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ON THE ACCESSIBILITY OF CONTROL SYSTEM $\dot{x} \in Q(x)$

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In this paper we present an algebraic condition under which the set of all points which are reachable from a fixed point ω at a constant time along solutions of a system (1) is a closed manifold whose dimension depends only on algebraic properties of ω. At the same time we present an explicit formula for this manifold.

Notations. E_n denotes a Euclidean n-dimensional space with a norm $\|.\|$. The dimension of a (finite dimensional) vector space V is written dim V. $\{p \in P;\ P(p)\}$ is the set of all points $p \in P$ with property $P(p)$. We use only Lebesgue measures and integrals.

In the space $\mathfrak{E}_n$ of all n-by-n matrices we define a "bracket" operation $[A, B] = BA - AB$, $A, B \in \mathfrak{E}_n$, which makes $\mathfrak{E}_n$ a Lie algebra. Remind, L is a Lie algebra if it is a linear space with a bilinear anticommutative operation $[.\,,.] : L \times L \to L$ such that

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0\,, \quad A, B, C \in L.$$

For $A_1, A_2, \ldots, A_r \in \mathfrak{E}_n$ we write $[A_1, A_2, \ldots, A_r] = [A_1, [A_2, \ldots, [A_{r-1}, A_r] \ldots]]$. We often meet a matrix $[A_1, A_2, \ldots, A_r]$, where $A_1 = A_2 = \ldots = A_{r-1}$. Then in the case that there is no danger of misunderstanding we write it simply $[A_1^{r-1} A_r]$. Zero matrix is denoted by O, unit matrix by I, and the inverse of a nonsingular matrix A by A^{-1}.

A connected set $S \subset E_n$ is called an r-dimensional manifold if for each $x \in S$ there is an open nonempty set $G \subset E_r$ and an injection $\varphi : G \to S$ such that:

1. $x \in \varphi(G)$,
2. $\varphi(G)$ is open in S,
3. Jacobian $\partial\varphi/\partial t$ is continuous and has rank r on G.

Given an r-dimensional manifold $S \subset E_n$ then the closure of S is called an r-dimensional closed manifold. A set $S \subset E_n$ which contains only one element is said to be a 0-dimensional manifold.

Definition 1. Let $\mathfrak{B} \subset \mathfrak{E}_n$ be a linear space. Then a mapping V defined on E_n by $V(x) = \{Ax; A \in \mathfrak{B}\}$ is called a linear distribution created by $\mathfrak{B}$. If among all linear subspaces of $\mathfrak{E}_n$ which create the same linear distribution V at least one is closed with respect to the bracket operation then we call V involutive.

Definition 2. Let V be a linear distribution and $S \subset E_n$ a manifold. If for each $x \in S$ the tangent space $T(x)$ to S at x equals to $V(x)$ then we call S an integral manifold of V.

Let V be a linear distribution. It was shown in [3] that then each $x \in E_n$ is contained in an integral manifold of V if and only if V is involutive. Moreover, if V is involutive then each $x \in E_n$ is contained in a unique integral manifold M_x of V which is maximal in the sense that any integral manifold M of V containing x is contained in M_x. Furthermore, let V be created by a Lie algebra $\mathfrak{B}$ and let $P_i \in \mathfrak{B}$, $i = 1, 2, \ldots, k$, be chosen so that $P_1x, P_2x, \ldots, P_kx$ form a base of $V(x)$. Then there exists an open set $G \subset E_k$ such that the mapping $\varphi(t) = e^{P_1t_1}e^{P_2t_2} \ldots e^{P_kt_k}x$, $t \in G$, describes an integral manifold $\varphi(G)$ of V.

Formulation of the problem. Let us have a compact, convex set $\mathfrak{A} \subset \mathfrak{E}_n$. Denote $Q(x) = \{Ax; A \in \mathfrak{A}\}$, $x \in E_n$. Then for each $x \in E_n$ $Q(x)$ is compact and convex and the mapping $Q(.)$ is continuous on E_n if we equip the image of Q with Hausdorff topology. Hence existence of solutions of an equation

$$\dot{x} \in Q(x)\,, \quad x(0) = \omega\,, \tag{1}$$

makes no problem. By a solution of (1) we mean any vector function $x(.)$, absolutely continuous on an interval $J \subset E_n$, which fulfils $\dot{x}(t) \in Q(x(t))$ for almost all $t \in J$.

Denote $\mathscr{U} = \{u : [0, \infty) \to \mathfrak{A};\ u \text{ measurable}\}$. Then to each $u \in \mathscr{U}$ and $\omega \in E_n$ it coresponds a unique solution of an equation

$$\dot{x} = ux\,, \quad x(0) = \omega\,. \tag{2}$$

Without any ambiguity we denote this solution by $x(., u, \omega)$. According to implicit function theorem [4] for any solution $x(.)$ of (1) there exists $u \in \mathscr{U}$ such that $x(t) = x(t, u, \omega)$, $t \in J$.

Let $\mathfrak{B}$ be the smallest Lie algebra which contains $\mathfrak{A}$ and $\mathfrak{W}$ the smallest linear space which contains the set $\mathfrak{B} = \{A - B;\ A, B \in \mathfrak{A}\}$ and is closed with respect to bracket multiplication by elements from $\mathfrak{B}$. Then evidently $\mathfrak{W} \subset \mathfrak{B}$ and $\mathfrak{W}$ is a Lie algebra. Hence, mappings V and $\mathscr{V}$, defined by $V(x) = \{Ax; A \in \mathfrak{B}\}$, $\mathscr{V}(x) = \{Bx; B \in \mathfrak{W}\}$, $x \in E_n$, are involutive linear distributions.

For a given $T \geqq 0$ write $\mathscr{S}_\omega(T) = \{x(T, u, \omega)); u \in \mathscr{U}\}$ and $S_\omega(T) = \bigcup_{t\in[0,T]} \mathscr{S}_\omega(t)$. According to [3] the reachable cone $\bigcup_{T\geqq 0} S_\omega(T)$ of (1) is contained in the maximal integral manifold of V which passes through ω. We are now looking for a condition which guarantees that for any $T > 0$ the set $S_\omega(T)$, resp. $\mathcal{S}_\omega(T)$, is a closure of an integral manifold of V, resp. $\mathcal{V}$.

Auxiliaries. Lemma 1. *Denote by Z_u the fundamental matrix solution of (2), corresponding to $u \in \mathcal{U}$, for which $Z_u(0) = I$. Then for any $A \in \mathfrak{V}$ and any $t \geqq 0$ we have $Z_u(t) A Z_u^{-1}(t) - A \in \mathfrak{W}$ and $Z_u^{-1}(t) A Z_u(t) - A \in \mathfrak{W}$.*

Proof. If $u(.)$ is piecewise constant then the assertion of Lemma 1 follows immediately from an identity $e^{-C} B e^{C} = \sum_{k=0}^{\infty} (1/k!) [\underbrace{C, C, \ldots, C}_{k\text{-times}}, B]$ which holds for any $B, C \in \mathfrak{E}_n$.

If $u(.)$ is not piecewise constant then we take a sequence $u_k \in \mathcal{U}$, $k = 1, 2, \ldots$, of piecewise constant functions which converges to u locally asymptotically on $[0, \infty)$. For any k and any $t \geqq 0$ we have $Z_{u_k}(t) A Z_{u_k}^{-1}(t) - A \in \mathfrak{W}$, $Z_{u_k}^{-1}(t) A Z_{u_k}(t) - A \in \mathfrak{W}$. The sequence Z_{u_k}, resp. $Z_{u_k}^{-1}$, $k = 1, 2, \ldots$, converges to Z_u, resp. Z_u^{-1}, locally uniformly on $[0, \infty)$. The space $\mathfrak{W}$ is finite dimensional, hence it is closed and the proof is complete.

Remark. We can similarly show that for $B \in \mathfrak{W}$, $u \in \mathcal{U}$ and $t \geqq 0$ it holds $Z_u(t) \,.\, B Z_u^{-1}(t) \in \mathfrak{W}$ and $Z_u^{-1}(t) B Z_u(t) \in \mathfrak{W}$.

Lemma 2. $\dim \mathcal{V}(x(t, u, \omega)) = \dim \mathcal{V}(\omega)$ *for any $t \geqq 0$ and any $u \in \mathcal{U}$.*

Proof. According to [3] all points $x(t, u, \omega)$ are contained in the maximal integral manifold of V which passes through ω. Therefore it suffices to prove an equivalence $A\omega \in \mathcal{V}(\omega)$ iff $A\, x(t, u, \omega) \in \mathcal{V}(x(t, u, \omega))$, where $A \in \mathfrak{V}$, $u \in \mathcal{U}$ and $t \geqq 0$.

Fix $A \in \mathfrak{V}$, $u \in \mathcal{U}$ and $t \geqq 0$. Put, for brevity, $B = Z_u(t) A Z_u^{-1}(t) - A$. Then we can write $A\omega = Z_u^{-1}(t) (Z_u(t) A Z_u^{-1}(t) - A + A) Z_u(t)\, \omega = Z_u^{-1}(t) (B + A) Z_u(t)\, \omega$.

Assume $A\omega \in \mathcal{V}(\omega)$. As $B \in \mathfrak{W}$ we have $Z_u^{-1}(t) B Z_u(t) \in \mathfrak{W}$ and $Z_u^{-1}(t) A Z_u(t)\, \omega = A\omega - Z_u^{-1}(t) B Z_u(t)\, \omega \in \mathcal{V}(\omega)$. This implies existence of such $B_i \in \mathfrak{W}$ and real numbers b_i, $i = 1, 2, \ldots, p$, that $Z_u^{-1}(t) A Z_u(t)\, \omega = \sum_{i=1}^{p} b_i B_i \omega$. Hence $A x(t, u, \omega) = Z_u(t) Z_u^{-1}(t) A Z_u(t)\, \omega = Z_u(t) \sum_{i=1}^{p} b_i B_i \omega = \sum_{i=1}^{p} b_i (Z_u(t)\, B_i Z_u^{-1}(t))\, x(t, u, \omega) \in \mathcal{V}(x(t, u, \omega))$, due to the remark to Lemma 1.

The inverse implication can be obtained by the same way.

Denote $\mathcal{U}_0$ the set of all $u \in \mathcal{U}$ which are piecewise continuous on $[0, \infty)$ and moreover at each point of discontinuity continuous from the right. Till the end of this paragraph fix $T > 0$ and $u_0 \in \mathcal{U}_0$. For any $v \in \Delta = \{u - u_0;\ u \in \mathcal{U}\}$ and any $\varepsilon \in [0, 1]$ we have $u_\varepsilon = u_0 + \varepsilon v \in \mathcal{U}$. The solution $(x., u_\varepsilon, \omega)$ of (2) is analytically dependent on ε and can be expanded into a power series

$$x(., u_\varepsilon, \omega) = \sum_{k=0}^{\infty} \varepsilon^k x_k(., v)\,, \tag{3}$$

where the coefficients $x_k(., v)$ solve an equation

$$\dot{x}_k = u_0 x_k + v x_{k-1}\,, \quad x_k(0, v) = 0\,, \quad k = 1, 2, \ldots \tag{4}$$

Here we write for brevity $x_0(t, v) = x(t, u_0, \omega)$.

If we put $a = \max\{\|A\|;\, A \in \mathfrak{A}\}$ then $\|x_0(t, v)\| \leqq \|\omega\|\, e^{at}$ and for $k = 1, 2, \ldots$, we have $(\mathrm{d}/\mathrm{d}t)\,\|x_k\| \leqq a\|x_k\| + 2a\|x_{k-1}\|$, which implies $\|x_k(t, v)\| \leqq 2a \int_0^t e^{a(t-\tau)}\,.\,\|x_{k-1}\|\,\mathrm{d}\tau$. Finally $\|x_k(t, v)\| \leqq \|\omega\|\,(2a)^k\,(1/k!)\,t^k e^{at}$, $t \geqq 0$. Hence $\|x(t, u_\varepsilon, \omega)\| \leqq \leqq \sum_{k \geqq 0} \varepsilon^k \|x_k(t, v)\| \leqq \|\omega\|\, e^{3at}$ and the series (3) is locally uniformly absolutely convergent on $[0, \infty)$.

Using the variation of constants formula we get $x_1(T, v) = Z(T) \int_0^T Z^{-1}(t)\,.\,v(t)\, x(t, u_0, \omega)\,\mathrm{d}t = Z(T) \int_0^T Z^{-1}(t)\, v(t)\, Z(t)\,\mathrm{d}t\,.\,Z^{-1}(T)\, x(T, u_0, \omega)$, where we, for brevity, write Z instead of Z_{u_0}.

Lemma 3. *The linear hull $\mathfrak{M}(T, u_0)$ of set $\{\int_0^T Z^{-1}(t)\, v(t)\, Z(t)\,\mathrm{d}t;\, v \in \Delta\}$ equals to the linear hull of $\{Z^{-1}(t)\, B\, Z(t);\, t \in [0, T],\, B \in \mathfrak{B}\}$.*

Proof. Take $B \in \mathfrak{B}$. There are matrices $A_{1,2} \in \mathfrak{A}$ such that $B = A_1 - A_2$. As $\mathfrak{A}$ is convex we have $u_i(t) = \frac{1}{2}(A_i + u_0(t)) \in \mathcal{U}$, $i = 1, 2$, and $v_i = u_i - u_0 \in \Delta$, $i = 1, 2$. The function $u_0 \in \mathcal{U}_0$ is everywhere continuous from the right therefore $v_{1,2}$ are continuous from the right too.

Take $t_0 \in [0, T)$ and for any $\alpha \in (0, T - t_0)$ denote

$$v_{i,\alpha}(t) = \begin{cases} v_i(t) & \text{for} \quad t \in [t_0, t_0 + \alpha) \\ 0 & \text{for} \quad t \notin [t_0, t_0 + \alpha) \end{cases}\,, \quad i = 1, 2\,.$$

Then $v_{i,\alpha} \in \Delta$. Due to continuity from the right of functins v_i we get

$$\lim_{\alpha \to 0+} \frac{1}{\alpha} \int_0^T Z^{-1}(t)\, v_{i,\alpha}(t)\, Z(t)\,\mathrm{d}t = Z^{-1}(t_0)\, v_i(t_0)\, Z(t_0) \in \mathfrak{M}(T, u_0)$$

and

$$Z^{-1}(t_0)\, B\, Z(t_0) = 2\, Z^{-1}(t_0)\, (v_1(t_0) - v_2(t_0))\, Z(t_0) \in \mathfrak{M}(T, u_0)\,.$$

Finally

$$\lim_{t_0 \to T-} Z^{-1}(t_0)\, B\, Z(t_0) = Z^{-1}(T)\, B\, Z(T) \in \mathfrak{M}(T, u_0)\,.$$

The inverse inclusion follows immediately from the fact that the values of any $v \in \Delta$ lie in $\mathfrak{B}$.

Definition. We say that a compact convex set $\mathfrak{A} \subset \mathfrak{C}_n$ has a property (A) if such matrices $A_i \in \mathfrak{A}$, $i = 1, 2, \ldots, p$, and $B_j \in \mathfrak{B}$, $j = 1, 2, \ldots, q$, exists that the linear space generated by matrices $[A_i^r B_j]$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, $r = 0, 1, 2, \ldots$, equals to the Lie algebra $\mathfrak{W}$.

Lemma 4. *If $\mathfrak{A} \subset \mathfrak{E}_n$ has property* (A) *then for any $T > 0$ there exists such $u_T \in \mathscr{U}_0$ that $\mathfrak{M}(T, u_T) = \mathfrak{W}$.*

Proof. Take $T > 0$ and a partition $0 = t_0 < t_1 < \ldots < t_{p+1} = T$ of interval $[0, T]$. Choose an arbitrary matrix from $\mathfrak{A}$ and denote it by A_{p+1}. Define $u(t) = A_i$, $t \in [t_{i-1}, t_i)$, $i = 1, 2, \ldots, p + 1$, $u(T) = A_{p+1}$. Evidently $u \in \mathscr{U}_0$ and $Z_u(t) = e^{A_i(t-t_{i-1})}e^{A_{i-1}(t_{i-1}-t_{i-2})} \ldots e^{A_1(t_1-t_0)}$, for $t \in [t_{i-1}, t_i]$.

For $t \in [t_{i-1}, t_i]$ and $B \in \mathfrak{B}$ we have

$$Z_u^{-1}(t)\, B\, Z_u(t) = Z_u^{-1}(t_{i-1})\, e^{-A_i(t-t_{i-1})} B e^{A_i(t-t_{i-1})}\, Z_u(t_{i-1}) =$$

$$= Z_u^{-1}(t_{i-1}) \sum_{r=0}^{\infty} \frac{(t - t_{i-1})^r}{r!} [A_i^r B]\, Z_u(t_{i-1})\,.$$

By differentiation with respect to t we get $Z_u^{-1}(t_{i-1})\,[A_i^r B]\, Z_u(t_{i-1}) \in \mathfrak{M}(T, u)$, $r = 0, 1, \ldots$.

If we take t_p sufficiently small then the dimension of a linear space generated by matrices $Z_u^{-1}(t_{i-1})\,[A_i^r B_j]\, Z_u(t_{i-1})$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, $r = 0, 1, \ldots$, equals to the dimension of a linear space L generated by matrices $[A_i^r B_j]$, $i = 1, 2, \ldots$, $j = 1, 2, \ldots, q$, $r = 0, 1, \ldots$. It implies $\dim \mathfrak{M}(T, u) \geqq \dim L$. Lut $\mathfrak{M}(T, u) \subset \mathfrak{W}$ and we have assumed $L = \mathfrak{W}$. This gives us the desired equality $\mathfrak{W}(T, u) = \mathfrak{W}$.

Lemma 5. *Let $A_0 \in \mathfrak{A}$ be an arbitrary matrix. Then $\mathfrak{B}$ is a linear hull of A_0 and $\mathfrak{W}$.*

Proof. One inclusion is trivial. As $\mathfrak{B}$ is a linear hull of matrices of a type $[A_1, A_2, \ldots, A_k]$, where $A_i \in \mathfrak{A}$, $i = 1, 2, \ldots, k$, it suffices to show that each such matrix belongs to the linear hull of A_0 and $\mathfrak{W}$. If $k = 1$ then $A_1 = A_0 + (A_1 - A_0)$, where $A_1 - A_0 \in \mathfrak{W}$. If $k > 1$ then $[A_1, \ldots, A_{k-1}, A_k] = [A_1, \ldots, A_{k-1}, A_k - A_{k-1}] \in \mathfrak{W}$ which completes the proof.

Main result. *Let $\mathfrak{A} \subset \mathfrak{E}_n$ have property* (A). *Construct distributions $\mathscr{V}$ and V. For given $T > 0$ and $\omega \in E_n$ let $\mathscr{S}_\omega(T)$, resp. $S_\omega(T)$, be the set of all points which are reachable at the time T, resp. at any time $t \in [0, T]$, from ω along solutions of* (1). *Denote $\dim \mathscr{V}(\omega) = q$ and $\dim V(\omega) = r$.*

Then $\mathscr{S}_\omega(T)$, resp. $S_\omega(T)$, is a closed q-, resp. r-, dimensional integral manifold of the distribution $\mathscr{V}$, resp. V.

Proof. Let $T > 0$. Take $x \in \mathscr{S}_\omega(T)$ and $\varepsilon > 0$. By implicit function theorem there exists such $u \in \mathscr{U}$ that $x = x(T, u, \omega)$. According to Lemma 4 for any $\lambda > 0$ there exists $\varphi_\lambda \in \mathscr{U}_0$ such that $\mathfrak{M}(\lambda, \varphi_\lambda) = \mathfrak{W}$. Define $u_\lambda(t) = \begin{Bmatrix} u(t), & t > \lambda \\ \varphi_\lambda(t), & t \in [0, \lambda] \end{Bmatrix}$. As $\mathfrak{A}$ is bounded the functions $u_\lambda(.)$ converge asymptotically to $u(.)$ on $[0, T]$ with $\lambda \to 0+$.

Therefore $\lim_{\lambda \to 0+} x(T, u_\lambda, \omega) = x(T, u, \omega) = x$ and we can fix such λ_0 that $\|x(T, u_{\lambda_0}, \omega) - - x\| < \varepsilon$.

Denote, for brevity, by $Z(.)$ the fundamental matrix solution of (2), corresponding to u_{λ_0}. According to Lemma 3 there are functions $v_i \in \{u - u_{\lambda_0};\ u \in \mathscr{U}\}$, $i = 1, 2, \ldots \ldots, k$, such that matrices $\int_0^{\lambda_0} Z^{-1}(t)\, v_i(t)\, Z(t)\, dt$, $i = 1, 2, \ldots, k$, form a base of $\mathfrak{W}$. Define $w_i(t) = \begin{Bmatrix} v_i(t), & t \in [0, \lambda_0] \\ 0, & t > \lambda_0 \end{Bmatrix}$, $i = 1, 2, \ldots, k$. Then evidently also $w_i(t) \in \in \{u - u_{\lambda_0};\ u \in \mathscr{U}\}$.

The matrices $B_i = Z(T) \int_0^T Z^{-1}(t)\, w_i(t)\, Z(t)\, dt\, Z^{-1}(T)$, $i = 1, 2, \ldots, k$, are linearly independent and according to Lemma 1 they belong into $\mathfrak{W}$. Hence they form a base of $\mathfrak{W}$ and vectors $B_i\, x(T, u_{\lambda_0}, \omega)$, $i = 1, 2, \ldots, k$, generate the linear space $\mathscr{V}(x(T, u_{\lambda_0}, \omega))$. According to Lemma 2 we have $\dim \mathscr{V}(x(T, u_{\lambda_0}, \omega)) = \dim \mathscr{V}(\omega) = = q$. Assume that vectors $B_i\, x(T, u_{\lambda_0}, \omega)$, $i = 1, 2, \ldots, q$, form a base of $\mathscr{V}(x(T, u_{\lambda_0}, \omega))$.

Now

$$(5) \qquad x\left(T, u_{\lambda_0} + \sum_{i=1}^{q} \tau_i w_i, \omega\right), \quad \text{where} \quad \tau \in G = \left\{\tau \in E_q;\ \sum_{i=1}^{q} |\tau_i| < 1\right\},$$

represents a mapping of an open set $G \subset E_q$ into $\mathscr{S}_\omega(T)$. It has continuous first partial derivatives with respect to τ, which are for $\tau = 0$ solutions of a corresponding equation

$$(6) \qquad \frac{d}{dt} \frac{\partial x}{\partial \tau_i} = u_{\lambda_0} \frac{\partial x}{\partial \tau_i} + w_i\, x(t, u_{\lambda_0}, \omega), \quad \left.\frac{\partial x}{\partial \tau_i}\right|_{\tau=0} = 0, \quad i = 1, 2, \ldots, q.$$

If we compare (6) with (4) we see that the vectors $B_i\, x(T, u_{\lambda_0}, \omega)$, $i = 1, 2, \ldots, q$, are columns of Jacobian $Dx/D\tau$ at $\tau = 0$. Therefore there exists a neighborhood $G_0 \subset G$ of origin in E_q such that $Dx/D\tau$ has rank q at any point of G_0.

Denote X_τ the fundamental matrix solution of (2), corresponding to $u_{\lambda_0} + \sum_{i=1}^{q} \tau_i w_i$, for which $X_\tau(0) = I$. Then we can write

$$\frac{\partial}{\partial \tau_i} x\left(T, u_{\lambda_0} + \sum_{i=1}^{q} \tau_i w_i, \omega\right) = X_\tau(T) \int_0^T X_\tau^{-1}(t) \sum_{i=1}^{q} \tau_i w_i\, X_\tau(t)\, dt\, X_\tau^{-1}(T)\,.$$

$$.\, x\left(T, u_{\lambda_0} + \sum_{i=1}^{q} \tau_i w_i, \omega\right) \in \mathscr{V}\left(x\left(T, u_{\lambda_0} + \sum_{i=1}^{q} \tau_i w_i, \omega\right)\right).$$

Thus, $\mathscr{S}_\omega(T)$ is a closure of a union of a family of q-dimensional integral manifolds of distribution $\mathscr{V}$.

Let $\mathscr{S}_{0,1}$ be two manifolds of this family. Choose $x(T, u_i, \omega) \in \mathscr{S}_i$, $i = 0, 1$, so that there exists $t_0 \in (0, T]$ such that $u_0(t) = u_1(t)$ for $t \in [0, t_0]$ and $\mathfrak{M}(t_0, u_0) = \mathfrak{W}$. Denote $u_\lambda = (1 - \lambda)\, u_0 + \lambda u_1$, $\lambda \in [0, 1]$. Then the curve $\Gamma(\lambda) = x(T, u_\lambda, \omega)$, $\lambda \in [0, 1]$ links points $x(T, u_i, \omega)$, $i = 0, 1$, and each its point is contained in $\mathscr{S}_\omega(T)$.

Let us prove that $\mathrm{d}\Gamma(\lambda)/\mathrm{d}\lambda \in \mathscr{V}(\Gamma(\lambda))$ for $\lambda \in [0, 1]$. It holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x(t, u_\lambda, \omega)}{\mathrm{d}\lambda} = u_\lambda(t)\frac{\mathrm{d}x(t, u_\lambda, \omega)}{\mathrm{d}\lambda} + (u,(t) - u_0(t))\, x(t, u_\lambda, \omega), \quad \left.\frac{\mathrm{d}x(t, u_\lambda, \omega)}{\mathrm{d}\lambda}\right|_{t=0} = 0.$$

Using variation of constants formula we get

$$\frac{\mathrm{d}\Gamma(\lambda)}{\mathrm{d}\lambda} = \frac{\mathrm{d}x(T, u_\lambda, \omega)}{\mathrm{d}\lambda} =$$

$$= Z_{u_\lambda}(T)\int_0^T Z_{u_\lambda}^{-1}(t)(u,(t) - u_0(t))\, Z_{u_\lambda}(t)\,\mathrm{d}t\; Z_{u_\lambda}^{-1}(T)\, x(T, u_\lambda, \omega) \in \mathscr{V}(\Gamma(\lambda)), \quad \lambda \in [0, 1].$$

We have proved that $\Gamma(\lambda)$, $\lambda \in [0, 1]$, is contained in the maximal integral manifold $\mathscr{M}$ of $\mathscr{V}$ which passes through $x(T, u_0, \omega)$. As $\mathscr{S}_\lambda \cap \mathscr{M} \neq \emptyset$ for any $\lambda \in [0, 1]$ it follows from [3] that $\mathscr{S}_\lambda \subset \mathscr{M}$, $\lambda \in [0, 1]$. Hence $\mathscr{S} = \bigcup_{\lambda \in [0,1]} \mathscr{S}_\lambda \subset \mathscr{S}_\omega(T)$ is an integral manifold of $\mathscr{V}$ which contains both points $x(T, u_i, \omega)$, $i = 0, 1$. First part of the theorem is proved.

Take $x \in S_\omega(T)$ and $\varepsilon > 0$. Again there exists $u \in \mathscr{U}$ and $t_0 \in [0, T]$ such that $x = x(t_0, u, \omega)$. By the same way we find functions $u_0 \in \mathscr{U}$, $w_i \in \{u - u_0;\ u \in \mathscr{U}\}$, $i = 1, 2, \ldots, q$, a number $t_1 > 0$ and a neighborhood $G \subset E_q$ of origin such that Jacobian of a mapping $x(t_1, u_0 + \sum_{i=1}^{q} \tau_i w_i, \omega) : G \to E_n$ has rank q at any point of G, and moreover $u_0(t_1) \neq 0$ and $\|x(t_1, u_0, \omega) - x\| < \varepsilon$.

If $\dim V(\omega) = q$ there is nothing to be proved. Assume $\dim V(\omega) > q$. Investigate a mapping

$$x(t, u_0 + \sum_{i=1}^{q} \tau_i w_i, \omega), \quad |t - t_1| < \delta, \quad \tau \in G. \tag{7}$$

We know that for any $i = 1, 2, \ldots, q$,

$$\frac{\partial}{\partial \tau_i} x(t, u_0 + \sum_{i=1}^{q} \tau_i w_i, \omega) \in \mathscr{V}(x(t, u_0 + \sum_{i=1}^{q} \tau_i w_i, \omega)).$$

But

$$\frac{\partial}{\partial t} x(t, u_0 + \sum_{i=1}^{q} \tau_i w_i, \omega) =$$

$$= (u_0 + \sum_{i=1}^{q} \tau_i w_i)\, x(t, u_0 + \sum_{i=1}^{q} \tau_i w_i, \omega) \in V(x(t, u_0 + \sum_{i=1}^{q} \tau_i w_i, \omega)).$$

If we take δ and an open set $G_0 \subset G$, $0 \in G_0$, so small that $A(t, \tau) = u_0(t) + \sum_{i=1}^{q} \tau_i w_i(t) \neq 0$ for $t \in (t_1 - \delta, t_1 + \delta)$, $\tau \in G_0$, then according to Lemma 5 the space $\mathfrak{V}$ is equal to the linear hull of $\mathfrak{W}$ and $A(t, \tau)$ for any $t \in (t_1 - \delta, t_1 + \delta)$, $\tau \in G_0$.

Hence Jacobian of the mapping (7) has at any point from $(t_1 - \delta, t_1 + \delta) \times G_0$ rank r.

Similarly as in the first part we find out that $S_\omega(T)$ is a closure of a union of a family of r-dimensional integral manifolds of the distribution V. Let again we have two manifolds S_0, S_1, of this family. Then we can choose points $x(t_i, u_i, \omega) \in S_i$, $i = 0, 1$, so that $t_2 = \min(t_0, t_1) > 0$ and there exists $t_3 \in (0, t_2]$ such that $u_0(t) = u_1(t)$ for $t \in [0, t_3]$ and $\mathfrak{M}(t_3, u_0) = \mathfrak{W}$. The case $t_0 = t_1$ has been already treated, therefore assume $t_0 < t_1$. Then a curve Γ consisting of arches $\Gamma_0(t) = x(t, u_0, \omega)$, $t \in [t_0, t_1]$, and $\Gamma_1(\lambda) = x(t_1, (1 - \lambda) u_0 + \lambda u_1, \omega)$, $\lambda \in [0, 1]$, again links points $x(t_i, u_i, \omega)$, $i = 0, 1$, is contained in $S_\omega(T)$, and through each of its points there passes an integral manifold of V. Hence the union of these manifolds is again an integral manifold of V, contains S_0 and S_1, and is contained in $S_\omega(T)$. Q.E.D.

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