Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 1, 34–38

Persistent URL: http://dml.cz/dmlcz/100942

ON A CLASS OF FUNCTIONS WITH STARSHAPED IMAGES

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(Received February 2, 1968)

1. Introduction. Let S be the class of functions

$$f(z) = z + \sum_2^\infty b_k z^k \tag{1}$$

regular in $|z| < 1$ and satisfying the condition

$$\operatorname{Re}\left[\frac{z f'(z)}{f(z)}\right] > \frac{1}{2} \quad \text{for} \quad |z| < 1 . \tag{2}$$

In [1] WU ZWAO-JEN proved the following theorem.

Theorem A. *If $f(z) \in S$, then any section*

$$f_n(z) = z + b_2 z^2 + \ldots + b_n z^n \quad (n \geqq 2)$$

of $f(z)$ is starshaped in $|z| < \frac{1}{2}$ for $n \neq 3, 4, 5$.

Later in [2] it was shown that the above theorem holds for $n = 3$.

Thus is has yet to be proved that Theorem A holds for $n = 4, 5$.

The purpose pf this paper is to show that the statement is true for $n = 4, 5$. We shall need the following Lemma

2. Lemma. *Let $f(z) = z + \sum_2^\infty b_k z^k \in S$, then the coefficients satisfy the following equalities*

$$b_2 = b \,, \tag{3}$$

$$2b_3 = 2b^2 + e_1(1 - |b|^2) \,, \tag{4}$$

$$6b_4 = 4b^3 + 3e_1 b(1 - |b|^2) + 2e_2 \,, \tag{5}$$

$$24b_5 = 10b^4 + 12b^2(1 - |b|^2)\, e_1 + 3(1 - |b|^2)^2\, e_1^2 + 8be_2 + 6e_3 \,, \tag{6}$$

$$|b| \leqq 1 \,, \quad |e_n| \leqq 1 \quad \textit{for} \quad n = 1, 2, 3 \,.$$

Proof. Let

$$G(z) = \frac{2z f'(z)}{f(z)} - 1 ,$$

then $G(0) = 1$ and $\operatorname{Re}[G(z)] > 0$ for $|z| < 1$. Hence, by Caratheodory – Toeplitz's theorem we can put

$$\tag{7} \frac{2z f'(z)}{f(z)} - 1 = \frac{2(1 + 2b_2 z + 3b_3 z^2 + 4b_4 z^3 + 5b_5 z^4 + \ldots)}{(1 + b_2 z + b_3 z^2 + b_4 z^3 + b_5 z^4 + \ldots)} - 1 = = 1 + \delta_1 z + \delta_2 z^2 + \delta_3 z^3 + \delta_4 z^4 + \ldots ,$$

$$\tag{8} |\delta_1| \leqq 2 , \quad |2\delta_2 - \delta_1| \leqq 4 - |\delta_1|^2 , \quad |\delta_3| \leqq 2 , \quad |\delta_4| \leqq 2 .$$

Equating coefficients of z, z^2, z^3, z^4 on both sides of (7) we get

$$\tag{9} 2b_2 = \delta_1 ,$$

$$\tag{10} 4b_3 = b_2\delta_1 + \delta_2 ,$$

$$\tag{11} 6b_4 = b_3\delta_1 + b_2\delta_2 + \delta_3 ,$$

$$\tag{12} 8b_5 = b_4\delta_1 + b_3\delta_2 + b_2\delta_3 + \delta_4 .$$

By the second inequality of (8) we can put

$$\tag{13} 2\delta_2 - \delta_1^2 = e_1(4 - |\delta_1|^2) \quad (|e_1| \leqq 1) .$$

Putting $\delta_1 = 2b(|b| \leqq 1)$, $\delta_2 = \frac{1}{2}\delta_1^2 + \frac{1}{2}e_1(4 - |\delta_1|^2)$, $\delta_3 = 2e_2(|e_2| \leqq 1)$ and $\delta_4 = 2e_3(|e_3| \leqq 1)$ in (9), (10), (11) and (12) we get (3), (4), (5) and (6).

3. Proof of theorem a for $n = 4$.

We shall show that $\operatorname{Re}[z f_4'(z)/f_4(z)] > 0$ for $|z| < \frac{1}{2}$.

$$\operatorname{Re}\left[\frac{z f_4'(z)}{f_4(z)}\right] = \operatorname{Re}\left[\frac{1 + 2b_2 z + 3b_3 z^2 + 4b_4 z^3}{1 + b_2 z + b_3 z^2 + b_4 z^3}\right] =$$

$$= 2 - \operatorname{Re}\left[\frac{1 - b_3 z^2 - 2b_4 z^3}{1 + b_2 z + b_3 z^2 + b_4 z^3}\right] \geqq$$

$$\geqq 2 - \left|\frac{1 - b_3 z^2 - 2b_4 z^3}{1 + b_2 z + b_3 z^2 + b_4 z^3}\right| .$$

It is easy to see that the denominator never vanishes. Hence, by the principle of minimum for harmonic functions, we have only to prove that $\operatorname{Re}[z f_4'(z)/f_4(z)] > 0$ for $|z| = \frac{1}{2}$. By considering $\bar{\varepsilon} f(\varepsilon z)$ in place of $f(z)$ with a suitable ε ($|\varepsilon| = 1$), the proof is reduced to the case $z = \frac{1}{2}$. Thus it is sufficient to show

$$\left|\frac{4 - b_3 - b_4}{8 + 4b_2 + 2b_3 + b_4}\right| < 1 . \tag{14}$$

By (3), (4) and (5)

$$\left|\frac{4 - b_3 - b_4}{8 + 4b_2 + 2b_3 + b_4}\right| = \left|\frac{4 - b^2 - \frac{2}{3}b^3 - \frac{1}{2}e_1(1 + b)(1 - |b|^2) - \frac{1}{3}e_2}{8 + 4b + 2b^2 + \frac{2}{3}b^3 + e_1(1 + \frac{1}{2}b)(1 - |b|^2) + \frac{1}{3}e_2}\right| .$$

Again, by the principle of minimum it is sufficient to prove (14) with $|b| = 1$, $|e_1| = 1$, $|e_2| = 1$. Thus we see that the inequality (14) is satisfied if

$$2|12 + 6b + 3b^2 + b^3| - |12 - 3b^2 - 2b^3| - 2 > 0 , \quad (|b| = 1) .$$

On putting $\operatorname{Re} b = x$, we have

$$\begin{aligned} P(x) &= 2(106 + 114x + 168x^2 + 96x^3)^{1/2} - \\ &- (229 + 156x - 144x^2 - 192x^3)^{1/2} - 2 > 0 , \quad (-1 \leqq x \leqq 1) . \end{aligned} \tag{15}$$

Differentiating (15), we obtain

$$\begin{aligned} P'(x) &= \frac{6(19 + 56x + 48x^2)}{(106 + 114x + 168x^2 + 96x^3)^{1/2}} - \\ &- \frac{6(13 - 24x - 48x^2)}{(229 + 156x - 144x^2 - 192x^3)^{1/2}} , \end{aligned} \tag{16}$$

and

$$\begin{aligned} P''(x) &= \frac{6(4853 + 10176x + 5472x^2 + 5376x^3 + 2304x^4)}{(106 + 114x + 168x^2 + 96x^3)^{3/2}} + \\ &+ \frac{6(6510 + 21984x + 7488x^2 - 4608x^3 - 4608x^4)}{(229 + 156x - 144x^2 - 192x^3)^{3/2}} . \end{aligned} \tag{17}$$

It is easy to see that $P'(x) > 0$ for $0 \leqq x \leqq 1$ and $P'(x) < 0$ for $-1 \leqq x \leqq -\frac{1}{4}$. Consequently, the minimum value of $P(x)$ for $-1 \leqq x \leqq 1$ is attained in the interval $-\frac{1}{4} < x < 0$. Moreover, from (17) we find by an easy calculation that $P''(x) > 0$ for $-\frac{1}{4} \leqq x \leqq 0$. From (15) we have

$$P(-\tfrac{1}{4}) = 2\sqrt{(86{\cdot}5)} - \sqrt{(184)} - 2 = 3{\cdot}04 ,$$

and from (16)

$$P'(-\tfrac{1}{4}) = \frac{48}{\sqrt{86{\cdot}5}} - \frac{96}{\sqrt{184}} = -1{\cdot}91 .$$

For $-\frac{1}{4} \leqq x \leqq 0$, noticing $P''(x) > 0$, we have by Taylor's theorem

$$P(x) > P(-\tfrac{1}{4}) - (-\tfrac{1}{4} - x)\, P'(-\tfrac{1}{4})\,. \tag{18}$$

Putting $x = 0$ in (18) we get

$$\min_{-1 \leqq x \leqq 1} P(x) > P(-\tfrac{1}{4}) + \tfrac{1}{4}P'(-\tfrac{1}{4}) = 3{\cdot}04 - \frac{1{\cdot}91}{4} > 0\,.$$

This completes the proof of the theorem when $n = 4$.

4. Proof of the theorem a for $n = 5$.

$$\operatorname{Re}\left[\frac{z f_5'(z)}{f_5(z)}\right] = \operatorname{Re}\left[\frac{1 + 2b_2 z + 3b_3 z^2 + 4b_4 z^3 + 5b_5 z^4}{1 + b_2 z + b_3 z^2 + b_4 z^3 + b_5 z^4}\right] =$$

$$= 2 - \left[\frac{1 - b_3 z^2 - 2b_4 z^3 - 3b_5 z^4}{1 + b_2 z + b_3 z^2 + b_4 z^3 + b_5 z^4}\right] =$$

$$= 2 - \left|\frac{1 - b_3 z^2 - 2b_4 z^3 - 3b_5 z^4}{1 + b_2 z + b_3 z^2 + b_4 z^3 + b_5 z^4}\right|.$$

As in the case $n = 4$, we may prove reducing the case to that with $z = \frac{1}{2}$. Thus it is sufficient to prove

$$\left|\frac{16 - 4b_3 - 4b_4 - 3b_5}{16 + 8b_2 + 4b_3 + 2b_4 + b_5}\right| < 2\,.$$

On using (3), (4), (5) and (6) the above inequality reduces to

$$2\left|192 + 96b + 48b^2 + 16b^3 + 5b^4\right| -$$
$$- \left|192 - 48b^2 - 32b^3 - 15b^4\right| - 67 > 0\,, \quad (|b| = 1)\,.$$

On putting $\operatorname{Re} b = x$, we have

$$Q(x) = 2(28601 + 26464x + 28608x^2 + 28416x^3 + 15360x^4)^{1/2} - \tag{19}$$
$$- (51649 + 40896x + 12096x^2 - 49152x^3 - 46080x^4)^{1/2} - 67 > 0\,, \quad (-1 \leqq x \leqq 1)\,.$$

Differentiating (19) we obtain

$$Q'(x) = \frac{32(827 + 1788x + 2664x^2 + 1920x^3)}{(28601 + 26464x + 28608x^2 + 28416x^3 + 15360x^4)^{1/2}} - \tag{20}$$
$$- \frac{288(71 + 42x - 256x^2 - 320x^3)}{(51649 + 40896x + 12096x^2 - 49152x^3 - 46080x^4)^{1/2}}$$

and

$$
(21)\quad Q''(x) = 1536\left(\frac{10048931}{12} + 3174711x + 4900872x^2 + 3175616x^3 + \right.
$$
$$
\left. + 2505024x^4 + 1704960x^5 + 614400x^6\right)(28601 + 26464x +
$$
$$
+ 28608x^2 + 28416x^3 + 15360x^4)^{-3/2} - 18432\left(\frac{358725}{32} - \right.
$$
$$
- 413192x - 938319x^2 - 441216x^3 + 7584x^4 + 368640x^5 +
$$
$$
\left. + 230400x^6\right)(51649 + 40896x + 12096x^2 - 49152x^3 - 46080x^4)^{-3/2}.
$$

It is easy to see that $Q'(x) > 0$ for $-\frac{1}{2} \leqq x \leqq 1$ and $Q'(x) < 0$ for $-1 \leqq x \leqq -\frac{3}{4}$. Consequently, the minimum value of $Q(x)$ for $-1 \leqq x \leqq 1$ is attained in the interval $-\frac{3}{4} < x < -\frac{1}{2}$. Moreover from (21) we find by an easy calculation that $Q''(x) > 0$ for $-\frac{3}{4} \leqq x \leqq -\frac{1}{2}$.

From (19) we have

$$
Q(-\tfrac{1}{2}) = 2\sqrt{(19929)} - \sqrt{(37449)} - 67 = 21{\cdot}83\,,
$$

and from (20)

$$
Q'(-\tfrac{1}{2}) = \frac{11488}{\sqrt{(19929)}} - \frac{7488}{\sqrt{(37449)}} = 42{\cdot}68\,.
$$

For $-\frac{3}{4} \leqq x \leqq -\frac{1}{2}$, noticing $Q''(x) > 0$, we have by Taylor's theorem

$$
(22)\qquad Q(x) > Q(-\tfrac{1}{2}) - (-\tfrac{1}{2} - x)\,Q'(-\tfrac{1}{2})\,.
$$

Taking $x = -\frac{3}{4}$ in (22), we get

$$
\operatorname*{Min}_{-1 \leqq x \leqq 1} Q(x) > Q(-\tfrac{1}{2}) - \tfrac{1}{4}Q'(-\tfrac{1}{2}) = 21{\cdot}83 - \frac{42{\cdot}68}{4} > 0\,.
$$

This completes the proof of the theorem.

Acknowledgements. At the end I would like to express my thanks to Professor Vikramaditva Singh for his kind encouragement and guidance.

References

[1] *Wu Zwao-jen:* A class of functions with starshaped images, Acta Math. Sinica 6 (1956) pp. 476—489.

[2] *Wu Zwao-jen:* A class of functions with starshaped images, Acta Math. Sinica 7 (1957) pp. 433—438.

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