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CYCLIC PRODUCTS AND AN INEQUALITY FOR DETERMINANTS

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Introduction. Let V be a matrix with positive diagonal elements and nonnegative off-diagonal elements. If all principal minors of V are positive, the matrix V belongs to an important class of matrices with many useful properties. In particular, V^{-1} exists and $V^{-1} \geqq 0$. This class of matrices, which the authors denote by $\mathbf{K}$, has been extensively studied by different authors. A systematic account of the properties of matrices of class $\mathbf{K}$ may be found in [1].

In a paper of KOTELJANSKIJ [4] the following comparison theorem has been proved. *Let $V \in \mathbf{K}$ and let U be a (complex) matrix such that $|u_{ii}| \geqq v_{ii}$ and $|u_{ik}| \leqq |v_{ik}|$; then $|\det U| \geqq \det V$.* The proof is based on an induction with respect to the dimension of the matrix. In the present note which belongs to a series of papers devoted to the systematic study of the class $\mathbf{K}$ and its generalizations we intend to investigate more closely inequalities of the above type. First of all, we use another method of proof based on an exponential expansion which is independent of the dimension and thus eliminates induction arguments. At the same time, this method suggests a weakening of the hypotheses. The inequalities $|u_{ik}| \leqq |v_{ik}|$ may be replaced by a weaker hypothesis, roughly speaking the term-by-term inequalities may be replaced by inequalities for cyclic products. Further, this method makes it possible to analyse completely the case of equality.

Cyclic products appear already in the work of OSTROWSKI [5]. GOLDBERG [2, 3] seems to be the first to observe that two matrices with the same cyclic products have the same principal minors.

1. Preliminaries. Throughout the whole paper, we shall denote by N the set $\{1, 2, \ldots, n\}$ where n is a fixed positive integer. A matrix is a complex-valued function on $N \times N$ or $N' \times N'$ where $N' \subset N$. If B is a matrix and $i, k \in N$, we shall denote by b_{ik} the value of this function at the point (i, k). The principal submatrix of a matrix A on $N \times N$ whose rows and columns correspond to indices from $N' \subset N$ is denoted by $A(N')$. Given a vector $(a_1, \ldots, a_n)$, we shall denote by diag $\{a_1, \ldots, a_n\}$ the diagonal matrix with a_i on the main diagonal.

Throughout the present paper we shall be using the properties of two classes of matrices, $\mathbf{K}$ and $\mathbf{K}_0$. We intend to list now the most important properties of these two classes; for the proofs, the reader is referred to [1].

The class $\mathbf{K}_0$ is defined as the class of those real-valued matrices $A = (a_{ik})$ which satisfy $a_{ik} \leqq 0$ for $i \neq k$ and such that all principal minors of A are nonnegative. Further, $\mathbf{K}$ is the class of those matrices from $\mathbf{K}_0$ whose principal minors are positive. It may be shown that a matrix $A \in \mathbf{K}_0$ which is nonsingular already belongs to $\mathbf{K}$.

We shall use the following abbreviations. With each matrix A we associate two real matrices $M(A)$ and $H(A)$ with elements m_{ik} and h_{ik} defined as follows

$$m_{ik} = |a_{ik}| \quad \text{for all} \quad i, k\,,$$
$$h_{ik} = |a_{ik}| \quad \text{if} \quad i = k \quad \text{and} \quad h_{ik} = -\,|a_{ik}| \quad \text{for} \quad i \neq k\,.$$

We denote by $\mathbf{H}$ the class of all matrices A such that $H(A) \in \mathbf{K}$. Similarly the class $\mathbf{H}_0$ is defined by the postulate $A \in \mathbf{H}_0$ if and only if $H(A) \in \mathbf{K}_0$.

A diagonal matrix D is one for which $d_{ik} = 0$ if $i \neq k$. If D is a diagonal matrix for which $d_{ii} > 0$, we shall frequently call D a positive diagonal matrix.

If x is a vector, we shall also denote by $M(x)$ that vector the coordinates of which are the moduli of the corresponding coordinates of x. The identity matrix will be denoted by I. If B is a matrix we denote by $\sigma(B)$ the set of all the eigenvalues of B, by $|B|_\sigma$ the spectral radius of B, i.e. max $|\lambda|$ for $\lambda \in \sigma(B)$. If B is nonnegative, we denote by $p(B)$ the Perron root of B so that $p(B) = |B|_\sigma$. We shall also use the following result on nonnegative matrices.

(1,1) *Let A be a nonnegative matrix. Let B be a (complex valued) matrix for which $M(B) \leqq A$. Then*

$$|B|_\sigma \leqq p(A)\,.$$

Suppose further that A is irreducible. Then equality is attained if and only if there exists a diagonal matrix D with $M(D) = I$ and a complex unit ε such that $B = \varepsilon DAD^{-1}$. In particular, equality implies $M(B) = A$, so that $B \geqq 0$ implies $B = A$.

Proof. To prove $|B|_\sigma \leqq p(A)$, take an arbitrary $\lambda \in \sigma(B)$ so that there exists a nonzero vector x for which $Bx = \lambda x$. Hence

$$(1) \qquad |\lambda|\, M(x) = M(\lambda x) = M(Bx) \leqq M(B)\, M(x) \leqq AM(x)\,.$$

Suppose further that $|\lambda| > p(A)$. Since $p(A) = |A|_\sigma$, the series $\sum_{k=0}^{\infty} (|\lambda|^{-1} A)^k$ is convergent, its sum is nonnegative and equals $(I - |\lambda|^{-1} A)^{-1}$. Since $(A - |\lambda|)\, M(x) \geqq 0$, we have also $-\,|\lambda|\, M(x) = (I - |\lambda|^{-1} A)^{-1} (A - |\lambda|)\, M(x) \geqq 0$ which is a contradiction.

Suppose now that $|B|_\sigma = p(A)$ and that A is irreducible. It follows from (1) that

(2) $$p(A)\,M(x) \leqq A\,M(x)\,.$$

According to the Perron-Frobenius theorem there exists a positive vector y with $y'A = p(A)\,y'$. Hence, using (2),

$$p(A)\,y'\,M(x) \leqq y'A\,M(x) = p(A)\,y'\,M(x)\,;$$

since $y' > 0$ and $p(A) > 0$ it follows that

(3) $$p(A)\,M(x) = A\,M(x)\,.$$

Since $M(x) \geqq 0$, $x \neq 0$ and A is irreducible, it follows that $M(x) > 0$. Further, by (1),

$$p(A)\,M(x) \leqq M(B)\,M(x) \leqq A\,M(x) = p(A)\,M(x)$$

whence $M(B)\,M(x) = A\,M(x)$. Since $M(B) \leqq A$ and $M(x) > 0$, it follows that $M(B) = A$.

Since $M(x) > 0$ there exists exactly one diagonal matrix D for which $x = D\,M(x)$; clearly $M(D) = I$. Hence

$$\varepsilon D^{-1}BD\,M(x) = p(A)\,M(x)$$

for a suitable ε with $|\varepsilon| = 1$.

Denote by W the matrix $\varepsilon D^{-1}BD$. We have $M(W) = M(B) = A$ and

$$W\,M(x) = p(A)\,M(x) = A\,M(x) = M(B)\,M(x) = M(W)\,M(x)\,.$$

Since $W\,M(x) = M(W)\,M(x)$ and $M(x) > 0$, it follows that $W = M(W) = A$. The proof is complete.

(1,2) *Let $A \in \mathbf{K}_0$ be irreducible. Then all proper principal minors of A are positive.*

Proof. If A happens to be nonsingular, the conclusion follows from the fact that A already belongs to $\mathbf{K}$.

Suppose now that A is singular. It follows that A is of the form $A = p(P) - P$ where P is an irreducible nonnegative matrix. Let M be a proper subset of the index set N. The present theorem will be proved if we show that $p(P(M)) < p(P)$. Indeed, $p(P) > p(P(M))$ implies $A(M) = p(P) - P(M) \in \mathbf{K}$. Let us denote by $\hat{P}$ the matrix defined as follows:

$$\hat{p}_{ik} = p_{ik} \quad \text{if both indices } i, k \text{ belong to } M\,,$$
$$\hat{p}_{ik} = 0 \quad \text{otherwise}\,.$$

It follows that $0 \leqq \hat{P} \leqq P$ and that $p(P(M)) = p(\hat{P})$. Since $\hat{P} \leqq P$, we have $p(\hat{P}) \leqq$

$\leqq p(P)$. Suppose that $p(P(M)) = p(P)$ so that $p(\hat{P}) = p(P)$. By (1,1), $\hat{P} = P$ which is a contradiction since $M \neq N$ and P is irreducible.

(1,3) Corollary. *Let $A \in \mathbf{K}_0$ and let M be a proper subset of the index set N. If $A(M)$ is singular then A is reducible and singular.*

2. Irreducible components. In the sequel we shall use frequently the well-known fact that every matrix may be rearranged into a block triangular form with indecomposable matrices on the block diagonal. Since these questions are usually treated rather casually in the literature and since we shall need a somewhat more precise statement of this and related facts we include, for the convenience of the reader, a section devoted to the precise description of these decompositions.

In the present section we intend to collect the combinatorial prerequisites which will be necessary for the formulation of the main result. We shall adopt the method of [6]. Everything which follows can, of course, be also expressed in the language of graph theory; the method using additive mappings of sets, however, seems to be more appropriate for the study of iterates of graphs.

Let φ be a relation on N (a subset of $N \times N$). If $A \subset N$ we denote by $\varphi(A)$ the set of those $k \in N$ which satisfy $[a, k] \in \varphi$ for a suitable $a \in A$. In this manner φ may be regarded as an additive mapping of $\exp N$ into $\exp N$. The set of all relations on N with a composition defined as the superposition of additive mappings of $\exp N$ into $\exp N$ will be denoted by $G(N)$.

We shall denote by δ the relation of identity, defined by the diagonal of $N \times N$, i.e.

$$[x, y] \in \delta \quad \text{if and only if} \quad x = y\,.$$

Also, it will be convenient to set $\varphi^0 = \delta$ for each $\varphi \in G(N)$. For every relation φ there exists a transitive relation $\tau(\varphi)$ such that (i) $\varphi^0 \cup \varphi \subset \tau(\varphi)$, (ii) if ψ is any transitive relation containing φ and φ^0, then $\psi \supset \tau(\varphi)$. It is easy to show that $\tau(\varphi)$ may be described as follows: $[x, y] \in \tau(\varphi)$ if and only if there exists a nonnegative integer p such that $y \in \varphi^p(x)$. If φ is a relation, we denote by φ^T the transpose of φ, defined as follows: $[x, y] \in \varphi^T$ if and only if $[y, x] \in \varphi$. It is easy to see that $\tau(\varphi^T) = \tau(\varphi)^T$ for any relation φ. The intersection $\tau(\varphi) \cap \tau(\varphi)^T = \tau(\varphi) \cap \tau(\varphi^T)$ is easily seen to be an equivalence. We shall denote it by $\varepsilon(\varphi)$.

A mapping $\varphi \in G(N)$ is said to be irreducible if $\varphi(P) \subset P$ implies that P is either void or $P = N$. Note that, for $n = 1$, every relation φ is irreducible.

First of all, we shall prove a lemma which will be useful in the next section.

(2,1) *Let $\varphi \in G(N)$ be irreducible, let z be a fixed element in N. Then there exists a relation $\psi \in G(N)$, $\psi \subset \varphi$ and a function h on N with the following properties:*

1° *$h(x)$ is a nonnegative integer for each $x \in N$;*
2° *$h(z) = 0$;*

3° *for each* $y \in N$, $y \neq z$ *there exists exactly one* x *such that* $[x, y] \in \psi$ *and* $h(x) = h(y) - 1$.

This relation ψ *has the property that whenever* $y \in N$, $y \neq z$ *then there exists a unique sequence* $k_0, k_1, \ldots, k_r$ *of elements in* N *such that* $k_0 = y$, $k_r = z$ *and*

$$[k_{i+1}, k_i] \in \psi\,, \quad i = 0, \ldots, r - 1\,.$$

The numbers $h(y)$ *and* r *satisfy the relation* $h(y) = r$.

Proof. We shall construct such h and ψ. We put $h(z) = 0$. Let $y \neq z$. It follows from the irreducibility of φ that $y \in \varphi^j(z)$ for some positive integer j since otherwise the set P consisting of all elements of the sets $\varphi^i(z)$, $i = 0, 1, \ldots$ which satisfies $\varphi(P) \subset P$ would be nonvoid and different from N. We define then $h(y)$ as the least integer j with that property. To construct ψ, denote first by N_j, $j = 0, 1, \ldots$ the set of all elements z of N with $h(z) = j$. Assign to each element $y \neq z$ of N exactly one pair $[x, y]$ in the following manner: if $y \in N_i$ then x is one of the elements in N_{i-1} for which $[x, y] \in \varphi$. All pairs $[x, y]$ for $y \in N$, $y \neq z$, form then the relation ψ. It is obvious that h and ψ have the properties 1°–3°. To prove the remaining assertions, observe that k_1 is uniquely determined; if $k_1 \neq z$, k_2 is uniquely determined etc. Since $h(k_0) = h(y)$, $h(k_1) = h(y) - 1$ etc., we obtain that this process will stop with $k_r = z$ where $r = h(y)$ since then $h(k_r) = h(y) - r = 0$. The proof is complete.

Let us turn now to the investigation of reducible relations. We shall begin with the definition:

(2,2) Definition. Let $\varphi \in G(N)$. A subset $M \subset N$ is said to be *an irreducible component* of φ if the following two conditions are satisfied:

1° the mapping φ is irreducible on M (there is no nonvoid proper subset P of M such that

$$\varphi(P) \cap M \subset (P)\,;$$

2° M is maximal with respect to property 1°.

(2,3) *Let* C_1 *and* C_2 *be irreducible components of* φ. *If* $C_1 \cap C_2 \neq \emptyset$ *then* $C_1 = C_2$.

Proof. It suffices to show that φ is irreducible on $C_1 \cup C_2$, i.e. satisfies condition 1° in Definition (2,2). Let P be a non-void proper subset of $C_1 \cup C_2$ such that $\varphi(P) \cap \cap (C_1 \cup C_2) \subset P$. Let us show that either $P \cap C_1$ is a non-void proper subset of C_1 or $P \cap C_2$ is a non-void proper subset of C_2. Since $P = (P \cap C_1) \cup (P \cap C_2)$, the case $P \cap C_1 = P \cap C_2 = \emptyset$ is impossible and so is the case $P \cap C_1 = C_1$, $P \cap C_2 = = C_2$. Let now $P \cap C_1 = \emptyset$, $P \cap C_2 = C_2$. Hence $C_1 \cap C_2 = C_1 \cap (P \cap C_2) = = (C_1 \cap P) \cap C_2 = \emptyset$ which is a contradiction. The case $P \cap C_2 = \emptyset$ and $P \cap C_1 = = C_1$ is similar. We have thus shown that one of the sets $P \cap C_i$ is a proper non-void subset of the corresponding C_i. Without loss of generality, assume that $0 \neq P \cap \cap C_1 \neq C_1$. Then

$$\varphi(P \cap C_1) \cap C_1 \subset \varphi(P) \cap C_1 = (\varphi(P) \cap (C_1 \cup C_2)) \cap C_1 \subset P \cap C_1,$$

which is a contradiction with the irreducibility of φ on C_1. Hence φ is irreducible on $C_1 \cup C_2$. The proof is complete.

(2,4) *Let M be a class of the equivalence $\varepsilon(\varphi)$. Then M is an irreducible component of φ.*

Proof. Let P be a non-void subset of M such that $\varphi(P) \cap M \subset P$. Let us show first that the inclusion $\varphi(P) \cap M \subset P$ implies $\varphi^k(P) \cap M \subset P$ for each natural k. We prove this by induction. The assertion is true for $k = 1$ by our assumption. Suppose now that $\varphi^k(P) \cap M \subset P$ and $x \in \varphi^{k+1}(P) \cap M$. Then there exists a $p \in P$ and a y such that $x \in \varphi(y)$, $y \in \varphi^k(p)$. Since $[x, p] \in \varepsilon(\varphi)$, there exists a nonnegative j such that $p \in \varphi^j(x)$; hence $y \in \varphi^{k+j}(p)$ so that $[x, y] \in \varepsilon(\varphi)$ and $y \in M$. It follows that $y \in \varphi^k(P) \cap M$ so that, according to the induction hypothesis, $y \in P$. Hence $x \in \varphi(y) \subset \varphi(P) \cap M \subset P$. Let $i \in M$. Take a $p \in P$. Since $[p, i] \in \varepsilon(\varphi)$, there exists an exponent k such that $i \in \varphi^k(p) \cap M \subset \varphi^k(P) \cap M \subset P$. Hence $P = M$ and M is irreducible. Suppose now that $M' \supset M$, $M' \neq M$. Let x be a fixed element in M, m a fixed element in $M' - M$. Since M is a class of $\varepsilon(\varphi) = \tau(\varphi) \cap \tau(\varphi)^T$, at least one of the relations $[x, m] \in \tau(\varphi)$ and $[m, x] \in \tau(\varphi)$ is violated. In the first case, the set $P = (\varphi^0(x) \cup \varphi^1(x) \cup \varphi^2(x) \cup \ldots) \cap M'$ does not contain m. Clearly $\varphi(P) \cap M' \subset \subset P$ so that φ is not irreducible on M'. Now let $[m, x] \in \tau(\varphi)$ be violated. Then $Q = (\varphi^0(m) \cup \varphi^1(m) \cup \varphi^2(m) \cup \ldots) \cap M'$ does not contain x and $\varphi(Q) \cap M' \subset \subset Q$. Hence φ is not irreducible on M' and the proof is complete.

(2,5) Theorem. *The set of all irreducible components of φ coincides with the system of all classes of the equivalence*

$$\varepsilon(\varphi) = \tau(\varphi) \cap \tau(\varphi)^T.$$

Proof. It follows from (2,4) that the set $\mathscr{E}$ of all classes of the equivalence $\varepsilon(\varphi)$ is contained in the set $\mathscr{C}$ of all irreducible components of φ.

Now let $C \in \mathscr{C}$. Since the union of the family $\mathscr{E}$ equals N, there exists a set $E \in \mathscr{E}$ which intersects C. By Lemma (2,3), $C = E$ and $\mathscr{C} \subset \mathscr{E}$. The proof is complete.

(2,6) Definition. Let $\varphi \in G(N)$. Let E be an equivalence on N. We shall denote by N/E the set of all classes of the equivalence E. If $a, b \in N/E$, we write $a \in \varphi^E(b)$ if there exists an element $x \in a$ and an element $y \in b$ such that $x \in \varphi(y)$. It is easy to see that $\varphi^E \in G(N/E)$.

We shall prove now a lemma.

(2,7) *Let* $\varphi \in G(N)$. *Then the following three conditions are equivalent:*

1° *for any* $x \in N$ *and any natural* k *we have*

$$x \cap \varphi^k(\varphi(x) - x) = \emptyset ,$$

2° *for any* $x \in N$ *and any natural numbers* k, m *we have*

$$x \cap \varphi^k(\varphi^m(x) - x) = \emptyset ,$$

3° $\varepsilon(\varphi)$ *is the identity.*

Proof. 1° → 2°. Suppose that condition 1° is satisfied and that $x \in \varphi^k(\varphi^m(x) - x)$ for some $x \in N$. Suppose that m is minimal with respect to this inclusion. It follows that $m \geqq 2$. We have $x \in \varphi^k(y)$ where $y \in \varphi^m(x)$ and y is different from x. Further, there exists a $z \in \varphi(x)$ such that $y \in \varphi^{m-1}(z)$. Now z must be different from x since otherwise $y \in \varphi^{m-1}(x) - x$ which contradicts the minimality of m. It follows that $z \in \varphi(x) - x$. At the same time $x \in \varphi^k(y) \subset \varphi^{k+m-1}(z)$ whence $x \in \varphi^{k+m-1}(\varphi(x) - x)$ which is a contradiction.

2° → 3°. Suppose that $[x, y] \in \varepsilon(\varphi)$ for some $x \neq y$. Since $[x, y] \in \tau(\varphi)$ and $y \neq x$, we have $y \in \varphi^p(x)$ for some $p \geqq 1$. Since $[y, x] \in \tau(\varphi)$, we have $x \in \varphi^q(y)$ for some $q \geqq 1$. Hence $x \in \varphi^q(\varphi^p(x) - x)$.

3° → 1°. Suppose that $\varepsilon(\varphi) = \delta$ and that $x \in \varphi^k(\varphi(x) - x)$ for some $x \in N$ and some natural k. It follows that $x \in \varphi^k(y)$ for some $y \in \varphi(x) - x$. We have thus $y \neq x$ and $x \in \varphi^k(y)$ so that $[y, x] \in \tau(\varphi)$. At the same time $y \in \varphi(x)$ so that $[x, y] \in \tau(\varphi)$. It follows that $[x, y] \in \varepsilon(\varphi)$ and $x \neq y$ which contradicts the assumption $\varepsilon(\varphi) = \delta$.

(2,8) Definition. A relation $\varphi \in G(N)$ is said to be *weakly acyclic* if it satisfies one of the conditions of the preceding proposition.

The next step in our considerations will be the proof of the fact that, for each $\varphi \in G(N)$, the quotient relation $\varphi^{\varepsilon(\varphi)}$ is weakly acyclic. First of all, we shall need two simple lemmas.

(2,9) *Let* $\varphi \in G(N)$ *and let* E *be an equivalence relation on* N. *Then, for each natural number* k,

$$(\varphi^k)^E \subset (\varphi^E)^k .$$

Proof. We shall use induction. The case $k = 1$ being obvious, suppose that $(\varphi^k)^E \subset (\varphi^E)^k$ and that $a \in (\varphi^{k+1})^E\, b$. It follows that there exist $x \in a$ and $y \in b$ such that $x \in \varphi^{k+1}(y)$. There exists a $z \in N$ with $x \in \varphi(z)$ and $z \in \varphi^k(y)$. Denote by c the class of the equivalence E which contains z. We have then $a \in \varphi^E c$ and $c \in (\varphi^k)^E\, b$; according to the induction hypothesis this implies $c \in (\varphi^E)^k\, b$ whence $a \in (\varphi^E)^{k+1}\, b$.

(2,10) *Let $\varphi \in G(N)$ and let $E = \varepsilon(\varphi)$. For each natural number k*

$$(\varphi^E)^k \subset (\varphi^k)^E \cup (\varphi^{k+1})^E \cup \dots$$

Proof. Obvious for $k = 1$. Suppose the theorem proved for k and let $a \in (\varphi^E)^{k+1} b$. It follows that there exists a $c \in N/E$ such that $a \in \varphi^E c$ and $c \in (\varphi^E)^k b$. According to the induction hypothesis this implies $c \in (\varphi^{k+m})^E b$ for a suitable $m \geqq 0$. Hence there exist elements $x \in a$, $z \in c$, $z' \in c$ and $y \in b$ such that $x \in \varphi(z)$, $z' \in \varphi^{k+m}(y)$; since $[z, z'] \in E = \varepsilon(\varphi)$, we have $z \in \varphi^j(z')$ for some $j \geqq 0$. Taken together, these yield $x \in \varphi^{j+1}(z') \subset \varphi^{j+1+k+m}(y)$ so that

$$a \in (\varphi^{k+1+(j+m)})^E b .$$

(2,11) *Let $\varphi \in G(N)$. Let E be the equivalence corresponding to the decomposition of φ into irreducible components, $E = \varepsilon(\varphi)$. Then φ^E is weakly acyclic.*

Proof. Using (2,10), it is easy to show that

$$\tau(\varphi^E) = \bigcup_{k \geqq 0} (\varphi^E)^k \subset \bigcup_{k \geqq 0} (\varphi^k)^E = (\tau(\varphi))^E .$$

To show that φ^E is weakly acyclic, it suffices to prove that $\varepsilon(\varphi^E)$ is the identity. We have the following inclusions:

$$\tau(\varphi^E) \cap (\tau(\varphi^E))^T = \tau(\varphi^E) \cap \tau((\varphi^T)^E) \subset (\tau(\varphi))^E \cap (\tau(\varphi^T))^E .$$

If we show that $(\tau(\varphi))^E \cap (\tau(\varphi^T))^E \subset (\tau(\varphi) \cap \tau(\varphi^T))^E$ the proof will be complete since $(\tau(\varphi) \cap \tau(\varphi^T))^E = \varepsilon(\varphi)^E$ and this is the identity. To prove the inclusion $(\tau(\varphi))^E \cap \cap (\tau(\varphi^T))^E \subset (\tau(\varphi) \cap \tau(\varphi^T))^E$ consider a pair $[a, b] \in (\tau(\varphi))^E \cap (\tau(\varphi^T))^E$. It follows that there exist elements $x \in a$ and $y \in b$ such that $[x, y] \in \tau(\varphi)$; further, there exist elements $x' \in a$ and $y' \in b$ such that $[y', x'] \in \tau(\varphi)$. We have thus $[y, y'] \in \varepsilon(\varphi) \subset \subset \tau(\varphi)$, $[y', x'] \in \tau(\varphi)$, $[x', x] \in \varepsilon(\varphi) \subset \tau(\varphi)$ whence $[y, x] \in \tau(\varphi)$, the relation $\tau(\varphi)$ being transitive. It follows that $[x, y] \in \varepsilon(\varphi)$ so that $[a, b] \in (\varepsilon(\varphi))^E$ so that $a = b$. The proof is complete.

(2,12) Theorem. *Let M be a finite set and let $\varphi \in G(M)$ be weakly acyclic. Then the elements of M may be arranged in a sequence $m_1, \dots, m_r$ in such a way that $i < j$ implies m_j* non $\in \varphi(m_i)$.

Proof. I. Observe first that for a weakly acyclic φ there exists at least one element m such that m non $\in \varphi(M - m)$. To see that suppose that $m \in \varphi(M - m)$ for all $m \in M$. Choose an element $p_0 \in M$. There exists a sequence $p_1, p_2, \dots$, such that $p_{i-1} \in \varphi(p_i)$ and $p_{i-1} \neq p_i$, $i = 1, 2, \dots$ Since M is finite, there exists an index j such that $p_j = p_k$ for some $k > j$. Since $p_j \neq p_{j+1}$, we have $k > j + 1$. Now $p_j \in \in \varphi(p_{j+1})$ so that $[p_{j+1}, p_j] \in \tau(\varphi)$; on the other hand, $p_{j+1} \in \varphi^{k-j-1}(p_k) = \varphi^{k-j-1}(p_j)$

whence $[p_j, p_{j+1}] \in \tau(\varphi)$. We have thus both $[p_{j+1}, p_j] \in \tau(\varphi)$ and $[p_j, p_{j+1}] \in \tau(\varphi)$. This is a contradiction with the assumption that φ is weakly acyclic.

II. The proof will proceed by induction with respect to the number r of elements in M. If $r = 1$, the assertion is true. Let now $r > 1$ and suppose the theorem proved for $r - 1$. According to I, there exists an m such that m non $\in \varphi(M - m)$. Denote by M' the set $M - m$ and by φ' the relation induced on M' by φ. Clearly φ' is weakly acyclic so that the elements of M' may be arranged in a sequence $m_1, \ldots, m_{r-1}$ in such a way that $i < j$ implies m_j non $\in \varphi'(m_i)$. Define $m_r = m$. It is easy to see that this ordering of M satisfies the conditions of the theorem. The proof is complete.

(2,13) Theorem. *Let $\varphi \in G(N)$. Then the set of all irreducible components of φ may be arranged in a sequence $M_1, \ldots, M_r$ in such a way that $i < j$ implies $\varphi(M_i) \cap \cap M_j = \emptyset$.*

Proof. An immediate consequence of the preceding two propositions.

Let us conclude this section with some terminological conventions concerning the combinatorial structure of matrices.

If A is a matrix of order n (a function on $N \times N$), we shall assign to it a certain $\varphi \in G(N)$ (a subset of $N \times N$) by the following postulate: $[i, k] \in \varphi$ if and only if $a_{ik} \neq 0$. This φ will be called the combinatorial structure of A. It is easy to see that the matrix A is irreducible (indecomposable) if and only if the corresponding φ is irreducible. According to (2,13), the irreducible components of φ may be arranged in a sequence $M_1, \ldots, M_r$ in such a way that $i < j$ implies $\varphi(M_i) \cap M_j = \emptyset$. This has the following meaning for the matrix A: If $p \in M_i$, $q \in M_j$, $i < j$ then $a_{pq} = 0$, i.e. A can be brought to the block triangular form by such a permutation of rows and columns in which the indices of M_j follow the indices of M_i for $j > i$. The diagonal blocks of A are then irreducible. In the next section, we shall use the obvious fact that det A is equal to the product $\prod_{i=1}^{r} \det A(M_i)$. The matrix $A_0 = (d_{pq})$ with $d_{pq} = a_{pq}$ if p, q belong to the same irreducible component of φ, $d_{pq} = 0$ otherwise, will be called the block-diagonal part of A. The sets M_i will be shortly called irreducible components of the matrix A. We shall also speak about the decomposition of the matrix A into irreducible components by which we mean the decomposition of the corresponding φ.

3. Cyclic products. In this section we shall define a new inequality for matrices. Instead of comparing them term by term we shall compare cyclic products. It is immediate that this inequality is invariant with respect to diagonal equivalence. The main result consists in showing that, under certain hypotheses, this new inequality reduces to the ordinary one after suitable diagonal transformations.

Let us begin with some terminology. An arbitrary sequence of indices $i_1, \ldots, i_m \in \in N$, $m > 1$ will be called a path. If $i_1 = i_m$ this path will be called a cycle, a cycle

$(i_1, \ldots, i_m)$ is said to be simple if the indices $i_1, \ldots, i_{m-1}$ are different from each other. If B is a matrix and P a path, $P = (i_1, \ldots, i_m)$, we define $B(P)$ as the product

$$B(P) = b_{i_1 i_2} b_{i_2 i_3} \ldots b_{i_{m-1} i_m} .$$

If P is a cycle, the product $B(P)$ will be called a *cyclic product*. If P_1 and P_2 are two paths such that the end point of P_1 coincides with the initial point of P_2 we define $P_1 P_2$ as the path $i_1 \ldots i_m j_2 \ldots j_n$ if $P_1 = (i_1, \ldots, i_m)$, $P_2 = (j_1, \ldots, j_n)$.

It follows that, for any matrix B,

$$B(P_1 P_2) = B(P_1)\, B(P_2) .$$

If $G \in G(N)$ we shall use the expression "a path P in G" for paths $(i_1, \ldots, i_m)$ such that all $[i_j, i_{j+1}]$ belong to G. Similarly, if A is a matrix, a path in A will be one for which all $[i_j, i_{j+1}]$ belong to the $\varphi \in G(N)$ corresponding to A. It follows that $A(P) \neq 0$ if P is a path in A.

(3,1) Definition. If A and B are two matrices we write $A \overset{w}{\geqq} B$ and we say that A *is weakly greater than or equal to* B if, given any cycle C, both $A(C)$ and $B(C)$ are real and $A(C) \geqq B(C)$.

In particular, it follows from this definition that, for each i, both a_{ii} and b_{ii} are real and $a_{ii} \geqq b_{ii}$.

The inverse relation will be denoted by $\overset{w}{\leqq}$.

The relation $\overset{w}{\geqq}$ is transitive but not reflexive. The intersection of the relation $\overset{w}{\geqq}$ and of its inverse will be denoted by $\overset{w}{=}$. It is easy to see that $\overset{w}{=}$ is an equivalence in the set of all matrices all cyclic products of which are real.

The relation $\overset{w}{\geqq}$ has the following properties.

(3,2) *If $A \overset{w}{\geqq} B$ and D is any nonsingular real diagonal matrix then*

$$AD \overset{w}{\geqq} BD\,, \quad DA \overset{w}{\geqq} DB\,, \quad AD \overset{w}{\geqq} DB\,, \quad DA \overset{w}{\geqq} BD\,, \quad D^{-1}AD \overset{w}{\geqq} B\,, \quad A \overset{w}{\geqq} D^{-1}BD\,.$$

If $A \overset{w}{\geqq} B$ then $A(M) \overset{w}{\geqq} B(M)$ for each $M \subset N$.

Proof. Obvious.

(3,3) *Let A and B be complex matrices such that $0 \leqq B \overset{w}{\leqq} A$. If B is irreducible then there exists a diagonal matrix D such that $B \leqq D^{-1}AD$.*

Moreover, if A itself is real then the matrix D may be chosen real, if A is nonnegative, D may be chosen nonnegative. If $M(A) = B$, the matrix D may be chosen in such a manner that $M(D) = I$.

Proof. Let B be irreducible. We shall show first that then $b_{ik} \neq 0$ implies $a_{ik} \neq 0$. This is clear for $i = k$. If $i \neq k$, it follows from the irreducibility of B that there exists a path $P = (k, \ldots, i)$ in B. If $P' = (i, k)$, we have $0 < B(PP') \leqq A(PP') = A(P)\, A(P')$ so that $a_{ik} = A(P') \neq 0$.

Denote by C the matrix with elements

$$c_{ik} = b_{ik}/a_{ik} \quad \text{if} \quad b_{ik} \neq 0\,,$$
$$c_{ik} = 0 \qquad \text{if} \quad b_{ik} = 0\,.$$

It follows that for any cycle P in C (and in B as well) the inequality $0 < C(P) \leqq 1$ is satisfied.

We shall construct a matrix $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ satisfying our condition. Define $d_1 = 1$. Let $k \neq 1$. Denote by S_k the set of all paths $(1, \ldots, k)$ in C in which every index occurs once at most. Since S_k is nonvoid, there exists a $P \in S_k$ such that $|C(P)| \geqq |C(P')|$ for any $P' \in S_k$. Let us show that $C(P)$ is uniquely determined. Assume that $|C(P)| = |C(P_1)|$ for some $P_1 \in S_k$. Since C is irreducible, there exists a path P_2 in C with the initial point k and endpoint 1. We have now $0 < C(PP_2) \leqq 1$, $0 < C(P_1P_2) \leqq 1$ whence

$$\begin{aligned} C(P) &= C(PP_2)/C(P_2) = |C(P)|\,|C(P_2)|/C(P_2) = \\ &= |C(P_1)|\,|C(P_2)|/C(P_2) = C(P_1P_2)/C(P_2) = C(P_1)\,. \end{aligned}$$

This enables us to define d_k by $d_k = C(P)$. It follows that $d_k \neq 0$ so that the matrix $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ is nonsingular.

To show that

$$0 \leqq B \leqq D^{-1}AD$$

or, in other words,

$$0 \leqq b_{uv} \leqq d_u^{-1} a_{uv} d_v\,,$$

observe that this last inequality is obvious if $a_{uv} = 0$ or if $u = v$. Suppose now that $a_{uv} \neq 0$ and $u \neq v$. If $u \neq 1$, $d_u = C(P)$ for some $P \in S_u$. Hence, if $E = (u, v)$

$$|d_u|\,|c_{uv}| = |C(PE)|\,.$$

Suppose first that v does not belong to P. Then,

$$|d_v| \geqq |C(PE)| = |d_u|\,|c_{uv}|\,.$$

This inequality is also satisfied if $u = 1$. Now, if v belongs to P, we can write $P = P_1P_2$ where $P_1 \in S_v$. Hence

$$|d_v| \geqq |C(P_1)| \geqq |C(P_1P_2E)| = |C(PE)| = |d_u|\,|c_{uv}|$$

as well.

From the inequality

$$|d_v| \geqq |d_u|\,|c_{uv}|$$

it follows that

$$0 \leqq b_{uv} \leqq |d_u|^{-1}\,|a_{uv}|\,|d_v|\,.$$

Let us show now that

$$|d_u|^{-1}\,|a_{uv}|\,|d_v| = d_u^{-1} a_{uv} d_v\,,$$

if $a_{uv} \neq 0$ and $u \neq v$.

Let first $u \neq 1 \neq v$. As before, let $E = (u, v)$, $P = (1, \ldots, u)$, $d_u = C(P)$. Further, let $\tilde{P}$ be an arbitrary path in C of the form $(v, \ldots, 1)$, P_1 a path in S_v such that $d_v = C(P_1)$. Now, $PE\tilde{P}$ being a cycle,

$$C(PE\tilde{P}) = |C(PE\tilde{P})|\,.$$

By the same reason,

$$C(P_1\tilde{P}) = |C(P_1\tilde{P})|$$

so that

$$C(PE)/C(P_1) = |C(PE)/C(P_1)|\,.$$

Hence

$$d_u c_{uv} d_v^{-1} = |d_u c_{uv} d_v^{-1}|$$

which implies

$$d_u^{-1} a_{uv} d_v = |d_u^{-1} a_{uv} d_v|\,.$$

An easy modification shows that this is true in the case $u = 1$ or $v = 1$ as well.

The proof of the first part of the theorem is complete.

To prove the remaining assertions, it suffices to observe that the matrix D defined above has the mentioned properties.

(3,4) *Let Q be a complex irreducible matrix. Then the following conditions are equivalent*:

1° $0 \overset{w}{\leqq} Q$,
2° $M(Q) \overset{w}{\leqq} Q$,
3° $Q = U^{-1}\,M(Q)\,U$ with $M(U) = I$,
4° $Q = D^{-1}\,M(Q)\,D$ for a nonsingular diagonal D,
5° $Q \overset{w}{=} M(Q)$.

Proof. If $0 \overset{w}{\leqq} Q$, the cyclic products of Q are real and nonnegative. It follows immediately that $M(Q) \overset{w}{\leqq} Q$. If 2° is satisfied, it follows from (3,3) that there exists a nonsingular diagonal D such that $M(Q) \leqq D^{-1}QD$. Hence $M(Q) \leqq D^{-1}QD = M(D^{-1}QD) = M(D)^{-1} M(Q) M(D)$. The first and the last matrices in this chain have the same Perron root. Since $M(Q)$ is irreducible, it follows from (1,1) that $M(Q) = M(D)^{-1} M(Q) M(D)$ so that $M(Q) = D^{-1}QD$ as well. Now $M(Q)$ being irreducible, the only diagonal matrices commuting with $M(Q)$ are scalar multiples of the unit matrix. We may thus suppose that $M(D) = I$. This proves 3°. The implication 3° → 4° is formal, 4° → 5° by definition. If 5° is satisfied, we have $Q \overset{w}{=} M(Q) \geqq 0$ hence $Q \overset{w}{\geqq} 0$.

(3,5) *Let P and Q be two complex matrices; suppose that Q is irreducible. Then the following two statements are equivalent:*

1° $0 \overset{w}{\leqq} Q \overset{w}{\leqq} P$;

2° *there exist two nonsingular diagonal matrices D_Q and D_P such that both $D_Q^{-1}QD_Q$ and $D_P^{-1}PD_P$ are real and*

$$0 \leqq D_Q^{-1}QD_Q \leqq D_P^{-1}PD_P\,.$$

Proof. Since $0 \overset{w}{\leqq} Q$, it follows from (3,4) that there exists a nonsingular diagonal matrix D such that $M(Q) = D^{-1}QD$. Since $Q \overset{w}{\leqq} P$, we have

$$D^{-1}QD \overset{w}{\leqq} D^{-1}PD$$

so that

$$0 \leqq M(Q) \overset{w}{\leqq} D^{-1}PD\,.$$

The matrix $M(Q)$ being irreducible it follows from (3,3) that there exists a nonsingular diagonal D_0 with

$$M(Q) \leqq D_0^{-1}D^{-1}PDD_0\,.$$

Together with the preceding inequalities this yields

$$0 \leqq D^{-1}QD \leqq D_0^{-1}D^{-1}PDD_0$$

so that it suffices to take $D_Q = D$, $D_P = DD_0$.

(3,6) *Let P, Q be two matrices such that $P \overset{w}{=} Q$. Suppose that one of them is irreducible. Then P and Q have the same combinatorial structure.*

Proof. The relation $P \overset{w}{=} Q$ clearly implies $M(P) \overset{w}{=} M(Q)$. Suppose that Q is irreducible. Then $M(Q)$ is irreducible as well and $0 \leqq M(Q) \overset{w}{\leqq} M(P)$. It follows from (3,3) that there exists a positive diagonal matrix D with

$$M(Q) \leqq D^{-1} M(P) D .$$

It follows that $q_{ik} \neq 0$ implies $p_{ik} \neq 0$. In particular, P is irreducible. Using what has been proved above, it follows that $p_{ik} \neq 0$ implies $q_{ik} \neq 0$ and the lemma is established.

(3,7) *Let A and B be matrices such that $A \overset{w}{=} B$. Then the decompositions of N into irreducible components with respect to A and B coincide. Moreover, for each $M \subset N$, we have* $\det A(M) = \det B(M)$.

Proof. Let $N = N_1 \cup \ldots \cup N_r$ be the decomposition with respect to A. It follows that the matrices $A(N_i)$ are irreducible. Further, $B(N_i) \overset{w}{=} A(N_i)$ so that $B(N_i)$ is irreducible by (3,6). It follows that, for a fixed i, N_i is contained in one irreducible component M of N with respect to B. The same consideration shows that $A(M)$ is irreducible so that $N_i = M$. Hence both decompositions coincide. The rest follows easily from the fact that the determinant may be expressed in terms of the cyclic products.

(3,8) *Let P and Q be two matrices such that at least one of them is irreducible. Let $P \geqq 0$. Then the following two statements are equivalent:*

1° $P \overset{w}{=} Q$,

2° *there exists a nonsingular diagonal matrix D such that*

$$P = D^{-1} Q D .$$

Moreover, if Q is nonnegative as well, D may be chosen positive. If $M(Q) = P$, then D may be chosen with $M(D) = I$.

Proof. The implication 2° → 1° being obvious, it suffices to prove 1° → 2°.

According to lemma (3,6) both P and Q are irreducible. Since $0 \leqq P \overset{w}{\leqq} Q$ and P irreducible, there exists, by (3,3), a diagonal matrix D such that

$$P \leqq D^{-1} Q D .$$

Further, we have

$$0 \leqq D^{-1} Q D \overset{w}{=} Q \overset{w}{\leqq} P$$

and $D^{-1}QD$ is irreducible. It follows, again from (3,3), that there exists a diagonal D_0 with $D^{-1}QD \leqq D_0^{-1}PD_0$. Combining this with the preceding inequalities, we obtain

$$0 \leqq P \leqq D^{-1} Q D \leqq D_0^{-1} P D_0 .$$

It follows from (1,1) that $P = D_0^{-1}PD_0$ so that $P = D^{-1}QD$ and the proof is complete.

If Q itself is nonnegative then, according to the corresponding assertion in (3,3), D may be chosen positive.

If $M(Q) = P$ then the last assertion in (3,3) yields that D may be chosen with $M(D) = I$.

(3,9) *Let P and Q be matrices such that Q is irreducible and $0 \leqq P \leqq Q$. If $P \overset{w}{=} Q$ then $P = Q$.*

Proof. According to (3,8), there exists a diagonal matrix D such that

$$P = D^{-1}QD\,.$$

Hence $p(P) = p(Q)$ so that, by (1,1), $P = Q$.

In the following theorems complex matrices with real cyclic products will be characterized.

(3,10) Theorem. *Let A be an irreducible complex matrix. Then the following properties of A are equivalent:*

1° *All cyclic products of A are real.*

2° *There exists a diagonal matrix D with $M(D) = I$ and a real matrix R such that*

$$A = D^{-1}RD\,.$$

3° *There exists a real matrix R such that*

$$A \overset{w}{=} R\,.$$

4° $$A \overset{w}{=} A\,.$$

Proof. Assume 1° and let us show that 2° is fulfilled. Denote by φ the relation in $G(N)$ corresponding to A. Since A is irreducible, it follows from (2,1) that there exists a relation $\varphi_1 \subset \varphi$ such that to each $y \in N$, $y \neq 1$ there is exactly one $x \in N$ such that $[x, y] \in \varphi_1$ while $[x, 1] \notin \varphi_1$ for any $x \in N$. Let h_1 be the corresponding function on N from (2,1). If we apply the same theorem to A^T and transpose the resulting relation, it follows that there exists a relation $\varphi_2 \subset \varphi$ such that for each $y \in N$, $y \neq 1$ there is exactly one $x \in N$ such that $[y, x] \in \varphi_2$ while $[1, x] \notin \varphi_2$ for any $x \in N$. Let h_2 be the corresponding function on N from (2,1).

We shall define step by step the diagonal elements d_i of D and elements r_{ik} of R in such a manner that the conditions

(a) $|d_i| = 1$ for $i \in N$,
(b) r_{ik} is real for $i, k \in N$,
(c) $a_{ik} = d_i^{-1}r_{ik}d_k$ for $i, k \in N$
will be satisfied.

Put $d_1 = 1$ and $r_{ii} = a_{ii}$ for all $i \in N$. To define d_j for $j > 1$ and r_{ij} for $(i, j) \in \varphi_1$, we shall use induction with respect to $h_1(j)$. If $h_1(j) = 1$ then $[1, j] \in \varphi_1$ and we put

$$r_{1j} = |a_{1j}|, \quad d_j = a_{1j} r_{1j}^{-1}.$$

If $h_1(j) > 1$ and d_k as well as r_{ik} has already been defined whenever $h_1(k) < h_1(j)$ and $(i, k) \in \varphi_1$, we put for $[m, j] \in \varphi_1$

$$r_{mj} = |a_{mj}|, \quad d_j = d_m a_{mj} r_{mj}^{-1}.$$

This can be done since $h_1(m) < h_1(j)$.

The definition of r_{pq} for $p \neq q$ and $[p, q] \notin \varphi_1$ as well as the checking of (b) and (c) will proceed by induction with respect to $h_2(q)$. If $h_2(q) = 0$, hence $q = 1$, we define

$$r_{p1} = a_{p1} A(P)/R(P)$$

where P is the unique path in φ_1 from 1 to p. It follows from the construction of $R(P)$ that

$$A(P)/R(P) = d_p$$

so that (c) as well as (b) is satisfied for $i = p$, $k = 1$.

Suppose now that $[p, q] \notin \varphi_1$, that $h_2(q) > 0$ and that all r_{ij} for which $h_2(j) < h_2(q)$ have already been defined. Then we put

$$r_{pq} = a_{pq} A(P_1) A(P_2) (R(P_1) R(P_2))^{-1}$$

where P_1 is the unique path in φ_1 from 1 to p and P_2 the unique path in φ_2 from q to 1. This is possible since all elements in $R(P_2)$ have already been defined. However, $[p, q]$ forms together with P_2 and P_1 a cycle in G. Since $a_{pq} A(P_1) A(P_2)$ is real and $R(P_1)$, $R(P_2)$ are real, r_{pq} is real as well. Further,

$$A(P_1)/R(P_1) = d_p, \quad A(P_2)/R(P_2) = d_q^{-1}$$

so that (c) is fulfilled for $i = p$, $k = q$.

This completes the proof of the implication $1° \to 2°$. The proof of $2° \to 3°$ as well as of $3° \to 4°$ and $4° \to 1°$ is an immediate consequence of the definition of $\overset{w}{=}$.

(3,11) *Let P and Q be two matrices such that at least one of them is irreducible. Then the following two statements are equivalent:*

1° $P \overset{w}{=} Q$;

2° *there exists a real matrix R such that both P and Q are diagonally similar to R.*

Proof. The implication $2° \to 1°$ being immediate, it suffices to prove $1° \to 2°$. If 1° is satisfied, P and Q have the same combinatorial structure by (3,6). According to (3,10) there exist real matrices R and S and diagonal matrices D_1 and D_2 such that $P = D_1^{-1}RD_1$ and $Q = D_2^{-1}SD_2$. The matrices R and S have also the same combinatorial structure. Denote by B the matrix defined as follows:

$$b_{ik} = r_{ik}/s_{ik} \quad \text{for} \quad s_{ik} \neq 0\,,$$
$$b_{ik} = 0 \qquad \text{for} \quad s_{ik} = 0\,.$$

According to our assumption 1° the cyclic products of the matrix B are equal either to 1 or 0. Denote by H the matrix for which $h_{ik} = 1$ if $q_{ik} \neq 0$ and $h_{ik} = 0$ otherwise. Clearly $B \overset{w}{=} H$ so that there exists, by (3,8), a diagonal D with $B = D^{-1}HD$. It follows that $r_{ik} = s_{ik}d_i^{-1}d_k$ whenever $s_{ik} \neq 0$. We have thus

$$P = D_1^{-1}RD_1\,, \quad Q = D_2^{-1}D(D^{-1}SD)\,D^{-1}D_2 = D_2^{-1}DRD^{-1}D_2\,.$$

The proof is complete.

(3,12) *Let P and Q be two matrices. Then the following two statements are equivalent*:

1° $P \overset{w}{=} Q$;

2° *P and Q have the same decomposition into irreducible components and there exists a real matrix R such that both P_0 and Q_0 are diagonally similar to R; here, P_0 and Q_0 are the block diagonal parts of P and Q.*

Proof. A consequence of (3,7) and (3,11).

(3,13) Corollary. *Let P and Q be two matrices such that $P \overset{w}{=} Q$. If one of the matrices P, Q is nonsingular then so is the other and $P^{-1} \overset{w}{=} Q^{-1}$.*

Proof. The first part follows from (3,7). Let P^{-1} exist and let the block diagonal parts $\tilde{P}$ and $\tilde{Q}$ be diagonally similar to a real matrix R. Then R^{-1} exists and all the three matrices P, Q and R have the same decomposition into irreducible components. The inverse matrices P^{-1} and Q^{-1} have then also the property that their block diagonal parts are diagonally similar to R^{-1}. By (3,12), $P^{-1} \overset{w}{=} Q^{-1}$ and the proof is complete.

4. Inequalities for determinants. Now we are ready to attack the main problem.

(4,1) *Let A and B be matrices such that $A \geqq 0$ and $M(B) \overset{w}{\leqq} A$. Then $|B|_\sigma \leqq p(A)$. If B is irreducible and $p(A) \in \sigma(B)$ then $B \overset{w}{=} A$.*

Proof. Let $N = N_1 \cup \ldots \cup N_r$ be the decomposition into irreducible components with respect to B. Using (1,1) it is easy to see that $p(A(N_i)) \leqq p(A)$ for each i. Since

$|B|_\sigma = \max |B(N_i)|_\sigma$, the first part of the theorem will be proved if we show that $|B(N_i)|_\sigma \leqq p(A(N_i))$ for each i.

Let i be fixed. Observe first that $M(B) \overset{w}{\leqq} A$ implies $M(B(N_i)) \overset{w}{\leqq} A(N_i)$ by (3,2). By (3,3), there exists a diagonal matrix D_i such that $0 \leqq D_i^{-1} M(B(N_i))\, D_i \leqq A(N_i)$. Hence

$$|B(N_i)|_\sigma \leqq |M(B(N_i))|_\sigma = |D_i^{-1} M(B(N_i))\, D_i|_\sigma \leqq p(A(N_i))$$

by (1,1).

Suppose now that B is irreducible and $p(A) \in \sigma(B)$.

Since $M(B) \overset{w}{\leqq} A$, it follows from (3,3) that there exists a positive diagonal matrix D such that

$$D^{-1}\, M(B)\, D \leqq A\,.$$

If we write $C = D^{-1} BD$, we have

$$0 \leqq M(C) = D^{-1}\, M(B)\, D \leqq A$$

and

$$|B|_\sigma = |C|_\sigma \leqq p(M(C)) \leqq p(A) = |B|_\sigma\,.$$

It follows from (1,1) that $M(C) = A$ and that there exists a diagonal matrix G and a complex unit ε such that

$$G^{-1} CG = \varepsilon A \quad \text{whence} \quad \varepsilon A = G^{-1} D^{-1} BDG\,.$$

Since $p(A) \in \sigma(B) = \sigma(C)$ it follows that $\varepsilon\, p(A) \in \sigma(A)$. If h is the index of imprimitivity of A, we have $\varepsilon^h = 1$. (See [7].) Since the lengths of all cycles in A are divisible by h we have finally $B \overset{w}{=} A$.

(4,2) Theorem. *Let A and B be matrices such that $A \geqq 0$ and $M(B) \overset{w}{\leqq} A$. Let $|A|_\sigma = p(A) \leqq 1$. Then*

$$|\det (I - B)| \geqq \det (I - A)\,.$$

If $p(A) < 1$ then equality is attained if and only if $B \overset{w}{=} A$.

If $p(A) = 1$, then the following conditions are equivalent:

1° *equality is attained (so that both determinants are zero);*

2° *there exists an irreducible component M of A such that $p(A(M)) = 1$ and $B(M) \overset{w}{=} A(M)$;*

3° *there exists an M which is an irreducible component for both A and B and $p(A(M)) = 1$, $B(M) \overset{w}{=} A(M)$.*

In particular, if A (or B) is irreducible then equality is attained if and only if $B \overset{w}{=} A$.

Proof. Consider first the case $p(A) < 1$. Since $M(B) \overset{w}{\leqq} A$ it follows from (4,1) that $|B|_\sigma \leqq p(A) < 1$. Hence the series

$$\log (I - B) = -(a_1 B + a_2 B^2 + a_3 B^3 + \ldots)$$

is convergent where $a_i = 1/i$. We note that $a_i > 0$ for $i = 1, 2, \ldots$

Let s be a natural number and let $k_1, \ldots, k_s$ be arbitrary indices in N. Since $M(B) \overset{w}{\leqq} A$, we have

(1) $$\operatorname{Re} B(k_1, \ldots, k_s, k_1) \leqq |B(k_1, \ldots, k_s, k_1)| \leqq A(k_1, \ldots, k_s, k_1) .$$

Further, let us recall that, for each s,

$$\operatorname{tr} B^s = \sum_{k_1, \ldots, k_s} B(k_1, \ldots, k_s, k_1) .$$

Using these facts, we obtain

$$\begin{aligned}
|\det (I - B)| &= |\exp (- \operatorname{tr} (a_1 B + a_2 B^2 + \ldots))| = \\
&= \exp (- \operatorname{Re} \operatorname{tr} (a_1 B + a_2 B^2 + \ldots)) = \\
&= \exp (- \sum_{s=1}^{\infty} a_s \sum_{k_1, \ldots, k_s} \operatorname{Re} B(k_1, \ldots, k_s, k_1)) \geqq \\
&\geqq \exp (- \sum_{s=1}^{\infty} a_s \sum_{k_1, \ldots, k_s} |B(k_1, \ldots, k_s, k_1)|) \geqq \\
&\geqq \exp (- \sum_{s=1}^{\infty} a_s \sum_{k_1, \ldots, k_s} A(k_1, \ldots, k_s, k_1)) = \\
&= \exp (- \operatorname{tr} (a_1 A + a_2 A^2 + \ldots)) = \det (I - A) .
\end{aligned}$$

Suppose now that $|\det (I - B)| = \det (I - A)$. It follows from the preceding chain of inequalities, from the fact that $a_i > 0$ and from the inequalities (1), that

$$\operatorname{Re} B(k_1, \ldots, k_s, k_1) = |B(k_1, \ldots, k_s, k_1)| = A(k_1, \ldots, k_s, k_1)$$

for each finite sequence $k_1, \ldots, k_s$, so that

$$0 \leqq B(k_1, \ldots, k_s, k_1) = A(k_1, \ldots, k_s, k_1) .$$

Hence

$$0 \overset{w}{\leqq} B \overset{w}{=} A .$$

Let us take up the case $p(A) = 1$ and prove now the equivalence of conditions 1°, 2° and 3°. The implication 3° → 2° being obvious assume 2° and let us prove 1°.

Since $M(B) \overset{w}{\leqq} A$ the decomposition of B into irreducible components is a refinement of the decomposition of A. Since $A(M)$ is irreducible and $B(M) \overset{w}{=} A(M)$, $B(M)$ is irreducible as well so that M is also an irreducible component for B.

Now it suffices to show that $I - B(M)$ is singular. This, however, follows from $B(M) \overset{w}{=} A(M)$ whence

$$\det(I - B(M)) = \det(I - A(M)) = 0 .$$

It remains to prove the implication $1° \to 3°$.

Suppose that $|\det(I - B)| = \det(I - A) = 0$. Let $N_1 \cup \ldots \cup N_r$ be the decomposition into irreducible components with respect to B. Since $\det(I - B)$ equals the product $\prod \det(I - B(N_i))$, it follows that there exists a component H such that $\det(I - B(H)) = 0$. We intend to show that H is also an irreducible component with respect to A and that $\det(I - A(H)) = 0$. Since $M(B(H)) \overset{w}{\leqq} A(H)$ and $p(A(H)) \leqq \leqq p(A) \leqq 1$ it follows from the first part of the theorem that

$$0 = |\det(I - B(H))| \geqq \det(I - A(H)) \geqq 0 .$$

Hence $\det(I - A(H)) = 0$. According to (3,3), $A(H)$ is irreducible. Denote by M' the irreducible component of A which contains H. Suppose that $M' \neq H$. The matrix $I - A(M')$ belongs to $\mathbf{K}_0$ since $p(A(M')) \leqq 1$. Since $I - A(M')$ is irreducible, it follows from (1,2) that $\det(I - A(M)) > 0$ which is a contradiction.

We have thus shown that $1 = p(A(H))$ belongs to $\sigma(B(H))$. By (4,1), it follows that $B(H) \overset{w}{=} A(H)$. The proof is complete.

(4,3) *Let A and B be matrices such that $A \geqq 0$ and $M(B) \leqq A$. Let $|A|_\sigma = p(A) \leqq \leqq 1$. Then*

$$|\det(I - B)| \geqq \det(I - A) .$$

If A is irreducible then equality is attained if and only if

$$M(B) = A$$

and

$$B = F^{-1}AF$$

where F is a diagonal matrix with $M(F) = I$.

Proof. Since $M(B) \leqq A$, we have $M(B) \overset{w}{\leqq} A$ so that theorem (4,2) can be applied. Hence

$$|\det(I - B)| \geqq \det(I - A) .$$

Suppose now that A is irreducible and that equality is attained.

It follows from (4,2) that $0 \overset{w}{\leqq} B \overset{w}{=} A$. According to condition 5° of (3,4), $B \overset{w}{=} M(B)$

so that $M(B) \overset{w}{=} A$. Since $M(B) \leqq A$ as well, it follows from (3,9) that $M(B) = A$. According to (3,8), $B = F^{-1}AF$ where F is a diagonal matrix with $M(F) = I$. The rest is easy.

(4,4) *Let P and Q be two matrices such that P is nonnegative and $M(Q) \overset{w}{\leqq} P$. Then*

1° $|Q|_\sigma \leqq p(P)$;

2° *if P is irreducible then $|Q|_\sigma = p(P)$ if and only if there exists a nonsingular diagonal matrix D and a number ε with $|\varepsilon| = 1$ such that $Q = \varepsilon D^{-1}PD$.*

Proof. The first part follows immediately from (4,1).

To prove 2°, suppose that P is irreducible. It suffices to prove the only if part. Accordingly, let $|Q|_\sigma = p(P)$. The matrix P being irreducible, $p(P) > 0$. Denote by A the matrix $(1/p(P))\, P$, by B the matrix $(1/\lambda)\, Q$ where λ is an eigenvalue of Q such that $|\lambda| = |Q|_\sigma$.

From $0 \leqq M(Q) \overset{w}{\leqq} P$ it follows that $0 \leqq M(B) \overset{w}{\leqq} A$. At the same time $\det(I - B) = \det(I - A) = 0$ and $p(A) = 1$. Since P is irreducible, A is irreducible as well so that, by (4,2), we have $0 \overset{w}{\leqq} B \overset{w}{=} A$. It follows from (3,8) that $A = D^{-1}BD$ for a suitable nonsingular diagonal matrix D. Hence $Q = \varepsilon DPD^{-1}$ where $\varepsilon = \lambda/p(P)$ is of modulus 1.

The main result forms a strengthening of a well known comparison theorem due to Koteljanskij [4]. If U is a complex matrix and V a (real) matrix, V in $\mathbf{K}$, such that $|u_{ii}| \geqq v_{ii}$ and $|u_{ik}| \leqq |v_{ik}|$ for $i \neq k$ then $|\det U| \geqq \det V$. We prove this under weaker hypotheses and present a complete discussion of the case of equality. To simplify notation, we shall use the following abbreviations: if A is a matrix, we write A^D and A^N for the diagonal and off-diagonal parts of A respectively.

(4,5) Theorem. *Let U be a complex matrix and let V be a matrix in $\mathbf{K}_0$. Suppose that*

$$M(U^D) \geqq M(V^D)\,, \quad M(U^N) \overset{w}{\leqq} M(V^N)\,.$$

Then

$$|\det U| \geqq \det V\,.$$

Denote by U_0 and V_0 the block-diagonal parts of U and V respectively corresponding to the decompositions into irreducible components.

If $\det V \neq 0$ *then the following conditions are equivalent*:

1° $|\det U| = \det V$;

2° *the decompositions of U and V into irreducible components coincide and*

$$U_0 = D_1 V_0 D_2$$

for a suitable pair of diagonal matrices D_1 and D_2 such that $M(D_1 D_2) = I$.

Moreover, if $M(U^N) \leqq M(V^N)$, *the matrices* D_1 *and* D_2 *may be chosen with* $M(D_1) = M(D_2) = I$.

If $\det V = 0$ *then the following conditions are equivalent*:

1° $|\det U| = \det V$;

2° *there exists an* M *which is an irreducible component for both* V *and* U *such that* $\det V(M) = 0$ *and*

$$U(M) = D_1 V(M) D_2$$

for suitable diagonal matrices D_1 *and* D_2 *such that* $M(D_1 D_2) = I$.

Moreover, if $M(U^N) \leqq M(V^N)$, *the matrices* D_1 *and* D_2 *may be chosen with* $M(D_1) = M(D_2) = I$.

Proof. Define the matrices A and B by the formulas

$$I - B = (U^D)^{-1} U \,, \quad I - A = D^{-1}V$$

where $D = M(U^D)$.

It follows that $b_{ii} = 0$ and $a_{ii} = 1 - (v_{ii}/|u_{ii}|) \geqq 0$ so that $A \geqq 0$. Since $M(U^N) \overset{w}{\leqq} M(V^N)$, we have obviously $M(B) \overset{w}{\leqq} A$.

Since $V \in \mathbf{K}_0$, $I - A$ belongs to $\mathbf{K}_0$ as well so that $p(A) \leqq 1$. By (4,2), we have

$$|\det (I - B)| \geqq \det (I - A)$$

whence

$$|\det (U^D)^{-1} U| \geqq \det D^{-1}V .$$

Since $|\det U^D| = \det D$, this implies

$$|\det U| \geqq \det V .$$

To investigate the case of equality, let us distinguish two cases. First, let $\det V \neq 0$ so that $p(A) < 1$. Assume that

$$|\det U| = \det V .$$

It follows that

$$|\det (I - B)| = \det (I - A) .$$

By (4,2), $0 \overset{w}{\leqq} B \overset{w}{=} A$. From (3,12) it follows that the matrices A and B have the same decompositions into irreducible components and, for some diagonal matrix F, we have

$$B_0 = F^{-1}A_0F$$

where A_0, B_0 are the corresponding block diagonal parts of A and B. Now,

$$(U^D)^{-1} U_0 = I - B_0 = F^{-1}(I - A_0) F = F^{-1}D^{-1}V_0F$$

whence

$$U_0 = U^D F^{-1} D^{-1} V_0 F .$$

It suffices therefore to take $D_1 = U^D F^{-1} D^{-1}$ and $D_2 = F$.

Moreover, if $M(U^N) \leqq M(V^N)$, it follows that $M(B) \leqq A$ and $M(B_0) \leqq A_0$ as well. By (3,9), $M(B_0) = A_0$ so that, by (3,8), F can be chosen in the manner that $M(F) = I$. If we take D_1 and D_2 as above, we shall have $M(D_1) = M(D_2) = I$ as asserted.

The converse implication is obvious.

Assume now that $\det V = 0$ so that $p(A) = 1$. Since the implication $2° \to 1°$ is obvious, assume that $1°$ is fulfilled. Then

$$\det (I - B) = \det (I - A) = 0 .$$

According to (4,2), there exists an M which is an irreducible component for both A and B such that $p(A(M)) = 1$ and $B(M) \overset{w}{=} A(M)$. Hence

$$I - B(M) \overset{w}{=} I - A(M)$$

so that

$$(U^D(M))^{-1} U(M) \overset{w}{=} (M(U^D(M)))^{-1} V(M) .$$

By (3,11), there exists a diagonal matrix D_2 such that

$$(U^D(M))^{-1} U(M) = D_2^{-1}(M(U^D(M)))^{-1} V(M) D_2 .$$

Hence

$$U(M) = D_1 V(M) D_2$$

where

$$D_1 = U^D(M) D_2^{-1}(M(U^D(M)))^{-1} .$$

Since $M(D_1 D_2) = I(M)$, $2°$ is fulfilled.

If $M(U^N) \leqq M(V^N)$, we have $M(B) \leqq A$ so that $M(B(M)) \leqq A(M)$. Since $p(A(M)) = |B(M)|_\sigma = 1$, it follows from (1,1) that $M(B(M)) = A(M)$. On the other hand, $B(M) \overset{w}{=} A(M) = M(B(M))$ implies, by (3,4), the existence of a diagonal matrix U with $M(U) = I$ such that $B(M) = U^{-1} M(B(M)) U = U^{-1} A(M) U$. Hence

$$I - B(M) = U^{-1}(I - A(M)) U ,$$

$$(U^D(M))^{-1} U(M) = U^{-1}(M(U^D(M)))^{-1} V(M) U$$

so that

$$U(M) = D_1 V(M) D_2$$

where

$$D_1 = U^{-1} U^D(M) M(U^D(M)))^{-1} ,$$

$$D_2 = U .$$

Since $M(D_1) = M(D_2) = I$, the proof is complete.

(4,6) Corollary. *Let $A \in \mathbf{H}_0$. Then*

$$|\det A| \geqq \det H(A) .$$

Let A be irreducible. Then, equality is attained if and only if

$$A = D_1 H(A) D_2$$

for some matrices D_1, D_2 with $M(D_1) = M(D_2) = I$.

If A is reducible and nonsingular then equality is attained if and only if

$$A_0 = D_{01} H(A_0) D_{02}$$

for some matrices D_{01}, D_{02} with $M(D_{01}) = M(D_{02}) = I$ where A_0 is the block diagonal part of A.

If A is reducible and singular then equality is attained if and only if there exists an irreducible component M and matrices D_1, D_2 such that $M(D_1) = M(D_2) = I(M)$ and

$$A(M) = D_1 H(A(M)) D_2 .$$

Proof. It suffices to take $U = A$, $V = H(A)$ in theorem (4,5).

Let us conclude with an easy consequence of the preceding theorem.

(4,7) Corollary. *Let $A \in \mathbf{H}_0$ be irreducible. Then all proper principal minors of A are different from zero.*

Proof. An immediate consequence of (4,5) and (1,2).

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