bigvee(\psi(I)) \neq 0 ,$$

for every $u \in S \cap U_L$.

Proof. It is sufficient to prove that $I \in \mathscr{I}^{\circ}_{S,L}$ implies $\psi(I) \in \mathscr{I}^{\circ}_{S,L}$. For, the opposite implication can be obtained in the same manner (considering the mapping ψ^{-1} in place of ψ) and the rest of the lemma follows easily from the preceding Proposition 2.

First, notice that, for every finite subset F of I and every $x \in F$,

$$\psi(F \setminus \{x\}) \cup \{x\} \in \mathscr{I}^{\circ}_{S,L} \quad \text{implies} \quad \psi(F) \in \mathscr{I}^{\circ}_{S,L} .$$

For, if $\psi(F) \notin \mathscr{I}^{\circ}_{S,L}$, then, in view of Proposition 2,

$$\psi(x) \wedge \bigvee(\psi(F \setminus \{x\})) \neq 0 ,$$

and thus

$$x \wedge \bigvee(\psi(F \setminus \{x\})) \neq 0 ,$$

because of $\psi(x) \overset{\mathfrak{e}}{\sim} x$. But the latter relation contradicts our hypothesis. Hence, by an induction argument, we can easily deduce that all finite subsets of $\psi(I)$ belong to $\mathscr{I}^{\circ}_{S,L}$; thus, $\psi(I) \in \mathscr{I}^{\circ}_{S,L}$ as well, as required.

The preceding lemma enables us to formulate several consequences. The following one expressing the "canonic zone" property of U_L is important.[3])

Proposition 3. *Let* $S \subseteq L^*$ *be a balanced subset. Let* $I^{\circ} \in \mathscr{I}^{\circ}_{S,L}$ *be finite and* $u \in S \cap U_L$ *such that*

$$u \wedge \bigvee(I^{\circ}) \neq 0 .$$

Let $a \in L$ *such that* $x \wedge a \neq 0$ *for each* $x \in I^{\circ}$. *Then, also* $u \wedge a \neq 0$.

Proof. But, for each $x \in I^{\circ}$, $\psi(x) = a \wedge x$; clearly, $\psi(x) \overset{\mathfrak{e}}{\sim} x$. Hence, by Lemma,

$$u \wedge \bigvee(\psi(I)) \neq 0 .$$

Since, obviously, $a \geqq \bigvee(\psi(I))$, we have $a \wedge a \neq 0$, as required.

The following proposition is of basic importance.

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[3]) In the terminology of [2] (cf. also [4]), $U_L \cap S \subseteq S$ possesses the properties of a canonic zone of $(S, \mathscr{I}_{S,L})$; the closure operation $\mathfrak{c} : \mathscr{I}_{S,L} \to 2^S$ of [4] is related to our present notation, evidently, by $\mathfrak{c}(I) = \{x \mid x \in S \;\&\; x \wedge \bigvee(F) \neq 0$ for a finite $F \subseteq I\}$ and Proposition 3 therefore represents the implication "$I^{\circ} \subseteq \mathfrak{c}(I)$ implies $\mathfrak{c}(I^{\circ}) \subseteq \mathfrak{c}(I)$".

Proposition 4. *Let $S \subseteq L^*$ be a balanced subset. Let M_1° and M_2° belong to $\mathscr{M}_{S,L}^\circ$. Then*

$$\operatorname{card}(M_1^\circ \setminus M_2^\circ) = \operatorname{card}(M_2^\circ \setminus M_1^\circ).$$

Proof. We propose to give a proof in three steps. Notice that, for every $x \in M_2^\circ \setminus M_1^\circ$, there is a finite subset F_x of M_1° such that

$$x \wedge \bigvee(F_x) \neq 0 .$$

Denote the union $\bigcup_{x \in M_2^0 \setminus M_1^0} F_x$ by I; hence, $I \subseteq M_1^\circ$.

(i) First, we show that

$$M_1^\circ \setminus M_2^\circ \subseteq I .$$

For, assuming that $u \in M_1^\circ \setminus (I \cup M_2^\circ)$, we can find a finite subset F of M_2° such that

$$u \wedge \bigvee(F) \neq 0 .$$

Also, writing $a = \bigvee(\bigcup_{x \in F} F_x)$, evidently

$$x \wedge a \neq 0 \quad \text{for each} \quad x \in F .$$

Hence, according to Proposition 3, $u \wedge a \neq 0$ – a contradiction of d-independence of M_1°.

(ii) Let $M_2^\circ \setminus M_1^\circ$ be infinite. In view of (i),

$$\operatorname{card}(M_1^\circ \setminus M_2^\circ) \leqq \operatorname{card}(I) .$$

On the other hand, evidently

$$\operatorname{card}(I) \leqq \aleph_0 \operatorname{card}(M_2^\circ \setminus M_1^\circ) = \operatorname{card}(M_2^\circ \setminus M_1^\circ) .$$

Hence

$$\operatorname{card}(M_1^\circ \setminus M_2^\circ) \leqq \operatorname{card}(M_2^\circ \setminus M_1^\circ) ,$$

and the equality follows because of symmetry.

(iii) If $M_2^\circ \setminus M_1^\circ$ is finite, then, by (ii), $M_1^\circ \setminus M_2^\circ$ is finite, as well. We shall prove the following statement equivalent to the required equality (the equivalence can be shown easily in a routine manner by induction):

Given an arbitrary $u_1 \in M_1^\circ \setminus M_2^\circ$, there is $u_2 \in M_2^\circ \setminus M_1^\circ$ such that

$$(M_1^\circ \setminus \{u_1\}) \cup \{u_2\} \in \mathscr{M}_{S,L}^\circ .$$

For, given $u_1 \in M_1^\circ \setminus M_2^\circ$, we know by (i) that there is $u_2 \in M_2^\circ \setminus M_1^\circ$ such that

$$u_2 \wedge \bigvee(M_1^\circ \setminus \{u_1\}) = 0 .$$

Hence, in view of Proposition 2, we conclude

$$M_0 = (M_1^\circ \setminus \{u_1\}) \cup \{u_2\} \in \mathscr{I}_{S,L}^\circ .$$

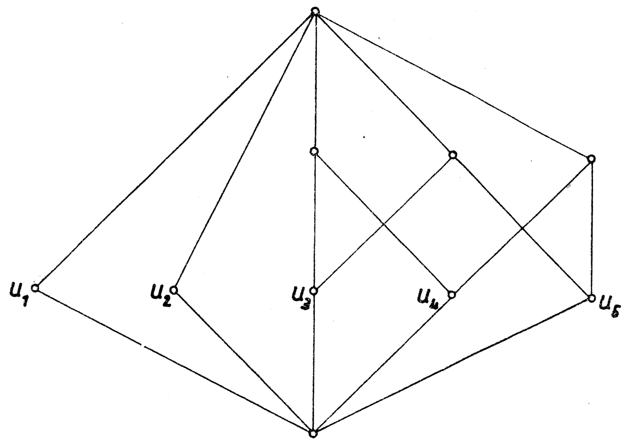
Notice that

$$u_1 \wedge \bigvee(M_0) \neq 0 ;$$

this follows from the fact that $u_2 \wedge (u_1 \vee \bigvee(M_1^\circ \setminus \{u_1\})) \neq 0$ and that S is balanced. But then we obtain, using Proposition 3 (taking $a = \bigvee(M_0)$) that $M_0 \in \mathscr{M}_{S,L}^\circ$.

The proof of Proposition 4 is completed.

The necessity for assuming in the above proposition the subset S to be balanced is obvious: The subsets $\{u_1, u_2\}$ and $\{u_3, u_4, u_5\}$ of the lattice given by the diagram



both belong to $\mathscr{M}_{L,L}^\circ$, but have different numbers of elements.

Now, we can introduce the following concept of rank of a lattice.

Definition 6. *Let S be a balanced subset of non-zero elements of L. Then the common cardinality of all maximal d-independent subsets of $S \cap U_L$ is called the S-rank* $r_S(L)$ *of the lattice L; in the case when $S = L$, we speak simply about the rank* $r(L)$ *of L.*[4])

Theorem. *If S is balanced tidy subset of non-zero elements of a lattice L, then*

$$\operatorname{card}(M) \leq \operatorname{card}(M_1^\circ) = \operatorname{card}(M_2^\circ) = r_S(L)$$

[4]) We can call this rank uniform (or irreducible; cf. [6]) in contrast to the (total) rank of L which can be defined as $\sup_{M \in \mathscr{M}_{L,L}} \operatorname{card}(M)$. The tidiness of L ensures equality of both these cardinals (cf. Theorem).

for all $M \in \mathcal{M}_{S,L}$ *and* $M_1^\circ, M_2^\circ \in \mathcal{M}_{S,L}^\circ$, *i.e.*

$$\mathrm{r}_S(L) = \sup_{M \in \mathcal{M}_{S,L}} \mathrm{card}(M).$$

Thus, if L *is a balanced tidy lattice, then*

$$\mathrm{r}(L) = \sup_{M \in \mathcal{M}_{L,L}} \mathrm{card}(M) = \mathrm{r}_S(L)$$

for every subset S *containing* U_L.[5])

Proof. Since S is tidy, there exists – for every $a \in M$ – an element $\psi(A) \in U_L$ such that $\psi(a) \leqq a$. Notice that in view of d-independence of M, the correspondence ψ is one-to-one. Moreover,

$$\psi(M) \in \mathcal{I}_{S,L}^\circ .$$

For, otherwise there is $a \in M$ and a finite subset $F \subseteq M \setminus \{a\}$ such that

$$\psi(a) \wedge \bigvee(\psi(F)) \neq 0,$$

and therefore

$$a \wedge \bigvee(F) \neq 0,$$

a contradiction of $M \in \mathcal{I}_{S,L}$. Now, the first part of Theorem follows easily from Propositions 1 and 4, and the second part is obvious.

Since the set of all atoms of a semimodular lattice is balanced, we get the following

Corollary. *If* L *is an (upper) simimodular lattice, then any two maximal d-independent subsets of atoms of* L *have the same cardinality. If* L *is moreover, atomistic, this cardinality equals to the rank* $\mathrm{r}(L)$ *of* L.

Let us conclude this section with a remark on the following stronger concept of independence in lattices (cf. J. VON NEUMANN [15]):

A subset I *of lattice* L *is said to be d*-independent if* $I \subseteq L^*$ *and if, for every two finite disjoint subsets* F_1 *and* F_2 *of* I,

$$\bigvee(F_1) \wedge \bigvee(F_2) = 0.$$

Evidently, if I is d*-independent, then it is also d-independent; in general, the converse is not true. However, here we like to show that, in a balanced tidy lattice, both concepts lead to the same rank invariant and that our Theorem interpreted

[5]) Or, more generally, for every subset S satisfying $\mathcal{M}_{S,L}^\circ \cap \mathcal{M}_{L,L}^\circ \neq \emptyset$. Notice also that if in a balanced lattice L the first equality does not hold, i.e. if L is not tidy, necessarily $\sup_{M \in \mathcal{M}_{L,L}} \mathrm{card}(M) \geqq \aleph_0$.

with respect to d*-independence holds, as well. This can immediately be deduced from the following

Proposition 5. *If L is a balanced tidy lattice and $I \subseteq U_L$, then I is d*-independent if and only if it is d-independent.*

Proof. Only sufficiency requires to be verified; we present an indirect argument. Let $I \in \mathscr{I}^{\circ}_{L,L}$ and let

$$\bigvee(F_1) \wedge \bigvee(F_2) = a \neq 0$$

for suitable disjoint finite subsets F_1 and F_2 of I. Since L is tidy, there is $u \in U_L$ such that $u \leqq a$. Thus, $F_1 \cup \{u\} \notin \mathscr{I}^{\circ}_{L,L}$. Let us take a minimal dependent subset X of $F_1 \cup \{u\}$; evidently,

$$X = F \cup \{u\} \quad \text{with} \quad \emptyset \neq F \subseteq F_1 .$$

Using Proposition 2, we obtain

$$u_0 \wedge \bigvee((F \cup \{u\}) \setminus \{u_0\}) \neq 0 \quad \text{with} \quad u_0 \in F .$$

Now, denoting the subset $(F \setminus \{u_0\}) \cup F_2$ of I by F_0, we have clearly

$$x \wedge \bigvee(F_0) \neq 0 \quad \text{for each} \quad x \in (F \cup \{u\}) \setminus \{u_0\} .$$

Therefore, applying Proposition 3, we deduce

$$u_0 \wedge \bigvee(F_0) \neq 0 ,$$

a contradiction of $I \in \mathscr{I}^{\circ}_{L,L}$.

4. ADMISSIBLE DECOMPOSITIONS AND RANKS

In this final section, we intend to deal briefly with a method of decomposition of a given invariant of a lattice into more refined ones in a certain admissible manner. Lattice-theoretical presentation of dependence enables us to treat this question quite easily. Let us point out that the results are useful for applications (cf. [6], [7]).

Definition 7 (cf. H. WHITNEY [19]). *By a circuit of a lattice L we understand a minimal d-dependent subset of L.*

Given a subset S, denote the set of all circuits of L contained in S, or in $S \cap U_L$, by $\mathscr{C}_{S,L}$, or by $\mathscr{C}^{\circ}_{S,L}$, respectively. Thus, $C \in \mathscr{C}_{S,L}$ if and only if $C \notin \mathscr{I}_{S,L}$, but $C \setminus \{x\} \in \mathscr{I}_{S,L}$ for each $x \in C$. Evidently, if $I \in \mathscr{I}_{S,L}$ and $a \in S$ such that $I \cup \{a\} \notin \mathscr{I}_{S,L}$, then there exists $C \in \mathscr{C}_{S,L}$ with $a \in C$ and $C \subseteq I \cup \{a\}$. Notice also that a circuit is always finite and that, if S is balanced, then every circuit $C^\circ \in \mathscr{C}^\circ_{S,L}$ satisfies

$$x \wedge (C^\circ \setminus \{x\}) \neq 0 \quad \text{for each} \quad x \in C^\circ$$

(cf. Proposition 2).

Definition 8. *Let S be a subset of L^*. A partition $\{S_\omega \mid \omega \in \Omega\}$ of S is said to be d-admissible if, for every $C^\circ \in \mathscr{C}^\circ_{S,L}$ and every $\omega \in \Omega$, the intersection $C^\circ \cap S_\omega$ equals C° or $\emptyset$.*

Notice that if S is balanced, then all S_ω are balanced. Also, if all S_ω are tidy, then S is tidy. The main result of this section reads as follows.

Theorem. *Let $\{S_\omega \mid \omega \in \Omega\}$ be a d-admissible partition of a balanced subset S of L^*. Let $X_\omega \subseteq S_\omega$ for each $\omega \in \Omega$ and $X = \bigcup_{\omega \in \Omega} X_\omega$. Then,*

$$X \in \mathscr{M}^\circ_{S,L} \quad \text{if and only if} \quad X_\omega \in \mathscr{M}^\circ_{S_\omega,L} \quad \text{for all} \quad \omega \in \Omega .$$

As a consequence,

$$\mathrm{r}_S(L) = \sum_{\omega \in \Omega} \mathrm{r}_{S_\omega}(L) .$$

Proof. In order to prove sufficiency notice first that X is obviously d-independent. For, otherwise there is a circuit C contained in X and thus $C \subseteq X_{\omega_0}$ for a suitable $\omega_0 \in \Omega$ – a contradiction. Maximality of X in $\mathscr{I}^\circ_{S,L}$ is trivial.

On the other hand, suppose $X \in \mathscr{M}^\circ_{S,L}$. Trivially, all $X_\omega = X \cap S_\omega$, $\omega \in \Omega$, are d-independent. Taking an arbitrary $\omega_0 \in \Omega$ and $u_0 \in (S_{\omega_0} \cap U_L) \setminus X_{\omega_0}$, we deduce $X \cup \{u_0\} \notin \mathscr{I}_{S,L}$ and, consequently, the existence of $C^\circ \in \mathscr{C}^\circ_{S,L}$ such that

$$u_0 \in C^\circ \quad \text{and} \quad C^\circ \subseteq X \cup \{u_0\} .$$

Thus, $C^\circ \cap S_{\omega_0} \neq \emptyset$ and therefore, $C^\circ \subseteq X_{\omega_0} \cup \{u_0\}$; the maximality of X_{ω_0} in $\mathscr{I}^\circ_{S_{\omega_0},L}$ follows.

There is always a particular d-admissible partition of a given subset $S \subseteq L^*$, namely

$$\{S_0, S \setminus S_0\}$$

with S_0 being defined as the subset of all *neutral* elements of S, i.e. of all elements not belonging to any circuit in S. Thus,

$$S \setminus S_0 = \bigcup_{C \in \mathscr{C}_{S,L}} C ,$$

or alternatively,

$$S_0 = \bigcap_{M \in \mathscr{M}_{S,L}} M .$$

The equivalence of both definitions is obvious; for,

$$S \setminus S_0 = S \setminus \bigcap_{M \in \mathcal{M}_{S,L}} M = \bigcup_{M \in \mathcal{M}_{S,L}} (S \setminus M) = \bigcup_{C \in \mathcal{C}_{S,L}} C .$$

Hence, for a balanced S, the rank $\mathrm{r}_S(L)$ always splits into $\mathrm{r}_{S_0}(L)$ and $\mathrm{r}_{S \setminus S_0}(L)$. Notice also that in the case when S is tidy, necessarily $S_0 \subseteq U_L$, and therefore

$$\mathrm{r}_{S_0}(L) = \mathrm{card}\,(S_0) .$$

Let us point out that, given $S \subseteq L^*$, one can very easily describe the finest d-admissible partition of S: Two elements a, b of S belong to the same subset of that partition if and only if there is a finite sequence of circuits $C_1, C_2, \ldots, C_n$ in S such that

$$C_i \cap C_{i+1} \neq \emptyset \quad \text{for} \quad 1 \leqq i \leqq n-1 \quad \text{and} \quad a \in C_1 , \quad b \in C_n$$

(notice that this condition defines an equivalence). Correspondingly, this partition yields the "finest" d-invariants of the lattice L derived from $\mathrm{r}_S(L)$.

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