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ON ACCESSIBILITY OF BILINEAR SYSTEMS

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In this paper we will present an explicit formula for solutions of a bilinear system

$$\dot{x} = \left(\sum_{i=1}^{\alpha} A_i u_i\right) x + \sum_{j=1}^{\beta} b_j v_j\,, \tag{1}$$

where A_i, $i = 1, 2, \ldots, \alpha$, are n-by-n matrices and b_j, $j = 1, 2, \ldots, \beta$, are vectors (both independent on time), and $w = (u, v) = (u_1, \ldots, u_\alpha, v_1, \ldots, v_\beta)$ ranges the set W of all vector-functions which are measurable on $[0, \infty)$ and have values in an interval $[-1, 1]^{\alpha+\beta} \subset E_{\alpha+\beta}$.

Further, we will construct an involutive distribution V on E_n (using the terminology of [2]) and show that the set of all points accessible along solutions of (1), which fulfil an initial condition $x(0) = \omega$, is just the maximal integral manifold of V which passes through ω.

Notations. We use Euclidean norm $\|.\|$ in E_n. Dimension of a finite-dimensional vector space $\mathscr{V}$ is denoted by $\dim \mathscr{V}$. Symbol $\{p \in P;\ P(p)\}$ represents the set of all elements $p \in P$ with property $P(p)$. Any solution $x(.)$ of (1) corresponding to $w \in W$ and fufilling an initial condition $x(0) = \omega$ is denoted by $x(., w, \omega)$. Finally, by I we denote a unit matrix.

A connected set $S \subset E_n$ is called an r-dimensional manifold if for each $x \in S$ there is an open nonempty set $G \subset E_r$ and an injection $\varphi : G \to S$ such that

1. $x \in \varphi(G)$,
2. $\varphi(G)$ is open in S,
3. Jacobian $D\varphi/Dt$ is continuous on G and its rank is equal to r for all $t \in G$.

A set $S \subset E_n$ which contains only one element is called 0-dimensional manifold.

The matrices $A_1, \ldots, A_\alpha$, and the vectors $b_1, \ldots, b_\beta$, from (1) are fixed throughout the whole paper. We denote by $\mathfrak{A}$ the smallest linear space which contains the matrices $A_1, \ldots, A_\alpha$, and which with any two matrices $P, Q \in \mathfrak{A}$ contains also $QP - PQ$. In other words, $\mathfrak{A}$ is the smallest Lie algebra, with a bracket operation $[P, Q] = QP - PQ$, which contains $A_1, \ldots, A_\alpha$. Further, denote by $\mathfrak{B}$ the smallest linear space which contains all $b_1, \ldots, b_\beta$, and fulfils an implication $A \in \mathfrak{A}$, $b \in \mathfrak{B} \Rightarrow Ab \in \mathfrak{B}$.

Distributional equation. Associate with each $x \in E_n$ a vector space $V(x) = \{Ax + b;\ A \in \mathfrak{A},\ b \in \mathfrak{B}\}$. Such mapping is in [2] called distribution. Let us form an equation

$$\dot{x} \in V(x) \tag{2}$$

and call it distributional equation corresponding to the bilinear system (1).

Solution of (2) is any function $x(.)$ absolutely continuous on an interval $J \subset E_1$ which for almost all $t \in J$ fulfils $\dot{x}(t) \in V(x(t))$. Beside this type of solution we define a "global" solution of (2) as any manifold $S \subset E_n$ whose tangent space $T(x)$ at each $x \in S$ equals to $V(x)$. Such manifold is in [2] called integral manifold of V.

It is proved in [2] that if V does not change its dimension in E_n then for any $x \in E_n$ there exists an integral manifold of V which contains x. This assumption is not necessarily true in our case. Nevertheless, we will prove that the statement of this theorem remains true for our distribution V.

Lemma 1. *Let $A_0 \in \mathfrak{A}$, $b_0 \in \mathfrak{B}$, $\omega \in E_n$. Let $x(.)$ be a solution of an equation*

$$\dot{x} = A_0 x + b_0\,, \quad x(0) = \omega\,. \tag{3}$$

Then $\dim V(x(t)) = \dim V(\omega)$ *for any* $t \geqq 0$.

Proof. Take a $t > 0$. Then $x(t) = e^{A_0 t}(\omega + \int_0^t e^{-A_0\tau}\,\mathrm{d}\tau\, b_0)$. For arbitrary $A \in \mathfrak{A}$, $b \in \mathfrak{B}$, we have

$$e^{-A_0 t}(Ax(t) + b) = e^{-A_0 t} A e^{A_0 t}\left(\omega + \int_0^t e^{-A_0\tau}\,\mathrm{d}\tau\, b_0\right) + e^{-A_0 t} b\,.$$

If we define $C_0 = A$, $C_{k+1} = [A_0, C_k]$, $k = 0, 1, \ldots$, then all $C_k \in \mathfrak{A}$ and hence $e^{-A_0 t} A e^{A_0 t} = \sum_{k \geqq 0} (t^k/k!)\, C_k \in \mathfrak{A}$. Evidently

$$\int_0^t e^{-A_0\tau}\,\mathrm{d}\tau\, b_0 = \sum_{k \geqq 0} \frac{(-1)^k\, t^{k+1}}{(k+1)!} A_0^k b_0 \in \mathfrak{B}$$

and

$$e^{-A_0 t} b = \sum_{k \geqq 0} \frac{(-t)^k}{k!} A_0^k b \in \mathfrak{B}\,.$$

Hence $e^{-A_0 t}(A\, x(t) + b) \in V(\omega)$ which implies $\dim V(x(t)) = \dim \{e^{-A_0 t}(A\, x(t) + b);\ A \in \mathfrak{A},\ b \in \mathfrak{B}\} \leqq \dim V(\omega)$.

Similarly, if we start at the point $x(t)$ and go back along $x(.)$ we get $\dim V(\omega) \leqq$ $\leqq \dim V(x(t))$.

Lemma 2. *Let $\omega \in E_n$. Take $P_i \in \mathfrak{A}$, $p_i \in \mathfrak{B}$, $i = 1, 2, \ldots, k$, so that $P_i\omega + p_i$, $i = 1, 2, \ldots, k$, form a base of $V(\omega)$. Define a mapping $\varphi : E_k \to E_n$ by*

$$(4) \qquad \varphi(t_1, \ldots, t_k) = e^{P_k t_k} \ldots e^{P_1 t_1}\omega + \sum_{j=1}^{k} \int_0^{t_j} e^{P_k t_k} \ldots e^{P_{j+1} t_{j+1}} e^{P_j(t_j - \tau_j)} \, \mathrm{d}\tau_j \, p_j \, .$$

Then there exists a neighborhood G of origin in E_k such that $\varphi(G)$ is an integral manifold of V passing through ω.

Proof. Take an integer j, $1 \leqq j \leqq k$, and $t \in E_k$. Then the function $\Phi(\tau) =$ $= \varphi(t_1, \ldots, t_{j-1}, \tau, 0, \ldots, 0)$ is a solution of (3), where $A_0 = P_j$, $b_0 = p_j$, and the initial condition is $\Phi(0) = \varphi(t_1, \ldots, t_{j-1}, 0, \ldots, 0)$. Hence according to Lemma 1 for every $t \in E_k$ we have $\dim V(\varphi(t)) = \dim V(\omega)$.

φ is an entire function on E_k. Let us write, for brevity, $F_s(t) = e^{P_k t_k} \ldots e^{P_s t_s}$, $t \in E_k$, $1 \leqq s \leqq k$, then

$$\frac{\partial \varphi(t)}{\partial t_s} = \frac{\partial}{\partial t_s}\left(F_1\omega + \sum_{j=1}^{s} F_j \int_0^{t_j} e^{-P_j \tau_j} \, \mathrm{d}\tau_j \, p_j\right) =$$

$$= F_s P_s F_s^{-1}\left(F_1\omega + \sum_{j=1}^{s} F_j \int_0^{t_j} e^{-P_j \tau_j} \, \mathrm{d}\tau_j \, p_j\right) + F_{s+1} p_s =$$

$$= F_s P_s F_s^{-1}\left(\varphi(t) - \sum_{j=s+1}^{n} F_j \int_0^{t_j} e^{-P_j \tau_j} \, \mathrm{d}\tau_j \, p_j\right) + F_{s+1} p_s \, .$$

As $F_s P_s F_s^{-1} \in \mathfrak{A}$ and $F_{s+1} p_s - F_s P_s F_s^{-1} \sum_{j=s+1}^{n} F_j \int_0^{t_j} e^{-P_j \tau_j} \, \mathrm{d}\tau_j \, p_j \in \mathfrak{B}$ we have got $\partial\varphi(t)/\partial t_s \in V(\varphi(t))$.

In particular $\partial\varphi(0)/\partial t_s = P_s \varphi(0) + p_s = P_s\omega + p_s$. Hence the Jacobian $\mathrm{D}\varphi/\mathrm{D}t$ has at $t = 0$ rank equal to k and the existence of a set G follows from the continuity of derivatives $\partial\varphi/\partial t_s$, $s = 1, 2, \ldots, k$.

Lemma 3. *Let $S_{1,2}$ be integral manifolds of V and $S_1 \cap S_2 \neq \emptyset$. Then for any $x \in S_1 \cap S_2$ it exists an integral manifold S of V which contains x and is contained in $S_1 \cap S_2$.*

Proof can be found in [3], Lemma 1.4.

Theorem 1. *Through each $x \in E_n$ it passes an integral manifold S_x of V which is maximal in the sense that any manifold S of V containing x is a subset of S_x.*

Proof. According to Lemma 2 through each $x \in E_n$ it passes an integral manifold of V. Fix this x and denote $\dim V(x) = r$, $Z = \{y \in E_n; \dim V(y) = r\}$. Let Z_0 be the connected component of Z which contains x. Then thanks to Lemma 3 we can define a new topology in Z_0 calling open all subsets of Z_0 which are representable as union of a family of integral manifolds of V. In this topology the connected component of Z_0 which contains x is the sought maximal integral manifold S_x.

Theorem 2. *For any $\omega \in E_n$ the maximal integral manifold S_ω of V is a set of all points which can be linked with ω by a solution of* (2).

Proof. It was shown in [3], Lemma 1.4, that all points on any solution of (2) which starts at ω are contained in S_ω. On the other hand, take $x \in S_\omega$. Then there are integral manifolds S_i, $i = 0, 1, 2, \ldots, k$, given by formula (4), such that $\omega \in S_0$, $x \in S_k$, and $S_{i-1} \cap S_i \neq \emptyset$, $i = 1, 2, \ldots, k$. It follows immediately from (4) that an arbitrary point $x_1 \in S_0 \cap S_1$ can be linked with ω by a solution of (2). The mathematical induction completes the proof.

Auxiliaries. We have defined a Lie algebra $\mathfrak{A}$, generated by matrices $A_1, \ldots, A_\alpha$, and a linear space $\mathfrak{B}$, generated by vectors $b_1, \ldots, b_\beta$, which is closed with respect to multiplication by elements from $\mathfrak{A}$. We will call elementary any vector $b \in \mathfrak{B}$ if there exist an index j, $1 \leqq j \leqq \beta$, and matrices $P_i \in \mathfrak{A}$, $i = 1, \ldots, k$, so that $b = P_k P_{k-1} \ldots \ldots P_1 b_j$. The index k will be called degree of b. Of course an elementary vector from $\mathfrak{B}$ can have different degrees.

Let us repeat Lemma 2 from [4] as

Lemma 4. *For any $A \in \mathfrak{A}$ there exist an integer $p > 0$, finite sequences $a_1, \ldots, a_s$, $\alpha_1, \ldots, \alpha_s$, of positive numbers and a sequence $i_1, \ldots, i_s$, of integers from interval $[1, \alpha]$ such that*

$$(5) \qquad \prod_{k=1}^{s} \exp\left(a_k t^{\alpha_k} A_{i_k}\right) = I + A t^p + O(t^{p+1}), \quad t \to 0.$$

We denote, for brevity, the matrix on the left-hand side of (5) by $F_A(t)$.

Lemma 5. *Be given $A \in \mathfrak{A}$ and an elementary vector $b \in \mathfrak{B}$. Then there exist an integer $s > 0$, a number $T > 0$, and a piecewise constant control $w \in W$, such that for any $x_0 \in E_n$ we have*

$$x(Tt, w(Tt), x_0) = x_0 + (A x_0 + b)\, t^s + O(t^{s+1}), \quad t \to 0.$$

Moreover, w can be taken so that each w_i, $1 \leqq i \leqq \alpha + \beta$, have only values equal to $-1, 0, 1$, and $\sum_{i=1}^{\alpha+\beta} |w_i| = 1$.

Proof. As $b \in \mathfrak{B}$ is elementary there are matrices $P_i \in \mathfrak{A}$, $i = 1, \ldots, k$, and an index j, $1 \leqq j \leqq \beta$, such that $b = P_k \ldots P_1 b_j$. To each P_i there corresponds a matrix function F_{P_i}, which we will in this proof denote simply by F_i, and an integer $p_i > 0$ such that $F_i(t) = I + P_i t^{p_i} + O(t^{p_i+1})$, $t \to 0$.

Now we distinguish two cases: 1. Assume $A = 0$. Put $f_1(\tau, t, x_0) = F_1(t) \,.\, (F_1^{-1}(t)\, x_0 + b_j \tau) - b_j \tau = x_0 + (F_1(t) - I)\, b_j \tau$. Having defined, by mathematical induction, $f_i(\tau, t, x_0) = x_0 + (F_i(t) - I) \ldots (F_1(t) - I)\, b_j \tau$, $i = 1, 2, \ldots, s - 1$, we put $f_s(\tau, t, x_0) = f_{s-1}(-\tau, t, F_s(t) f_{s-1}(\tau, t, F_s^{-1}(t)\, x_0)) = F_s(t) f_{s-1}(\tau, t, F_s^{-1}(t)\, x_0) - (F_{s-1}(t) - I) \ldots (F_1(t) - I)\, b_j \tau = x_0 + (F_s(t) - I) \ldots (F_1(t) - I)\, b_j \tau$.

If we denote $p = \sum_{i=1}^{k} p_i$ and $g_b(t, x_0) = f_k(t, t, x_0)$ then we get

$$g_b(t, x_0) = x_0 + bt^p + O(t^{p+1}) \,, \quad t \to 0 \,. \tag{6}$$

2. Let $A \in \mathfrak{A}$ be arbitrary. Then there exists an integer $q > 0$ such that $F_A(t) = I + At^q + O(t^{q+1})$, $t \to 0$. Put $s = \max(p, q)$. Then there exist a number $T > 0$ and a control $w \in W$, fulfilling all restriction on its range, so that

$$\begin{aligned} x(Tt, w(Tt), x_0) &= F_A(t^{s/q})\, g_b(t^{s/p}, x_0) = \\ &= (I + At^s + O(t^{s+1}))\,(x_0 + bt^s + O(t^{s+1})) = \\ &= x_0 + (Ax_0 + b)\, t^s + O(t^{s+1}) \,, \quad t \to 0 \,. \end{aligned}$$

Lemma 6. *Be given $A \in \mathfrak{A}$ and an elementary $b \in \mathfrak{B}$. Then for any $\varepsilon > 0$, $\lambda \in (0, 1]$, and $\omega \in E_n$ there exist $w \in W$ and $T > 0$ so that*

$$\left\| x(T, w, \omega) - e^{A\lambda} \left(\omega + \int_0^\lambda e^{-A\tau}\, d\tau\, b \right) \right\| \leqq \varepsilon \,.$$

Moreover, the control w can be taken so that it is piecewise constant and its coordinates w_i, $1 \leqq i \leqq \alpha + \beta$, have only values $-1, 0, 1$, and $\sum_{i=1}^{\alpha+\beta} |w_i| = 1$.

Proof. Take an integer $m > 0$ and put $x_0 = y_0 = \omega$,

$$x_i = e^{A(\lambda/m)} \left(x_{i-1} + \int_0^{\lambda/m} e^{-A\tau}\, d\tau\, b \right), \quad i = 1, 2, \ldots, m \,,$$

$$y_i = \left(I + \frac{\lambda}{m} A \right) y_{i-1} + \frac{\lambda}{m} b \,, \quad i = 1, 2, \ldots, m \,,$$

$$\varkappa = \max_{t \in [0,1]} \left\| e^{At} \left(\omega + \int_0^t e^{-A\tau}\, d\tau\, b \right) \right\| .$$

Then

$$\|y_i - x_i\| \leqq \left(1 + \frac{\lambda}{m}\|A\|\right)\|y_{i-1} - x_{i-1}\| + \left(\frac{\lambda}{m}\right)^2 \|A\|\, e^{(\lambda/m)\|A\|}(\|A\|\,\varkappa + \|b\|)\,.$$

This implies

$$\|y_m - x_m\| \leqq \left(\frac{\lambda}{m}\right)^2 \|A\|\, e^{(\lambda/m)\|A\|}(\|A\|\,\varkappa + \|b\|)\sum_{i=1}^{m}\left(1 + \frac{\lambda}{m}\|A\|\right)^{i-1} \leqq$$

$$\leqq \frac{\lambda}{m}(\|A\|\,\varkappa + \|b\|)\, e^{(i+m^{-1})\lambda\|A\|} \leqq C\frac{\lambda}{m}\,,$$

where

$$C = (\|A\|\,\varkappa + \|b\|)\, e^{2\|A\|}\,.$$

There are matrix function F_A and a vector function g_b and indices p, q, corresponding to A and b. Put $s = \max(p, q)$ and $h(t, x) = F_A(t^{1/q})\, g_b(t^{1/p}, x)$ and define points $z_0 = \omega$, $z_i = h(\lambda/m, z_{i-1})$, $i = 1, \ldots, m$. Then there exists a constant $K > 0$, which depends only on A and b, such that for all i, $1 \leqq i \leqq m$, we have

$$\left\|z_i - \left(I + \frac{\lambda}{m}A\right)z_{i-1} - \frac{\lambda}{m}b\right\| \leqq K(1 + \|z_{i-1}\|)\left(\frac{\lambda}{m}\right)^{1+1/s}.$$

Further,

$$\|z_i\| \leqq K(1 + \|z_{i-1}\|)\frac{\lambda}{m} + \left\|\left(I + \frac{\lambda}{m}A\right)z_{i-1} + \frac{\lambda}{m}b\right\| \leqq$$

$$\leqq \left(1 + \frac{\lambda}{m}(\|A\| + K)\right)\|z_{i-1}\| + \frac{\lambda}{m}(\|b\| + K)\,.$$

Hence

$$\|z_i\| \leqq \left(1 + \frac{\lambda}{m}(\|A\| + K)\right)^i \|\omega\| + \frac{\lambda}{m}(\|b\| + K)\sum_{j=1}^{i}\left(1 + \frac{\lambda}{m}(\|A\| + K)\right)^{j-1} \leqq$$

$$\leqq \left(\|\omega\| + \frac{\|b\| + K}{\|A\| + K}\right)e^{\lambda(\|A\| + K)} = L - 1\,.$$

For any $i = 1, 2, \ldots, m$, we have

$$\|z_i - y_i\| \leqq \left\|\left(I + \frac{\lambda}{m}A\right)z_{i-1} + \frac{\lambda}{m}b - \left(I + \frac{\lambda}{m}A\right)y_{i-1} - \frac{\lambda}{m}b\right\| +$$

$$+ KL\left(\frac{\lambda}{m}\right)^{1+1/s} \leqq \left(1 + \frac{\lambda}{m}\|A\|\right)\|z_{i-1} - y_{i-1}\| + KL\left(\frac{\lambda}{m}\right)^{1+1/s},$$

$$\|z_m - y_m\| \leqq KL\left(\frac{\lambda}{m}\right)^{1+1/s}\sum_{i=1}^{m}\left(1 + \frac{\lambda}{m}\|A\|\right)^{i-1} \leqq \lambda K L e^{\lambda\|A\|}\left(\frac{\lambda}{m}\right)^{1/s}.$$

Finally,

$$\|z_m - x_m\| \leqq \|z_m - y_m\| + \|y_m - x_m\| \leqq$$
$$\leqq \lambda K L e^{\lambda\|A\|} \left(\frac{\lambda}{m}\right)^{1/s} + C\frac{\lambda}{m} = K_1 \left(\frac{\lambda}{m}\right)^{1/s}.$$

It remains to take m so that $K_1(\lambda/m)^{1/s} < \varepsilon$.

Main result. Theorem 3. *Given $\omega \in E_n$. Then the maximal integral manifold S_ω of V is equal to the set of all points in E_n which are accessible from ω along solutions of the bilinear system* (1).

Moreover, each $x \in E_n$ can be reached from ω along a solution $x(., w, \omega)$, where w is piecewise constant, its coordinates have only values $-1, 0, 1$, *and* $\sum_{i=1}^{\alpha+\beta} |w_i| = 1$.

Proof. Evidently all points on any solution $x(., w, \omega)$ are contained in S_ω. On the other hand take $x \in S_\omega$. Then there exist integral manifolds $\varphi_i(G_i)$, $i = 0, 1, \ldots, k$, which have form (4), such that $\omega \in \varphi_0(G_0)$, $x \in \varphi_k(G_k)$, $\varphi_{i-1}(G_{i-1}) \cap \varphi_i(G_i) \neq \emptyset$, $i = 1, 2, \ldots, k$.

According to Lemma 6 a point $x_1 \in \varphi_0(G_0) \cap \varphi_1(G_1)$ can be reached from ω along a solution of (1) which corresponds to a piecewise constant control, satisfying restrictions on its values. By mathematical induction we conclude that for any $\varepsilon > 0$ there exists a piecewise constant control w_ε, satisfying restrictions on its values, and a number T_ε such that $\|x(T_\varepsilon, w_\varepsilon, \omega) - x\| < \varepsilon$.

Let $\dim V(x) = k$. We can choose matrices $P_i \in \mathfrak{A}$ and elementary vectors $p_i \in \mathfrak{B}$ so that $P_i x + p_i$, $i = 1, 2, \ldots, k$, form a base of $V(x)$. According to Lemma 5 for any $i = 1, 2, \ldots, k$, there exist a matrix function F_{P_i} and a vector function g_{p_i}, denote them for brevity F_i and g_i, respectively, and indices q_i, r_i such that

$$h_i(t, x) = F_i(t^{1/q_i})\, g_i(t^{1/r_i}, x) = x + (P_i x + p_i)\, t + O(t^{1+\lambda_i}), \quad t \to 0, \quad \lambda_i > 0,$$
$$i = 1, 2, \ldots, k.$$

Now, define $H_1(t_1, x) = h_1(t_1, x)$, $t_1 \in E_1$, $H_i(t_1, \ldots, t_i, x) = h_i(t_i, H_{i-1}(t_1, \ldots \ldots, t_{i-1}, x))$, $(t_1, \ldots, t_i) \in E_i$, $i = 1, 2, \ldots, k$. Then $H_k(t, x) = H_k(t_1, \ldots, t_k, x)$ has all derivatives of the first order continuous on E_k, $H_k(0, x) = x$, and $\partial H_k(0, x)/\partial t_i = P_i x + p_i$, $i = 1, 2, \ldots, k$. Hence there exists a neighborhood $G \subset E_k$ of origin such that $H_k(G, x)$ is an integral manifold of V.

Each point in $H_k(G, x)$ can be reached from x along a solution of (1), corresponding to a piecewise constant control w, which fulfils the restrictions on its values. But for sufficiently small $\varepsilon > 0$ we have $x(T_\varepsilon, w_\varepsilon, \omega) \in H_k(G, x)$. This completes the proof.

We have actually proved a slightly more general

Theorem 3a. *Let $\mathfrak{A}_0$ be a set of n-by-n matrices and $\mathfrak{B}_0$ a set of n-dimensional vectors with a property:*

1. $0 \in \mathfrak{A}$, $0 \in \mathfrak{B}_0$,
2. $A \in \mathfrak{A}_0 \Rightarrow -A \in \mathfrak{A}_0$; $b \in \mathfrak{B}_0 \Rightarrow -b \in \mathfrak{B}_0$.

Let $\mathfrak{A}$ be the smallest Lie algebra containing $\mathfrak{A}_0$ and $\mathfrak{B}$ the smallest linear space containing $\mathfrak{B}_0$ which with $b \in \mathfrak{B}$ contains also Ab for any $A \in \mathfrak{A}$. Put $V_0(x) = \{Ax + b; A \in \mathfrak{A}_0, b \in \mathfrak{B}_0\}$, $V(x) = \{Ax + b; A \in \mathfrak{A}, b \in \mathfrak{B}\}$ and form an equation

$$\dot{x} \in V_0(x)\,. \tag{7}$$

Then for any $\omega \in E_n$ the set of all points accessible from ω along solutions of (7) is equal to the maximal integral manifold S_ω of (2).

Example. Consider an equation

$$\dot{x} = Ax + Bu\,, \tag{8}$$

where A, B are constant matrices of type n-by-n, n-by-m, respectively, and u ranges the set of all vector functions, measurable on $[0, \infty)$, with values in $[-1, 1]^m$.

Then the set of all points accessible from 0 along solutions of (8) is contained in the maximal integral manifold of the distribution V which passes through 0, where V is generated by matrix A and columns b_j, $j = 1, 2, \ldots, m$, of the matrix B.

Hence a classical necessary condition for controllability of (8), see [6], which reads:

(9) "a matrix with columns $A^k b_j$, $j = 1, \ldots, m$, $k = 0, 1, \ldots, n$, has rank n";

follows from Theorem 3.

If we instead of (8) have an equation

$$\dot{x} = \chi A x + Bu\,, \tag{10}$$

where all symbols have the same meaning as in (8) and χ ranges the set of all piecewise constant functions which assume only values $-1, 0, 1$, then it follows from Theorem 3 that (9) is also a sufficient condition for controllability of (10).

References

[1] *E. A. Coddington, N. Levinson:* Theory of ordinary differential equations, McGraw-Hill 1955.

[2] *C. Chevalley:* Theory of Lie Groups I, Princeton 1946.

[3] *J. Kučera:* Solution in Large of Control Problem $\dot{x} = (A(1 - u) + Bu)\,x$, Czechoslovak Math. J., Vol. *16 (91)*, 1966, pp. 600–623.

[4] *J. Kučera:* Solution in Large of Control Problem $\dot{x} = (Au + Bv)\,x$, Czechoslovak Math. J., Vol. *17* (*92*), 1967, pp. 91–96.

[5] *R. E. Rink, R. R. Mohler:* Completely Controllable Bilinear Systems, SIAM J. Control, Vol. *6*, No. 3, 1968, pp. 477–486.

[6] *J. P. La Salle:* The Time Optimal Control Problem, Contributions to the theory of nonlinear oscillations, Vol. *V*, 1960, pp. 1–24.

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