Oldřich Kowalski

Immersions of Riemannian manifolds with a given normal bundle structure. I

Czechoslovak Mathematical Journal, Vol. 19 (1969), No. 4, 676–696

Persistent URL: http://dml.cz/dmlcz/100928

Czechoslovak Mathematical Journal, 19 (94) 1969, Praha

IMMERSIONS OF RIEMANNIAN MANIFOLDS WITH A GIVEN NORMAL BUNDLE STRUCTURE

Part I

OLDŘICH KOWALSKI, Praha

(Received November 5, 1968)

A start point for this investigation has been given by L. BOČEK in his paper [1]. There the author characterizes the isometric immersion of a Riemannian manifold M into a manifold N with a constant curvature. This is done by a system of tensors given on M, called *fundamental tensors of the* 1^{st}, 2^{nd}, ..., etc. *order*. The main point is to give a full system of integrability conditions for the problem. In [1] a rather complicated system of analytic conditions is found, some of them being formulated only implicitly.

Here we show that, under the assumption of maximality of the osculation spaces, we can replace the compatibility and integrability conditions of [1] by a single system of geometrical conditions called *generalized Gaussian equations*. Moreover, another immersion problem is solved without any restriction on the dimension of the osculation spaces. Here the fundamental tensors are replaced by *fundamental forms*, and the integrability conditions consist of a system of *generalized Gaussian equations* and a system of *generalized Mainardi-Codazzi equations*. Let us remark that the integrability conditions are not identical with those studied in the book [2].

Throughout this paper the reader is expected to have a basic knowledge about vector bundles, principal fibre bundles and connections in principal bundles. For the sake of simplicity all manifolds, fibre bundles, sections, mappings, etc. are supposed to be of the class C^∞.

CONNECTIONS IN VECTOR BUNDLES

Let $p : E \to M$ be a vector bundle over M, and $\mathscr{D}^1 E$ the vector bundle over M of 1-jets of all local sections $s : M \to E$. *An (infinitesimal) connection* in E is a bundle morphism $D : E \to \mathscr{D}^1 E$ compatible with the projection $\mathscr{D}^1 E \to E$. (Cf [4].)

A covariant derivative in E, ∇, is given by the following properties:

a) for $x \in M$, $t \in T(M)_x$ and for any local section $U : M \to E$ defined at x we have $\nabla_t U \in E_x$,

b) $\nabla_{at_1+bt_2} U = a \,.\, \nabla_{t_1} U + b \,.\, \nabla_{t_2} U$,

c) $\nabla_t(U_1 + U_2) = \nabla_t U_1 + \nabla_t U_2$,

d) $\nabla_t(fU) = (tf)\, U(x) + f(x)\, \nabla_t U$.

Proposition 1. *There is a* $(1-1)$ *correspondence between the connections and the covariant derivatives in* E.

Proof. If V is a vector space, $v \in V$, let us denote by σ_v the canonical isomorphism $V \to T_v(V)$. Let a connection D be given. For any $t \in T(M)_x$ and for any local section $Y: M \to E$, $Y(x) = e$ we put $\nabla_t Y = \sigma_e^{-1}(dY(t) - ds(t))$ where $s : M \to E$ is a local section representing the 1-jet $D(e)$. (Here $dY(t) - ds(t) \in T(E_x)_e$. Conversely, if ∇ is given and $e \in E_x$, we choose an arbitrary section $Y: M \to E$ passing through e and represent De by a section $s : M \to E$ such that $ds(t) = dY(t) - \sigma_e \nabla_t Y$ for any $t \in \in T(M)_x$. It suffices to verify that if we replace Y by the section $f \,.\, Y$ with $f(x) = 1$ then the expression $\sigma_e^{-1}(dY(t) - ds(t))$ is increased by $(tf)\; Y(x)$ and the expression $dY(t) - \sigma_e \nabla_t Y$ does not change. But this can be checked easily by means of the formula $d(fY)(t) = df(t)\, \sigma_e Y + dY(t)$. The rest is trivial.

We shall often speak about "connection ∇" instead of "covariant derivative ∇". Let us remind also the well-known fact that *the connections in a vector bundle* $E \to M$ *are in a* $(1-1)$ *correspondence with the connections in the associated principal frame bundle* $B(E) \to M$.

Let $E \to M$ be a vector bundle, $E = E_1 \oplus E_2$ (the direct sum). Then $\mathscr{D}^1 E = \mathscr{D}^1 E_1 \oplus \oplus \mathscr{D}^1 E_2$ and we have canonical projections $\tilde{\pi}_i : \mathscr{D}^1 E \to \mathscr{D}^1 E_i$ induced by the projections $\pi_i : E \to E_i$, $i = 1, 2$. Hence to any connection $D : E \to \mathscr{D}^1 E$ we obtain the *projection connections* $D^i = \tilde{\pi}_i \circ D$.

Proposition 2. *For the corresponding covariant derivatives we have* $\nabla^i = \pi_i \circ \nabla$, $i = 1, 2$.

Proof. Let be given $x \in M$, $Y^i : M \to E_i$, $Y^i(x) = e^i$, and a section $s^i : M \to E$ representing De^i. Then $(\pi_i \circ \nabla_t)(Y^i) = (\pi_i \circ \sigma_{ei}^{-1})(dY^i(t) - ds^i(t)) = (\sigma_{ei}^{-1} \circ d\pi_i) \,.\, (dY^i(t) - ds^i(t)) = \sigma_{ei}^{-1}[d(\pi_i \circ Y^i)(t) - d(\pi_i \circ s^i)(t))$. But $\pi_i \circ Y^i = Y^i$ and $\pi_i \circ s^i$ represents the jet $D^i e^i$. Hence $\pi_i \circ \nabla = \nabla^i$, q.e.d.

Note. Let $E \to M$ be a vector bundle, $E = E_1 \oplus E_2$, and let $B(E)$, $B(E_1)$, $B(E_2)$ be the associated principal frame bundles over M. Then any connection $\tilde{D}$ in $B(E)$ can be reduced to the subbundle $B(E_1, E_2)$ of adapted frames, i.e., of all frames of the form $(e_1, \ldots, e_k, e_{k+1}, \ldots, e_r)$ where $e_1, \ldots, e_k \in E_1$, $e_{k+1}, \ldots, e_r \in E_2$. Using the canonical projections $B(E_1, E_2) \begin{smallmatrix} \nearrow B(E_1) \\ \searrow B(E_2) \end{smallmatrix}$ we obtain *the projection connections* $\tilde{D}^1$, $\tilde{D}^2$

in $B(E_1)$, $B(E_2)$ respectively. (Cf [3].) It is not difficult to show that if D is a connection in E associated with $\tilde{D}$, then its projection connections D^1, D^2 are associated with $\tilde{D}^1$, $\tilde{D}^2$ respectively.

Let us introduce the following notation: If $\pi : E \to M$ is a vector bundle, dim $E = d$, and if $b = (f_1, \ldots, f_d)$ is a frame at $x \in M$, then we define *the isomorphism* $b : R^d \to E_x$ by the rule

$$b \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix} = \sum_{i=1}^{d} u_i f_i \quad \text{for any } \textit{column} \quad \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} \in R^d .$$

Similarly we define the inverse mapping $b^{-1} : E_x \to R^d$. Now let D be a connection in E and $\tilde{D}$ the associated connection in the principal bundle $\tilde{\pi} : B(E) \to M$. Then *the connection form*, φ, and *the curvature form*, Φ, of $\tilde{D}$ are defined and they are $(R^d \otimes R^d)$-valued. If a point $x \in M$ and two vectors $u, t \in T(M)_x$ are given, then *the curvature transformation* $R_{ut} : E_x \to E_x$ will be defined by the formula

$$R_{ut}(v) = b\, \Phi(\bar{u}, \bar{t})\, b^{-1}(v)\,, \quad v \in E_x \tag{1}$$

where $b \in B(E)_x$ is an arbitrary frame and $\bar{u}, \bar{t} \in T(B)_b$ are arbitrary vectors such that $d\tilde{\pi}(\bar{u}) = u$, $d\tilde{\pi}(\bar{t}) = t$. It is easy to verify that the above definition is independent of the choice of b, $\bar{u}$ and $\bar{t}$. (Cf. Bishop-Crittenden, [3]).

Proposition 3. *Let ∇ be the covariant derivative corresponding to D. Then for any local vectorfields U, T on M and for any local section $V : M \to E$ defined in the same domain $\Omega \subset M$ we have*

$$R_{UT}(V) = \nabla_U \nabla_T V - \nabla_T \nabla_U V - \nabla_{[U,T]} V . \tag{2}$$

Proof is the same as in the classical case $E = T(M)$.

A soldered vector bundle, (E, j), is a couple consisting of a vector bundle $\pi : E \to M$ and a bundle injection $j : T(M) \to E$. Particularly, for a soldered vector bundle E we have dim $E \geqq$ dim M. Let us remark that this definition is a natural modification of that given in paper [5].

Let $B(E) \to M$ (canonical projection $\tilde{\pi}$) be a principal frame bundle associated to a soldered vector bundle (E, j). *The solder form* ω is defined on $B(E)$ by the formula

$$\omega_b(\bar{u}) = b^{-1}(j \circ d\tilde{\pi}(\bar{u})) \tag{3}$$

for any $b \in B(E)$, $\bar{u} \in T(B)_b$. (Cf. [3].)

Clearly the solder form is an R^d-form, $d = \dim E$.

Let be given a connection $\tilde{D}$ in $B(E, j)$. *The torsion form* Ω of the connection $\tilde{D}$ is defined by the formula

$$(4) \qquad \Omega(\bar{u}, \bar{t}) = d\omega(H\bar{u}, H\bar{t})$$

for any two $\bar{u}, \bar{t} \in T(B)_b$, where H means the horizontal projection with respect to the connection $\tilde{D}$. Ω is an R^d-form, too.

Let A, B be two matrix forms on M such that the matrix product $A \, . \, B$ exists. If $A = (a_{ij})$, $B = (b_{jk})$ then we define the exterior product $A \wedge B$ by the formula $A \wedge B = (\sum_j a_{ij} \wedge b_{jk})$.

Proposition 4. *If $B(E, j) \to M$ is a principal bundle associated to a soldered vector bundle and if a connection $\tilde{D}$ is given in $B(E, j)$ then we have two structural equations*

$$(5) \qquad \begin{aligned} &\text{a)} \quad d\omega = -\varphi \wedge \omega + \Omega \\ &\text{b)} \quad d\varphi = -\varphi \wedge \varphi + \Phi \end{aligned}$$

Here the latter equation is classical, the former is analogous to the classical case. (Cf. [3], [6].)

Let (E, j) be a soldered vector bundle with a connection D, so that the torsion form Ω of the associated connection $\tilde{D}$ in $B(E, j) \to M$ (projection $\tilde{\pi}$) is defined. *The torsion translation* T_{ut} is defined for any $x \in M$ and any two vectors $u, t \in T(M)_x$ as follows: let $b \in B(E)$, $\bar{u}, \bar{t} \in T(B)_b$ be such that $d\tilde{\pi}(\bar{u}) = u$, $d\tilde{\pi}(\bar{t}) = t$. Then put

$$(6) \qquad T_{ut} = b \, \Omega(\bar{u}, \bar{t}) \, .$$

We can see that the above definition is independent of the choice of $b, \bar{u}, \bar{t}$. We always have $T_{ut} \in E_x$.

Proposition 5. *For any two local vector fields X, Y on M we have*

$$(7) \qquad T_{XY} = \nabla_X(jY) - \nabla_Y(jX) - j[X, Y] \, .$$

Proof. We use the first structural equation and the following Lemma:

Lemma. *Let X be a vector field on M, $t \in T(M)_x$, $\bar{t}$ a lift of t to $b \in B(E)$, $\bar{X}$* a lift *of X. Then*

$$(8) \qquad \nabla_t(jX) = b(\bar{t}\omega(\bar{X}) + \varphi(\bar{t}) \, \omega(\bar{X}(b))) \, .$$

Proof is the same as the proof of Theorem 11, [3].

A vector bundle $E \to M$ is called *Riemannian* if on each fibre E_x there is defined a symmetric positively definite bilinear form $\langle\ ,\ \rangle_x$ (inner product) and if the real function f induced on the bundle $E \otimes E$ (tensor product) by all $\langle\ ,\ \rangle_x$, $x \in M$, is of class C^∞. If ∇ is an arbitrary connection in E then in any subbundle $F \subset E$ we have *a canonical projection connection* ∇^F because of the canonical splitting $E = F \oplus F^\perp$. A connection ∇ defined in E will be called *semi-Riemannian* if the parallel translation by ∇ preserves the inner product. Clearly a connection ∇ is semi-Riemannian if and only if

$$(9) \qquad t\langle U, V\rangle = \langle \nabla_t U, V\rangle + \langle U, \nabla_t V\rangle$$

for any vector t on M and for any two local sections U, V of E. Hence we see that any projection of a semi-Riemannian connection is semi-Riemannian.

Let now $E \to M$ be a Riemannian vector bundle over a Riemannian manifold M. Let us denote by ∇ the canonical Riemannian connection on M and let ∇^E be a fixed semi-Riemannian connection in E. Consider a function $H(U, T, X, Y)$ induced by a bundle morphism $H : T(M) \otimes T(M) \otimes E \otimes E \to M \times R$, i.e., $H(U, T, X, Y) = H(U \otimes T \otimes X \otimes Y)$. *The covariant derivative* $\nabla_Z H$ *of* H is a function of the same type defined by the formula

$$(10) \qquad (\nabla_Z H)(U, T, X, Y) = Z\{H(U, T, X, Y)\} - H(\nabla_Z U, T, X, Y) - H(U, \nabla_Z T, X, Y) - H(U, T, \nabla^E_Z X, Y) - H(U, T, X, \nabla^E_Z Y).$$

We say that H *satisfies the Bianchi identity* if

$$(11) \qquad (\nabla_Z H)(U, T, X, Y) + (\nabla_T H)(Z, U, X, Y) + (\nabla_U H)(T, Z, X, Y) = 0$$

holds for any local sections Z, U, T of $T(M)$ and X, Y of E.

Proposition 6. *Let* R^E_{UT} *denote the curvature transformation of* ∇^E. *Then the function* $R^E(U, T, X, Y) = \langle R^E_{UT}X, Y\rangle$ *satisfies the Bianchi identity.*

Proof is the same as in the classical case (see [6]).

A Riemannian vector bundle over a Riemannian manifold, $E \to M$, will be called *soldered* if it is a soldered vector bundle (E, j) and if the solder map $j : T(M) \to jT(M) \subset E$ is an isometry. A semi-Riemannian connection D in (E, j) is called *Riemannian* if its torsion form vanishes.

Now, any projection of a Riemannian connection is also Riemannian (assuming that the bundle $F \subset E$ is still soldered by the mapping j). Indeed, let ∇^E, ∇^F be the corresponding covariant derivatives, T^E, T^F the corresponding torsion translations.

According to (7) we have

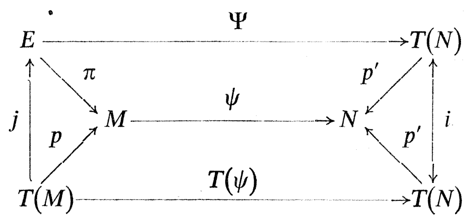
$$T^F_{XY} = \nabla^F_X(jY) - \nabla^F_Y(jX) - j[X, Y] \in F$$
$$T^E_{XY} = \nabla^E_X(jY) - \nabla^E_Y(jX) - j[X, Y] = 0 ,$$

hence $T^F_{XY} = \text{proj}_F(T^E_{XY}) = 0$ and $\Omega^F = 0$ as required. Particularly, if we project a Riemannian connection ∇^E from (E, j) onto $jT(M)$ and then onto $T(M)$, we obtain the usual Riemannian connection ∇ of the Riemannian manifold M.

*

Let M, N be two Riemannian manifolds, $\dim M \leqq \dim N$, and $\psi : M \to N$ an isometric immersion. The induced bundle $E = \psi^* T(N)$ is a soldered Riemannian vector bundle over M, the solder map j being defined by the commutative diagram

(12)



Here π, p, p' denote the canonical projections, Ψ a canonical bundle map, i the identity map and $T(\psi)$ the tangent map to ψ. Moreover, a Riemannian connection D is induced in E. Remark that if N has a constant curvature C then the identity $\Phi = C(\omega \wedge \omega^t)$ holds in the frame bundle $B(N)$ (cf. [3], p. 184), and consequently, in the frame bundle $B(E)$. (Here ω^t denotes the transpose of the column matrix ω, i.e., a row matrix).

Let $E \to M$, $E' \to M'$ be two fibre bundles of the same type fibre. A bundle map $E \to E'$ will be called a *fibre isomorphism* if any fibre is mapped isomorphically onto a fibre.

Theorem 1. (First Immersion Theorem). *Let (E, j) be a soldered Riemannian vector bundle over a simply connected Riemannian manifold M,* $\dim M = m \leqq \leqq \dim E = d$. *Let D be a Riemannian connection in E such that the equality $\Phi = C(\omega \wedge \omega^t)$ holds in $B(E)$. Further let N be a complete Riemannian d-manifold with the constant curvature C; denote by D^N the canonical Riemannian connection in $T(N)$. Then there exists exactly one fibre isomorphism Ψ of E into $T(N)$ such that*

a) $\Psi(e_1, \ldots, e_d) = (f_1, \ldots, f_d)$ where $(e_1, \ldots, e_d) \in B(E)$, $(f_1, \ldots, f_d) \in B(N)$ are prescribed orthonormal frames, *b) a commutative diagram* (12) *holds,*

c) $D = \Psi^* D^N$.

Moreover, Ψ *is an isometry of Riemannian bundles and* ψ *is an isometric immersion of* M *into* N.

Proof. If $\Psi : E \to T(N)$ is a fibre isomorphism as required, then we have an induced mapping $\tilde{\Psi} : B(E) \to B(N)$ of principal bundles. From *c)* we obtain $\tilde{D} = \tilde{\Psi}^* \tilde{D}^N$ for the corresponding connections, i.e., $\varphi = \tilde{\Psi}^* \varphi^N$ for the connection forms. From *b)* we obtain $\omega = \tilde{\Psi}^* \omega^N$ for the corresponding solder forms. Consider the diagram $P = B(E) \times B(N) \begin{smallmatrix} \overset{p_1}{\nearrow} B(E) \\ \underset{p_2}{\searrow} B(N) \end{smallmatrix}$. The matrix equations $p_1^* \varphi - p_2^* \varphi^N = 0$, $p_1^* \omega - p_2^* \omega^N = 0$ represent a system of $d + d^2$ independent linear equations and hence a linear distribution of dimension $m + d^2$ on P. The system of equations is completely integrable because of the structure equations and because $\Omega = \Omega^N = 0$,

$\Phi^N = C(\omega^N \wedge \omega^{N^t})$. Hence the distribution is involutive and there is exactly one maximal integral manifold $\tilde{B}$ through any point (e, f) of P. The mappings p_1, p_2 are regular on $\tilde{B}$ and $\tilde{B}$ becomes a covering manifold over $B(E)$ with the covering map p_1. From the relation $p_1^* \varphi = p_2^* \varphi^N$ we see that the full linear group $GL(d)$ acts freely on the integral manifold $\tilde{B}$. $\tilde{B}$ is a principal fibre bundle over the factor manifold $\tilde{B}/GL(d)$ and p_1, p_2 are fibre isomorphisms. Here p_1 induces a covering map $\pi_1 : \tilde{B}/GL(d) \to M$. Because M is simply connected, π_1 is a diffeomorphism; thus p_1 is a bundle isomorphism and consequently a diffeomorphism. We obtain an immersion $\tilde{\Psi} = p_2 p_1^{-1} : B(E) \to B(N)$ which is a fibre isomorphism and induces an immersion $\psi : M \to N$. Now it is easy to complete the proof.

CANONICAL GRADUATIONS

Let $(E, j) \to M$ be a soldered Riemannian vector bundle with a Riemannian connection ∇. We can identify $T(M)$ with a subbundle $E^1 \subset E$ via j. At any point $x \in M$ put $S_x^1 = E_x^1$, $S_x^2 =$ the subspace of E_x generated by S_x^1 and by all vectors of the form $\nabla_t X^{[1]}$ where $t \in E_x^1$ and $X^{[1]} : M \to S^1 = E^1$ is a local section at x. Put $S^2 = \bigcup_{x \in M} S_x^2$. Further, let S_x^3 be the subspace of E_x generated by S_x^2 and by all vectors of the form $\nabla_t X^{[2]}$ where $t \in E_x^1$ and $X^{[2]} : M \to E$ is a local section at x such that $\operatorname{Im} X^{[2]} \subset S^2$. Put $S^3 = \bigcup_{x \in M} S_x^3$, etc. Finally, we obtain at any point $x \in M$ an orthogonal decomposition $E_x = S_x^r + Z_x$ where $\nabla_t X^{[r]} \in S_x^r$ for any $t \in E_x^1$ and for any local section $X^{[r]} : M \to E$ at x such that $\operatorname{Im} X^{[r]} \subset S^r$. The number r depends, in general, on x. Moreover, we have the orthogonal decomposition

$$S_x^k = E_x^1 + E_x^2 + \ldots + E_x^k \quad \text{for any} \quad k \leqq r\,, \quad x \in M\,.$$

Let us put $E^k = \bigcup_{x \in M} E_x^k$ for $k = 1, \ldots, r$. Now, the connection ∇ will be called *stable* if $\dim S_x^k$ is constant on M for any $k = 1, \ldots, r$, i.e., if the subsets S^k are subbundles of E. Then E^k, $k = 1, \ldots, r$ are subbundles of E as well and we have the canonical orthogonal splitting

$$E = E^1 \oplus E^2 \ldots \oplus E^r \oplus Z \tag{13}$$

where $Z = \bigcup_{x \in M} Z_x$. Each subspace S_x^{k+1} is generated by S_x^k and by all vectors of the form $\nabla_t X^{(k)}$ where $t \in T(M)_x$ and $X^{(k)}$ is a local section at x of the bundle E^k.

Each subspace S_x^k is called *the osculation space of order* k at x and each subbundle S^k is called *the osculation bundle of order* k. The elements of E^k will be called *vectors of grade* k and the local sections $X^{(k)} : M \to E^k$ *sections of grade* k. The orthogonal splitting (13) will be called *the canonical graduation.* An immersion $\psi : M \to N$ will be called *stable* if the induced connection ∇ in $E = \psi^* T(N)$ is stable. In the following we suppose that Riemannian connections and immersions under consideration are stable.

It is well-known that any local section in a vector bundle can be prolonged to the whole base. For the sake of simplicity (to avoid speaking about definition domains) we shall work with global sections throughout this paper.

Proposition 7. *Let $E = E^1 \oplus E^2 \oplus \ldots \oplus E^r \oplus Z$ be the canonical graduation induced by ∇ in (E, j). Then there exist bundle morphisms*

$$P_k : E^1 \otimes E^k \to E^{k+1} \quad (k = 1, \ldots, r - 1)$$
$$L_k : E^1 \otimes E^k \to E^{k-1} \quad (k = 2, \ldots, r)$$

such that for any vector $t \in E^1$ and any section $X^{(k)}$ of grade k we have the orthogonal decomposition

$$\nabla_t X^{(k)} = L_k(t \otimes X^{(k)}) + \nabla_t^{(k)} X^{(k)} + P_k(t \otimes X^{(k)}) . \tag{14}$$

Here $\nabla^{(k)}$ is the projection connection of ∇ onto E^k and $P_r = L_1 = 0$ by definition.

All P_k are bundle epimorphisms and P_1 is symmetric. Further, for any section $U : M \to Z$ and for any vector $t \in E_x^1$ we have $\nabla_t U \in Z$.

Proof. According to the definition we have $\nabla_t X^{(k)} \in S_x^{k+1}$. If $i \leqq k - 2$, then $\langle X^{(k)}, X^{(i)} \rangle = 0$ for any $X^{(i)}$ and $t\langle X^{(k)}, X^{(i)} \rangle = \langle \nabla_t X^{(k)}, X^{(i)} \rangle + \langle X^{(k)}, \nabla_t X^{(i)} \rangle = 0$. Since $\nabla_t X^{(i)} \in S_x^{k-1}$ we have $\langle X^{(k)}, \nabla_t X^{(i)} \rangle = 0$ and hence $\langle \nabla_t X^{(k)}, X \rangle = 0$ for any $X \in S_x^{k-2}$. We obtain $\nabla_t X^{(k)} \in E_x^{k-1} \dotplus E_x^k \dotplus E_x^{k+1}$ for any $k = 1, \ldots, r$. (Here put $E_x^0 = E_x^{r+1} = 0$ by definition). Now, $\text{proj}_{E^k}(\nabla_t X^{(k)}) = \nabla_t^{(k)} X^{(k)}$ and clearly $\nabla - \nabla^{(k)}$ induces a bilinear mapping $E_x^1 \times E_x^k \to E_x^{k-1} \dotplus E_x^{k+1}$ at any $x \in M$. Hence we obtain a bundle morphism

$$L_k \oplus P_k : E^1 \otimes E^k \to E^{k-1} \oplus E^{k+1}$$

as required. Further, ∇ being Riemannian, we have for any two sections X, Y of E^1 $\nabla_X(Y) - \nabla_Y(X) - [X, Y] = \nabla_X^{(1)}(Y) - \nabla_Y^{(1)}(X) - [X, Y] + P_1(X \otimes Y) - P_1(Y \otimes X) = 0$, and because $\nabla^{(1)}$ is Riemannian, we have $P_1(X \otimes Y) = P_1(Y \otimes X)$. The rest of the proof is trivial. We shall write also $P_k(U, X^{(k)})$, $L_k(U, X^{(k)})$ instead of $P_k(U \otimes X^{(k)})$, $L_k(U \otimes X^{(k)})$.

Remark that we have identically

$$(15) \qquad \langle L_k(T, X^{(k)}), Y^{(k-1)} \rangle = - \langle X^{(k)}, P_{k-1}(T, Y^{(k-1)}) \rangle$$

by differentiation of the identity $\langle X^{(k)}, Y^{(k-1)} \rangle = 0$. Let us denote by $[X]^k$ the projection of a vector $X \in E$ into E^k.

Proposition 8. *Denote by R_{UT}, $R_{UT}^{(k)}$ the curvature transformations of ∇, $\nabla^{(k)}$ respectively. Then*

a) $R_{UT}X^{(k)} \in (E^{k-2} \oplus E^{k-1} \oplus E^k \oplus E^{k+1} \oplus E^{k+2})$,

b)
$$\begin{aligned}
[R_{UT}X^{(k)}]^{k-2} &= L_{k-1}(U, L_k(T, X^{(k)})) - L_{k-1}(T, L_k(U, X^{(k)})),\\
[R_{UT}X^{(k)}]^{k-1} &= \nabla_U^{(k-1)} L_k(T, X^{(k)}) - \nabla_T^{(k-1)} L_k(U, X^{(k)}) +\\
&\quad + L_k(U, \nabla_T^{(k)} X^{(k)}) - L_k(T, \nabla_U^{(k)} X^{(k)}) - L_k([U, T], X^{(k)}),\\
[R_{UT}X^{(k)}]^{k} &= R_{UT}^{(k)} X^{(k)} + P_{k-1}(U, L_k(T, X^{(k)})) - P_{k-1}(T, L_k(U, X^{(k)})) +\\
&\quad + L_{k+1}(U, P_k(T, X^{(k)})) - L_{k+1}(T, P_k(U, X^{(k)})),\\
[R_{UT}X^{(k)}]^{k+1} &= \nabla_U^{(k+1)} P_k(T, X^{(k)}) - \nabla_T^{(k+1)} P_k(U, X^{(k)}) +\\
&\quad + P_k(U, \nabla_T^{(k)} X^{(k)}) - P_k(T, \nabla_U^{(k)} X^{(k)}) - P_k([U, T], X^{(k)}),\\
[R_{UT}X^{(k)}]^{k+2} &= P_{k+1}(U, P_k(T, X^{(k)})) - (P_{k+1}(T, P_k(U, X^{(k)})).
\end{aligned}$$

Here we put $P_r = 0$, $L_1 = 0$, $\nabla^{(0)} = \nabla^{(r+1)} = 0$.

Proof. By direct calculation using (2) and (14).

In the following we are going to study in detail the immersions of the form $\psi : M \to N$, where N has a constant curvature C. As we know, the identity $\Phi = C(\omega \wedge \omega^t)$ holds on the frame bundle $B(E)$, where $E = \psi^* T(N)$ is the induced vector bundle of the immersion. Using an adapted orthonormal frame b in formula (1) we obtain

$$R_{UT}X^{(k)} = C\{\langle T, X^{(k)} \rangle U - \langle U, X^{(k)} \rangle T\}.$$

Hence $R_{UT}X^{(k)} = 0$ for $k > 1$ and $[R_{UT}X^{(1)}]^2 = [R_{UT}X^{(1)}]^3 = 0$.

Proposition 9. *Let the connection ∇ be such that $\Phi = C(\omega \wedge \omega^t)$ on $B(E)$. Then the composed mappings $P^k = P_{k-1} \circ \ldots \circ P_2 \circ P_1$, $k > 1$, of $\underbrace{E^1 \otimes E^1 \otimes \ldots \otimes E^1}_{k \text{ times}}$ into E^k are symmetric in all variables.*

Proof. We use the last formula of Proposition 8 and the symmetry of P_1 (Proposition 7).

Let once again $\psi : M \to N$, $E = \psi^* T(N)$ be given; suppose $E = E^1 \oplus \ldots \oplus E^r \oplus \oplus Z$. If N has a constant curvature, it is not difficult to show the following: if $Z \not\equiv 0$, then there is a totally geodesic submanifold $\tilde{N} \subset N$ such that $\dim \tilde{N} = \dim (E^1 \oplus \ldots \ldots \oplus E^r)$ and $\psi(M) \subset \tilde{N}$. Thus we can always reduce our problem to the case $Z \equiv 0$, $E = E^1 \oplus \ldots \oplus E^r$. In the following we shall consider the reduced problem only.

For the further considerations it will be convenient to introduce the following definition:

Definition 1. A graded Riemannian bundle $\{E^k, P_k\}^r$ over a Riemannian manifold M is a soldered Riemannian bundle $(E, j) \to M$ provided by an orthogonal splitting (graduation) $E = E^1 \oplus \ldots \oplus E^r$ and by a system of bundle epimorphisms $P_k : E^1 \otimes \otimes E^k \to E^{k+1}$, $k = 1, \ldots, r - 1$. Moreover, we require that $E^1 = j\,T(M)$ and that the composed mappings $P^k : \underbrace{E^1 \otimes \ldots \otimes E^1}_{k \text{ times}} \to E^k$ be all symmetric.

Thus each P^k induces an epimorphism $\underbrace{E^1 \circ \ldots \circ E^1}_{k \text{ times}} \to E^k$ where $\circ$ denotes the symmetric tensor product.

Proposition 10. *Let a graded Riemannian bundle* $\{E^k, P_k\}^r$ *be given. Define bundle morphisms*

$$L_k : E^1 \otimes E^k \to E^{k-1} \quad (k = 2, \ldots, r) \tag{16}$$

by relations (15). *Then*

$$L_{k-1}(U, L_k(T, X^{(k)})) = L_{k-1}(T, L_k(U, X^{(k)})) \tag{17}$$

for any $U, T, X^{(k)}$.

Proof. Using (15) twice we obtain

$$\langle L_{k-1}(U, L_k(T, X^{(k)})), Y^{(k-2)} \rangle = \langle P_{k-1}(T, P_{k-2}(U, Y^{(k-2)})), X^{(k)} \rangle .$$

But the composed mapping $P_{k-1} \circ P_{k-2}$ is symmetric with respect to T, U because P^k is symmetric in all variables.

Definition 2. Let $\{E^k, P_k\}^r$ be a graded Riemannian vector bundle over M, and let l be an integer, $1 \leqq l \leqq r$. A sequence of canonical connections in the subbundle $S^l = E^1 \oplus \ldots \oplus E^l$ is a sequence of semi-Riemannian connections $\nabla^{(1)}, \ldots, \nabla^{(l)}$ in the bundles $E^1, \ldots, E^l$ respectively such that

a) $j^*\nabla^{(1)} = \nabla =$ the canonical Riemannian connection in $T(M)$,

b) for any k, $1 \leqq k \leqq l - 1$, we have

$$\nabla_U^{(k+1)} P_k(T, X^{(k)}) - \nabla_T^{(k+1)} P_k(U, X^{(k)}) + P_k(U, \nabla_T^{(k)} X^{(k)}) - - P_k(T, \nabla_U^{(k)} X^{(k)}) - P_k([U, T], X^{(k)}) = 0 \tag{18}$$

for arbitrary sections $U, T, X^{(k)}$.

Proposition 11. *Let be given a sequence of canonical connections in* $E^1 \oplus \ldots \oplus E^l$. *Then*

a) *For* $k = 1, \ldots, l-1$ *and for any sections* $U, T, Y^{(k+1)}$ *we have*

$$\nabla_U^{(k)} L_{k+1}(T, Y^{(k+1)}) - \nabla_T^{(k)} L_{k+1}(U, Y^{(k+1)}) + \\ + L_{k+1}(U, \nabla_T^{(k+1)} Y^{(k+1)}) - L_{k+1}(T, \nabla_U^{(k+1)} Y^{(k+1)}) - L_{k+1}([U, T], Y^{(k+1)}) = 0\,, \tag{19}$$

b) *The Bianchi identity holds for the functions*

$$L_k(U, T, X^{(k)}, Y^{(k)}) = \langle L_k(T, X^{(k)}), L_k(U, Y^{(k)})\rangle - \langle L_k(U, X^{(k)}), L_k(T, Y^{(k)})\rangle\,, \tag{20}$$

$2 \leqq k \leqq l$, *and also for the functions*

$$P_k(U, T, X^{(k)}, Y^{(k)}) = \langle P_k(T, X^{(k)}), P_k(U, Y^{(k)})\rangle - \langle P_k(U, X^{(k)}), P_k(T, Y^{(k)})\rangle \tag{21}$$

where $1 \leqq k \leqq l - 1$.

Proof. Using (9) and (15) we obtain

$$\langle \nabla_U^{(k)} L_{k+1}(T, Y^{(k+1)}) - L_{k+1}(T, \nabla_U^{(k+1)} Y^{(k+1)}), X^{(k)}\rangle = \\ = -\langle \nabla_U^{(k+1)} P_k(T, X^{(k)}) - P_k(T, \nabla_U^{(k)} X^{(k)}), Y^{(k+1)}\rangle\,, \quad k = 1, \ldots, l-1\,.$$

Denoting by $\Delta_{k+1}(L)$ the left hand side of (19) and by $\Delta_k(P)$ the left hand side of (18), we obtain easily

$$\langle \Delta_{k+1}(L), X^{(k)}\rangle = -\langle \Delta_k(P), Y^{(k+1)}\rangle = 0 \quad \text{and} \quad \text{hence} \quad \Delta_{k+1}(L) = 0$$

for $k = 1, \ldots, l-1$.

The Bianchi identity may be checked directly using (9), (10), (11), (18) and (19).

*

In the following we shall denote by P the bundle epimorphism of $E^1 \oplus (E^1 \otimes \otimes E^1) \oplus \ldots \oplus (\overset{r}{\otimes} E^1)$ onto $E^1 \oplus E^2 \oplus \ldots \oplus E^r$ given by $P = \mathrm{id}_{E^1} \oplus P^2 \oplus \ldots \oplus P^r$ (see Definition 1).

We shall always write $P(X_1, \ldots, X_l)$ instead of $P(X_1 \otimes \ldots \otimes X_l)$. Further, we shall put $P^1 = \mathrm{id}_{E^1}$.

*

Proposition 12. *Let* $\{E^k, P_k\}^r$ *be a graded Riemannian vector bundle over* M. *If a sequence* $\nabla^{(1)}, \ldots, \nabla^{(r)}$ *of canonical connections in* $E^1 \oplus \ldots \oplus E^r$ *exists, then it is unique.*

Proof. Suppose that $\nabla^{(1)}, \ldots, \nabla^{(r)}$ is a canonical sequence. From (9) and (18) we obtain easily the following Formula

$$
\begin{aligned}
(22)\qquad & 2\langle \nabla_Z^{(l)} P(X_1, \ldots, X_l), P(Y_1, \ldots, Y_l)\rangle = \\
&= Z\langle P(X_1, \ldots, X_l), P(Y_1, \ldots, Y_l)\rangle + \\
&+ \sum_{i=1}^{l} X_i\langle P(X_{i+1}, \ldots, X_l, Y_1, \ldots, Y_i), P(Y_{i+1}, \ldots, Y_l, Z, X_1, \ldots, X_{i-1})\rangle - \\
&- \sum_{i=1}^{l} Y_i\langle P(Y_{i+1}, \ldots, Y_l, Z, X_1, \ldots, X_{i-1}), P(X_i, \ldots, X_l, Y_1, \ldots, Y_{i-1})\rangle + \\
&+ h(Z, P(X_1, \ldots, X_{l-1}), X_l, P(Y_1, \ldots, Y_{l-1}), Y_l) - \\
&- \sum_{i=1}^{l} h(X_i, P(X_{i+1}, \ldots, X_l, Y_1, \ldots, Y_{i-1}), Y_i, P(Y_{i+1}, \ldots, Y_l, Z, X_1, \ldots \\
&\quad \ldots, X_{i-2}), X_{i-1}) + \\
&+ \sum_{i=1}^{l} h(Y_i, P(Y_{i+1}, \ldots, Y_l, Z, X_1, \ldots, X_{i-2}), X_{i-1}, P(X_i, \ldots, X_l, Y_1, \ldots \\
&\quad \ldots, Y_{i-2}), Y_{i-1})
\end{aligned}
$$

where

$$
\begin{aligned}
(23)\qquad & h(U, X^{(l-1)}, T, Y^{(l-1)}, S) = \langle P_{l-1}(\nabla_U^{(l-1)} X^{(l-1)}, T) - \\
&- P_{l-1}(\nabla_T^{(l-1)} X^{(l-1)}, U) + P_{l-1}(X^{(l-1)}, [U, T]), P_{l-1}(Y^{(l-1)}, S)\rangle
\end{aligned}
$$

for any sections $X^{(l-1)}, Y^{(l-1)}, U, T, S$. (Cf. [1].) Hence each $\nabla^{(l)}$ is determined uniquely by $\nabla^{(l-1)}$, and $\nabla^{(1)}$ is determined uniquely by the Riemannian connection of $T(M)$ and by the solder map j. Hence the sequence is unique.

Let $\psi : M \to N$ be a stable immersion, ∇ the induced connection in the Riemannian bundle $E = \psi^* T(N)$. The bundle epimorphisms P_k given by Proposition 7 will be called *the normal operators of the immersion* ψ.

Theorem 2. *Let $\{E^k, P_k\}^r$ be a graded Riemannian vector bundle over a simply connected Riemannian manifold M, and let N be a complete Riemannian d-manifold with a constant curvature C, $d = \dim E$. Suppose that there exists a sequence $\nabla^{(1)}, \ldots, \nabla^{(r)}$ of canonical connections in the bundle $E^1 \oplus \ldots \oplus E^r$ and that a system of "generalized Gaussian equations"*

$$
\begin{aligned}
(24)\qquad & P_k(U, T, X^{(k)}, Y^{(k)}) + L_k(U, T, X^{(k)}, Y^{(k)}) = \\
&= R^{(k)}(U, T, X^{(k)}, Y^{(k)}) + C\{\langle U, X^{(k)}\rangle \langle T, Y^{(k)}\rangle - \langle U, Y^{(k)}\rangle \langle T, X^{(k)}\rangle\}
\end{aligned}
$$

for $k = 1, \ldots, r$ holds. Then there is exactly one isometric fibre isomorphism $\Psi : E \to T(N)$ such that

a) $\Psi(e_1, \ldots, e_d) = (f_1, \ldots, f_d)$ *where* $(e_1, \ldots, e_d) \in B(E)$, $(f_1, \ldots, f_d) \in B(N)$ *are prescribed orthonormal frames,*

b) *a commutative diagram* (12) *holds,*

c) *under the canonical identification* $E = \psi^* T(N)$, *the mappings* P_k *are the normal operators of the isometric immersion* ψ.

Proof. *A*. Existence. Define a semi-Riemannian connection ∇ on E by formula (14). Since the mappings P^k are all symmetric, and $P_1 = P^2$, we have $[R_{UT}X^{(k)}]^{k+2} = 0$ for $k = 1, \ldots, r$, and the torsion of ∇ is zero. Relations (17) imply $[R_{UT}X^{(k)}]^{k-2} = 0$ and (18), (19) mean that $[R_{UT}X^{(k)}]^{k-1} = [R_{UT}X^{(k)}]^{k+1} = 0$. The Gaussian equations (24) and Formula (15) imply $[R_{UT}X^{(k)}]^k = C\{\langle T, X^{(k)}\rangle U - \langle U, X^{(k)}\rangle T\}$. Hence we have $\Phi = C(\omega \wedge \omega^t)$ on $B(E)$. Now we can apply the existence part of Theorem 1.

B. Uniqueness. Let be given an isometric fibre isomorphism satisfying *a*), *b*), *c*). Then the induced Riemannian connection $\bar{\nabla}$ in E satisfies the relation $\Phi = C(\omega \wedge \omega^t)$ and the projection connections $\bar{\nabla}^{(1)}, \ldots, \bar{\nabla}^{(r)}$ of $\bar{\nabla}$ form a sequence of canonical connections in $\{E^k, P_k\}^r$. According to Proposition 12 such sequence is unique and hence $\bar{\nabla}^i = \nabla^i$, $\bar{\nabla} = \nabla$. Now we can use the uniqueness part of Theorem 1.

MAXIMAL IMMERSIONS

A graded Riemannian bundle $\{E^k, P_k\}^r$ will be called *maximal* if the mappings P_k are of maximal ranks. It requires that the composed mappings P^k induce isomorphisms and hence *any* E^k *is isomorphic to the symmetric tensor product of k copies of* E^1. (See Definition 1.) If $\psi : M \to N$ is a stable immersion into a manifold with a constant curvature, then $E = \psi^* T(N)$ is a graded bundle in the sense of Definition 1. The immersion ψ will be called *maximal* if the graded bundle E is maximal. The maximality property means that all osculation spaces of the immersed manifold are of maximal dimensions.

Proposition 13. *If* $\{E^k, P_k\}^r$ *is maximal then all mappings* L_k *are epimorphisms.*

Proof. Let $x \in M$ be given and let $\{Y_1^{(k-1)}, \ldots, Y_f^{(k-1)}\}$ be a basis of E_x^{k-1}. Choose a vector $U \in E_x^1$, $U \neq 0$; then the vectors $P_{k-1}(U, Y_1^{(k-1)}), \ldots, P_{k-1}(U, Y_f^{(k-1)})$ are linearly independent because P^k, P^{k-1} induce isomorphisms of E^k, E^{k-1} with symmetric tensor powers of E^1. Let be given $Z \in E_x^{k-1}$ and denote $a_j = \langle Z, Y_j^{(k-1)}\rangle$. Then there is an element $X^{(k)} \in E_x^k$ such that $\langle X^{(k)}, P_{k-1}(U, Y^{(k-1)}_j)\rangle = -a_j$, $j = 1, \ldots, f$. Hence $\langle L_k(U, X^{(k)}), Y_j^{(k-1)}\rangle = a_j$ for each j, and $L_k(U, X^{(k)}) = Z$.

The study of the maximal immersions is based on the following „Prolongation Theorem".

Theorem 3. *Let $\{E^k, P_k\}^r$ be a maximal graded Riemannian bundle over M. Let be given a sequence $\nabla^{(1)}, \ldots, \nabla^{(l-1)}$ of canonical connections in $E^1 \oplus \ldots \oplus E^{l-1}$, $l \leqq r$. Then the sequence $\nabla^{(1)}, \ldots, \nabla^{(l-1)}$ can be prolonged to the bundle $E^1 \oplus \ldots \oplus E^{l-1} \oplus E^l$ if and only if the function $P_{l-1}(U, T, X^{(l-1)}, Y^{(l-1)})$ satisfies the Bianchi identity. If such prolongation exists, it is unique.*

Proof. The necessity of the Bianchi identity follows from Proposition 11 and the uniqueness from Proposition 12. The sufficiency will be proved in several steps. We must prove that under the above assumption for P_{l-1}, system (22) is solvable and its solution determines a canonical connection. Let us denote the right hand side of (22) by $S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_l)$.

Lemma 1. *We have*

$$S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_l) + S_l(Z \mid Y_l, \ldots, Y_1 \mid X_l, \ldots, X_1) = \\ = 2Z\langle P(X_1, \ldots, X_l), P(Y_1, \ldots, Y_l)\rangle . \tag{25}$$

Proof. If we perform the permutation $(X_1, \ldots, X_l) \leftrightarrow (Y_l, \ldots, Y_1)$ then all summands of S_l are transformed into one another:

$$\left.\begin{aligned} X_i\langle P(\ldots), P(\ldots)\rangle &\to Y_{l+1-i}\langle P(\ldots), P(\ldots)\rangle \\ Y_i\langle P(\ldots), P(\ldots)\rangle &\to X_{l+1-i}\langle P(\ldots), P(\ldots)\rangle \\ h(X_i, \ldots) &\to -h(X_{l+1-i}, \ldots) \end{aligned}\right\} \quad i = 1, \ldots, l$$
$$\begin{aligned} h(Y_i, \ldots) &\to -h(Y_{l+2-i}, \ldots) \qquad i = 2, \ldots, l\,, \\ h(Y_1, \ldots) &\to -h(Z, \ldots) \\ h(Z, \ldots) &\to -h(Y_1, \ldots) \end{aligned}$$

Hence our result follows.

Lemma 2. *We have*

$$S_l(Z \mid X_1, \ldots, X_{l-1}, X_l \mid Y_1, \ldots, Y_l) - S_l(X_l \mid X_{l-1}, \ldots, X_1, Z \mid Y_l, \ldots, Y_1) = \\ = 2h(Z, P(X_1, \ldots, X_{l-1}), X_l, P(Y_1, \ldots, Y_{l-1}), Y_l)\,. \tag{26}$$

Proof. If we perform the permutations $Z \leftrightarrow X_l$,

$$(X_1, \ldots, X_{l-1}) \to (X_{l-1}, \ldots, X_1), (Y_1, \ldots, Y_l) \to (Y_l, \ldots, Y_1)\,,$$

all summands of S_l are transformed into one another:

$$\begin{aligned} Z\langle P(\ldots), P(\ldots)\rangle &\leftrightarrow X_l\langle P(\ldots), P(\ldots)\rangle\,, \\ X_i\langle P(\ldots), P(\ldots)\rangle &\to X_{l-i}\langle P(\ldots), P(\ldots)\rangle\,, \qquad i = 1, \ldots, l-1\,, \end{aligned}$$
$$\left.\begin{aligned} Y_i\langle P(\ldots), P(\ldots)\rangle &\to Y_{l+1-i}\langle P(\ldots), P(\ldots)\rangle \\ h(X_i, \ldots) &\to -h(Y_{l+1-i}, \ldots) \\ h(Y_i, \ldots) &\to -h(X_{l+1-i}, \ldots) \end{aligned}\right\} \quad i = 1, \ldots, l\,,$$
$$h(Z, \ldots) \to -h(Z, \ldots)$$

Hence our result follows.

Lemma 3. *If the function* $P_{l-1}(U, T, X^{(l-1)}, Y^{(l-1)})$ *satisfies the Bianchi identity, then* $S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_l)$ *is symmetric in the variables* X_i *and also in the variables* Y_i.

Proof. First of all we obtain easily

$$\begin{aligned}
&S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_i, Y_{i+1}, \ldots, Y_l) - \\
&\quad - S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_{i+1}, Y_i, \ldots, Y_l) = \\
&= X_i P_{l-1}(Y_i, Y_{i+1}, A_i, B_i) + Y_i P_{l-1}(Y_{i+1}, X_i, A_i, B_i) + \\
&\quad + Y_{i+1} P_{l-1}(X_i, Y_i, A_i, B_i) + \\
&\quad + h(X_i, B_i, Y_{i+1}, A_i, Y_i) - h(X_i, B_i, Y_i, A_i, Y_{i+1}) + \\
&\quad + h(Y_{i+1}, A_i, X_i, B_i, Y_i) - h(Y_i, A_i, X_i, B_i, Y_{i+1}) + \\
&\quad + h(X_{i+1}, B_i'', Y_i, A_i, X_i) - h(X_{i+1}, B_i', Y_{i+1}, A_i, X_i) + \\
&\quad + h(Y_i, A_i', X_{i-1}, B_i, X_i) - h(Y_{i+1}, A_i'', X_{i-1}, B_i, X_i)
\end{aligned}$$

where

$$\begin{aligned}
A_i &= P(Y_{i+2}, \ldots, Y_l, Z, X_1, \ldots, X_{i-1}) & B_i &= P(X_{i+1}, \ldots, X_l, Y_1, \ldots, Y_{i-1}) \\
A_i' &= P(Y_{i+1}, \ldots, Y_l, Z, X_1, \ldots, X_{i-2}) & B_i' &= P(X_{i+2}, \ldots, X_l, Y_1, \ldots, Y_{i-1}, Y_i) \\
A_i'' &= P(Y_i, Y_{i+2}, \ldots, Y_l, Z, X_1, \ldots, X_{i-2}) & B_i'' &= P(X_{i+2}, \ldots, X_l, Y_1, \ldots, Y_{i-1}, Y_{i+1}) .
\end{aligned}$$

After a routine calculation we obtain

$$\begin{aligned}
&S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_i, Y_{i+1}, \ldots, Y_l) - \\
&\quad - S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_{i+1}, Y_i, \ldots, Y_l) = \\
&= (\nabla_{X_i} P_{l-1})(Y_i, Y_{i+1}, A_i, B_i) + (\nabla_{Y_i} P_{l-1})(Y_{i+1}, X_i, A_i, B_i) + \\
&\quad + (\nabla_{Y_{i+1}} P_{l-1})(X_i, Y_i, A_i, B_i) + \langle H_1, P(X_i, B_i)\rangle + \langle H_2, P(X_i, A_i)\rangle ,
\end{aligned}$$

where H_1, H_2 are certain expressions. Here H_1 involves three functions of grade $(l-1)$, namely A_i, A_i', A_i'', and H_2 involves B_i, B_i', B_i''. We can apply identities (18) in order to replace all functions of grade $l-1$ by functions of grade $l-2$. Then H_1 involves only one function of order $l-2$, $A_i^0 = P(Y_{i+1}, \ldots, Y_l, Z, X_1, \ldots, X_{i-2})$, and similarly, H_2 involves only one function of order $l-2$, $B_i^0 = P(X_{i+2}, \ldots, X_l, Y_1, \ldots, Y_{i-1})$. After the substitutions we can see that $H_1 = H_2 = 0$. Thus the function $S_l(Z \mid X_1, \ldots, X_l \mid Y_1, \ldots, Y_l)$ is symmetric with respect to any transposition (Y_i, Y_{i+1}), and consequently, it is symmetric with respect to all variables Y_i. Now, using (25) we obtain

$$\begin{aligned}
&S_l(Z \mid X_1, \ldots, X_i, X_{i+1}, \ldots, X_l \mid Y_1, \ldots, Y) - \\
&\quad - S_l(Z \mid X_1, \ldots, X_{i+1}, X_i, \ldots, X_l \mid Y_1, \ldots, Y_l) = \\
&= S_l(Z \mid Y_l, \ldots, Y_1 \mid X_l, \ldots, X_i, X_{i+1}, \ldots, X_1) - \\
&\quad - S_l(Z \mid Y_l, \ldots, Y_1 \mid X_l, \ldots, X_{i+1}, X_i, \ldots, X_1) = 0 .
\end{aligned}$$

Hence S_l is symmetric in all variables X_i. This completes the proof.

Lemma 4. *We have*

(27) $S_l(Z \mid X_1, \dots, X_l \mid Y_1, \dots, fY_i, \dots, Y_l) = f \, . \, S_l(Z \mid X_1, \dots, X_l \mid Y_1, \dots, Y_i, \dots, Y_l)$

(28) $S_l(fZ \mid X_1, \dots, X_l \mid Y_1, \dots, Y_l) = f \, . \, S_l(Z \mid X_1, \dots, X_l \mid Y_1, \dots, Y_l)$

(29) $S_l(Z \mid X_1, \dots, fX_i, \dots, X_l \mid Y_1, \dots, Y_l) =$
$= f \, . \, S_l(Z \mid X_1, \dots, X_l \mid Y_1, \dots, Y_l) + 2(Zf) \langle P(X_1, \dots, X_l), P(Y_1, \dots, Y_l) \rangle$

for any function f on M.

Proof. (27), (28) follows by direct calculation and (29) follows from (27) and (25).

From Lemma 3 and (27), (28) we see that the value $S_l(Z \mid X_1, \dots, X_l \mid Y_1, \dots, Y_l)_x$ at any point $x \in M$ depends only on the vector Z_x, on the section $X_1 \bigcirc \dots \bigcirc X_l$ and on the vector $(Y_1 \bigcirc \dots \bigcirc Y_l)_x$. Now, the function P_x^l induces an isomorphism $\underbrace{E_x^1 \bigcirc \dots \bigcirc E_x^1}_{l\text{-times}} \to E_x^l$. Hence, if $Z_x, X_1, \dots, X_l$ are prescribed. there is exactly one vector $W_x \in E_x$ such that

$$\langle W_x, P_x(Y_1, \dots, Y_l) \rangle = (1/2) \, S_l(Z_x \mid X_1, \dots, X_l \mid Y_1, \dots, Y_l)_x$$

for any vector $(Y_1 \bigcirc \dots \bigcirc Y_l)_x$. Let us denote $W_x = \chi(Z_x, P(X_1 \bigcirc \dots \bigcirc X_l))$. From (29) we obtain

$$\chi(Z_x, f \, . \, P_l(X_1 \bigcirc \dots \bigcirc X_l)) =$$
$$= f(x) \, \chi(Z_x, P_l(X_1 \bigcirc \dots \bigcirc X_l)) + (Z_x f) \, P_l(X_1 \bigcirc \dots \bigcirc X_l)_x \, .$$

Hence we can extend the function χ to all sections of the bundle E^l in such a way that $\chi(Z, X^{(l)})$ is a covariant derivative on E^l. We put $\nabla_Z^{(l)} X^{(l)} = \chi(Z, X^{(l)})$. From (25) there follows that $\nabla^{(l)}$ is semi-Riemannian and (26) induces relation (18) of order l. This completes the proof of Theorem.

Corollary. *Let $\{E^k, P_k\}^r$ be a maximal graded Riemannian bundle over M. Let be given a sequence $\nabla^{(1)}, \dots, \nabla^{(l)}$ of canonical connections in $E^1 \oplus \dots \oplus E^l$, $l < r$. If the "generalized Gaussian equation"* (*)

(30) $$P_l(U, T, X^{(l)}, Y^{(l)}) + L_l(U, T, X^{(l)}, Y^{(l)}) =$$
$$= R^{(l)}(U, T, X^{(l)}, Y^{(l)}) + C\{\langle U, X^{(l)} \rangle \langle T, Y^{(l)} \rangle - \langle U, Y^{(l)} \rangle \langle T, X^{(l)} \rangle\}$$

holds then we have the unique prolongation to a canonical sequence $\nabla^{(1)}, \dots$ $\dots, \nabla^{(l)}, \nabla^{(l+1)}$.

[1]) For $l = 1$, $\dim M = 2$, $\dim E = 3$, $C = 0$ equation (30) is reduced to the classical Gaussian equation

$$\langle P_1(U, T), P_1(U, T) \rangle - \langle P_1(U, U), P_1(T, T) \rangle = R^{(1)}(U, T, U, T) \, .$$

Proof. L_l satisfies the Bianchi identity according to Proposition 11 and so does $R^{(l)}$ according to Proposition 6. The second term on the right-hand side is easily proved to have the same property. Hence P_l satisfies the Bianchi identity.

*

If P_k, $k = 1, \ldots, r$ are the normal operators of an immersion $\psi : M \to N$, then the scalar product

$$h_k(X_1, \ldots, X_k, Y_1, \ldots, Y_k) = \langle P^k(X_1, \ldots, X_k), P^k(Y_1, \ldots, Y_k)\rangle$$

is called *the k-th metric tensor of the immersion* ψ. (See [1].) Let $\{E^k, P_k\}^r$ be a maximal graded Riemannian bundle over M and let us identify each E^k with the symmetric tensor product $\bigcirc_k E^1 = \bigcirc_k T(M)$ via P^k. We obtain a maximal graded Riemannian bundle $\{\bigcirc_k T(M), P_k^0\}^r$, the "normal form" of $\{E^k, P_k\}^r$. Here the mappings P_k^0 are defined as follows: if $X^{(k)} \in \bigcirc_k T(M)$, $X^{(k)} = \sum_{i=1}^{s} \lambda_i(X_{i1} \bigcirc \ldots \bigcirc X_{ik})$ and $Z \in T(M)$, we put $P_k^0(Z, X^{(k)}) = \sum_{i=1}^{s} \lambda_i(Z \bigcirc X_{i1} \bigcirc \ldots \bigcirc X_{ik})$.

Now, Theorem 2 and Corollary of Theorem 3 imply

Theorem 4. (Second Immersion Theorem). *Let M be a simply connected manifold, and let be given a Riemannian inner product $\langle\ ,\ \rangle_k$ on each vector bundle $E^k = \underbrace{T(M) \bigcirc \ldots \bigcirc T(M)}_{k\text{—times}}$ for $k = 1, \ldots, r$. Let N be a complete Riemannian manifold with a constant curvature C and of dimension $d = \dim(E^1 \oplus \ldots \oplus E^r)$. Suppose that the generalized Gaussian equations (24) are satisfied for $k = 1, \ldots, r$. Then there is, exact up to an isometry of N, a unique immersion $\psi : M \to N$ such that*

$$h_k(X_1, \ldots, X_k; Y_1, \ldots, Y_k) = \langle X_1 \bigcirc \ldots \bigcirc X_k, Y_1 \bigcirc \ldots \bigcirc Y_k\rangle_k\,, \quad k = 1, \ldots, r\,.$$

Particularly, the immersion ψ is isometric and maximal.

FLAT IMMERSIONS

Let $(E, j) \to M$ be a soldered Riemannian vector bundle with a Riemannian connection ∇. Suppose that ∇ is stable and induces a canonical graduation $E = E^1 \oplus E^2 \oplus \ldots \oplus E^r$ in E. The connection ∇ will be called *flat* if all projection connections $\nabla^{(i)}$ for $i \geqq 2$ are locally flat; i.e., if the curvatures $R^{(2)}, \ldots, R^{(r)}$ vanish. Also a stable isometric immersion $\psi : M \to N$ will be called *flat* if it induces a flat connection in the bundle $E = \psi^* T(N)$.

Let ∇ be a flat Riemannian connection in $(E, j) \to M$, $E = E^1 \oplus \ldots \oplus E^r$, and suppose the manifold M to be simply connected. Then the conections $\nabla^{(k)}$, $k \geqq 2$

are all flat and each bundle E^k, $k \geqq 2$ is canonically a product bundle $\mathscr{R}^{d_k} = M \times R^{d_k}$. Let us choose for any $k = 2, \ldots, r$ an orthonormal frame $b^{(k)} = \{f_1^{(k)}, \ldots, f_{d_k}^{(k)}\}$ of R^{d_k}. Then for any section $Z^{(k)} = (x, \sum_{i=1}^{d_k} z_i^{(k)}(x) f_i^{(k)})$ we have $\nabla_U^{(k)} Z^{(k)} = (x, \sum_{i=1}^{d_k} (U z_i^{(k)}) . f_i^{(k)})$. We shall represent $Z^{(k)}$ by a column vector $\mathbf{z}^{(k)} \in R^{d_k}$, $\mathbf{z}^{(k)} = \begin{pmatrix} z_1^{(k)}(x) \\ \vdots \\ z_{d_k}^{(k)}(x) \end{pmatrix}$. Then we can write briefly

$$(31) \qquad Z^{(k)} = (x, b^{(k)} . \mathbf{z}^{(k)}) , \quad \nabla_U^{(k)} Z^{(k)} = (x, b^{(k)} . (U\mathbf{z}^{(k)})) .$$

Clearly, there is an R^{d_2}-valued symmetric form $\omega^{(2)}(U, Z)$ on M such that

$$(32) \qquad P_1(U, Z) = (x, b^{(2)} . \omega^{(2)}(U, Z))$$

and, for $k \geqq 2$, an $(R^{d_{k+1}} \otimes R^{d_k})$-valued 1-form

$$\omega^{(k+1)}(U) = \begin{pmatrix} \omega_{1,1}^{(k+1)}(U), & \ldots, & \omega_{1,d_k}^{(k+1)}(U) \\ \vdots & & \\ \omega_{d_{k+1},1}^{(k+1)}(U), & \ldots, & \omega_{d_{k+1},d_k}^{(k+1)}(U) \end{pmatrix}$$

such that

$$(33) \qquad P_k(U, Z^{(k)}) = (x, b^{(k+1)} . \omega^{(k+1)}(U)\, \mathbf{z}^{(k)}) , \quad k = 2, \ldots, r .$$

Each form $\omega^{(k)}$ will be called *the k-th fundamental form of a flat Riemannian connection in E*, or else, if ∇ is induced by an immersion $\psi : M \to N$, *the k-th fundamental form of the flat immersion* ψ. We can see that any isometric immersion $\psi : M \to$ $\to N$ where M is a hypersurface of N, $\dim N = \dim M + 1$, is flat. If M is simply connected, the second fundamental form is defined and it has the classical meaning.

Let us remark that each fundamental form $\omega^{(k)}$ is determined uniquely exact up to a constant matrix factor belonging to the orthogonal group $O(d_k)$, and a constant matrix factor from $O(d_{k-1})$. We check easily that

$$(34) \qquad L_k(U, Z^{(k)}) = (x, -b^{(k-1)} \omega^{(k)t}(U)\, \mathbf{z}^{(k)})$$

for $k \geqq 3$ where $\omega^{(k)t}$ denotes the transpose of $\omega^{(k)}$. Indeed, we obtain

$$\begin{aligned} \langle L_k(U, Z^{(k)}), Y^{(k-1)} \rangle &= - \langle P_{k-1}(U, Y^{(k-1)}), Z^{(k)} \rangle = \\ = - \langle b^{(k)} \omega^{(k)}(U)\, \mathbf{y}^{(k-1)}, b^{(k)} \mathbf{z}^{(k)} \rangle &= - \mathbf{z}^{(k)t} \omega^{(k)}(U)\, \mathbf{y}^{(k-1)} = \\ = - (\omega^{(k)t}(U)\, \mathbf{z}^{(k)})^t\, \mathbf{y}^{(k-1)} &= \langle -b^{(k-1)} \omega^{(k)t}(U)\, \mathbf{z}^{(k)}, b^{(k-1)} \mathbf{y}^{(k-1)} \rangle = \\ &= \langle -b^{(k-1)} \omega^{(k)t}(U)\, \mathbf{z}^{(k)}, Y^{(k-1)} \rangle . \end{aligned}$$

Finally

$$(35) \qquad L_2(U, Z^{(2)}) = \sum_{i=1}^{m} \mathbf{z}^{(2)t} \omega^{(2)}(U, T_i)\, T_i$$

where $\{T_1, \ldots, T_m\}$ is an arbitrary orthonormal frame of the tangent vectors at $x \in M$, x being the initial point of U.

Now, the symmetry conditions

$$P_{k+1}(U, P_k(T, Z^{(k)})) = P_{k+1}(T, P_k(U, Z^{(k)})) \quad k = 1, \ldots, r-2$$
$$L_{k-1}(U, L_k(T, Z^{(k)})) = L_{k-1}(T, L_k(U, Z^{(k)})) \quad k = 3, \ldots, r$$

are equivalent to the system

(36)
$$\omega^{(3)}(U)\,\omega^{(2)}(T, Z) - \omega^{(3)}(T)\,\omega^{(2)}(U, Z) = 0$$
$$\omega^{(k+1)} \wedge \omega^{(k)} = 0 \quad k = 3, \ldots, r-1\,.$$

We only substitute into (32)–(35).

Similarly, conditions (18), (19) are equivalent to the system

(37)
$$(\nabla_U \omega^{(2)})(T, Z) = (\nabla_T \omega^{(2)})(U, Z)$$
$$(\nabla_U \omega^{(k)})(T) = (\nabla_T \omega^{(k)})(U) \quad k = 3, \ldots, r\,.$$

Here, of course,

$$(\nabla_U \omega^{(2)})(T, Z) = U[\omega^{(2)}(T, Z)] - \omega^{(2)}(\nabla_U T, Z) - \omega^{(2)}(T, \nabla_U Z)\,,$$
$$(\nabla_U \omega^{(k)})(T) = U[\omega^{(k)}(T)] - \omega^{(k)}(\nabla_U T)\,,$$

are usual covariant derivatives. For example, condition (18), $k \geqq 2$ can be written in the form

$$b^{(k+1)}\{U(\omega^{(k+1)}(T)\,\mathbf{z}^{(k)}) - T(\omega^{(k+1)}(U)\,\mathbf{z}^{(k)}) + \omega^{(k+1)}(U)\,T\mathbf{z}^{(k)} -$$
$$- \omega^{(k+1)}(T)\,U\mathbf{z}^{(k)} - \omega^{(k+1)}([U, T])\,\mathbf{z}^{(k)}\} = 0$$

and hence

$$b^{(k+1)}\{U\omega^{(k+1)}(T) - T\omega^{(k+1)}(U) - \omega^{(k+1)}([U, T])\}\,\mathbf{z}^{(k)} = 0\,,$$
$$U\omega^{(k+1)}(T) - T\omega^{(k+1)}(U) - \omega^{(k+1)}(\nabla_U T) + \omega^{(k+1)}(\nabla_T U) = 0\,.$$

Finally, we obtain $(\nabla_U \omega^{(k+1)})(T) = (\nabla_T \omega^{(k+1)})(U)$ as required.

The Gaussian equations (24) assume the form

(38)
$$\omega^{(2)t}(U, X)\,\omega^{(2)}(T, Y) - \omega^{(2)t}(U, Y)\,\omega^{(2)}(T, X) =$$
$$= C\{\langle U, X\rangle \langle T, Y\rangle - \langle U, Y\rangle \langle T, X\rangle\} + \langle R_{UT}X, Y\rangle$$
$$\omega^{(k)} \wedge \omega^{(k)t} + \omega^{(k+1)t} \wedge \omega^{(k+1)} = 0\,, \quad k = 2, \ldots, r\,.$$

Here we define the anti-symmetric 2-form $\omega^{(2)} \wedge \omega^{(2)t}$ by

$$(\omega^{(2)} \wedge \omega^{(2)t})(U, T) = \sum_{i=1}^{m} \{\omega^{(2)}(U, T_i)\,\omega^{(2)t}(T, T_i) - \omega^{(2)}(T, T_i)\,\omega^{(2)t}(U, T_i)\}$$

where $\{T_1, \ldots, T_m\}$ is an orthonormal frame $B(M)$, and we put $\omega^{(r+1)} = 0$.

*

Theorem 5. (Third Immersion Theorem). *Let M be a simply connected Riemannian manifold,* $\dim M = m$, *and $d_2, \ldots, d_r$ positive integers. Let $\omega^{(2)}(U, T)$ be a symmetric R^{d_2}-valued form on M and let $\omega^{(k+1)}(U)$, $k = 2, \ldots, r-1$, be a $(R^{d_{k+1}} \otimes R^{d_k})$-valued 1-form on M. Let N be a complete d-manifold with a constant curvature C, $d = m + d_2 + \ldots + d_r$. Suppose that the following conditions are satisfied.*

(I) *Dimension conditions: at any point $x \in M$ the vectors $\omega^{(2)}(U, T)$, U, $T \in T(M)_x$, span R^{d_2} and the vectors $\omega^{(k+1)}(U)\,\mathbf{z}^{(k)}$, where $U \in T(M)_x$, $\mathbf{z}^{(k)} \in R^{d_k}$, span $R^{d_{k+1}}$.*

(II) *Symmetry conditions:*

$$\omega^{(3)}(U)\,\omega^{(2)}(T, Z) - \omega^{(3)}(T)\,\omega^{(2)}(U, Z) = 0,$$
$$\omega^{(k+1)} \wedge \omega^{(k)} = 0, \quad k = 3, \ldots, r-1.$$

(III) *Generalized Gaussian equations:*

$$\omega^{(2)t}(U, X)\,\omega^{(2)}(T, Y) - \omega^{(2)t}(U, Y)\,\omega^{(2)}(T, X) =$$
$$= C\{\langle U, X\rangle \langle T, Y\rangle - \langle U, Y\rangle \langle T, X\rangle\} + \langle R_{UT}X, Y\rangle$$
$$\omega^{(k)} \wedge \omega^{(k)t} + \omega^{(k+1)t} \wedge \omega^{(k+1)} = 0\,, \quad k = 2, \ldots, r\,.$$

(IV) *Generalized Mainardi-Codazzi equations:*

$$(\nabla_U \omega^{(2)})(T, Z) = (\nabla_T \omega^{(2)})(U, Z)\,,$$
$$(\nabla_U \omega^{(k)})(T) = (\nabla_T \omega^{(k)})(U)\,, \quad k = 3, \ldots, r\,.$$

Under these conditions there is, exact up to an isometry of N, a unique isometric flat immersion $\psi : M \to N$ such that $\omega^{(k)}$, $k = 2, \ldots, r$, are fundamental forms of the immersion.

Proof. Let us provide each R^{d_k} with the Euclidean inner product. We define a graded Riemannian vector bundle $\{E^k, P_k\}^r$ over M putting $E^1 = T(M)$, $E^k = M \times R^{d_k}$, $P_1(U, T) = (x, \omega^{(2)}(U, T))$, $P_k(U, (x, \mathbf{z}^{(k)})) = (x, \omega^{(k+1)}(U)\,\mathbf{z}^{(k)})$, $k = 2, \ldots, r$. Moreover, we define a sequence of connections in $E^1, \ldots, E^r$ respectively as follows: $\nabla^{(1)}$ = the canonical Riemannian connection in M, $\nabla^{(k)}_U(y, z^{(k)}(y)) = (x, Uz^{(k)})$, ($x$ = the origin of U, $k = 2, \ldots, r$). According to (IV) the sequence $\nabla^{(1)}, \nabla^{(2)}, \ldots, \nabla^{(r)}$ is a canonical one. Now we can apply Theorem 2.

(To be continued)

References

[1] *L. Boček:* Globaldifferentialgeometrie der Untermannigfaltigkeiten in E_n und S_n, Czech. Math. Journal, *17* (*92*) 1967, Praha.

[2] *J. A. Schouten, D. J. Struik:* Einführung in die neueren Methoden der Differentialgeometrie, Zweiter Band, 1938, Groningen—Batavia.

[3] *R. L. Bishop, R. J. Crittenden:* Geometry of manifolds, 1964. Academic Press, New York and London.

[4] Sur la Géométrie des prolongements des espaces fibrés vectoriels, P. Libermann, Ann. Inst. Fourier, Grenoble *14*, 1 (1964), 145—172.

[5] *S. Kobayashi:* Theory of Connections, Annali di matematica pura ed applicata *43* (1957), 119—185.

[6] *Kobayashi, Nomizn:* Foundations of Differential Geometry, 1963, Interscience Publishers, New York—London.

Author's address: Praha 8-Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta KU.)