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SOME THEOREMS ON THE PROJECTIVE DERIVATIVE OF CERTAIN ENTITIES IN CONFORMAL FINSLER SPACES

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1. Conformal Finsler space. Let two distinct metric functions $F(x, \dot{x})$ and $\bar{F}(x, \dot{x})$ be defined over an n-dimensional Finsler space F_n which satisfy the requisite conditions for a Finsler space [1][1]). The two metrics resulting from them are called conformal if the corresponding metric tensors $g_{ij}(x, \dot{x})$ and $\bar{g}_{ij}(x, \dot{x})$ are proportional to each other. It has been shown that the factor of proportionality between them is at most a point function. Thus, we have [1]

(1.1a) $$\bar{g}_{ij}(x, \dot{x}) = e^{2\sigma} g_{ij}(x, \dot{x}),$$

(1.1b) $$\bar{g}^{ij}(x, \dot{x}) = \bar{e}^{2\sigma} g^{ij}(x, \dot{x}),$$

(1.1c) $$\bar{F}(x, \dot{x}) = e^{\sigma} F(x, \dot{x}),$$

where $\sigma = \sigma(x)$, $g^{ij}(x, \dot{x})$ being the contravariant components of the metric tensor of F_n. The space $\bar{F}_n$ with the entities $\bar{F}$, $\bar{g}_{ij}$ etc. is called a conformal Finsler space.

The covariant derivative of a vector $X^i(x, \dot{x})$ with respect to x^k in the sense of Berwald is given by

(1.2) $$X^i_{(k)}(x, \dot{x}) = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j G^i_{jk}\,,^2)$$

where $G^i_j(x, \dot{x}) \stackrel{\text{def}}{=} G^i_{mj}(x, \dot{x})\, \dot{x}^m$, $G^i_{jk}(x, \dot{x})$ being the Berwald's connection coefficients which are homogeneous of degree zero in their directional arguments. We have the following entities of the conformal Finsler space [2], [3]

(1.3) $$\bar{G}^i(x, \dot{x}) = G^i(x, \dot{x}) - \sigma_m B^{im}(x, \dot{x}),$$

(1.4) $$\bar{G}^i_j(x, \dot{x}) = G^i_j(x, \dot{x}) - \sigma_m \dot{\partial}_j B^{im}(x, \dot{x}),$$

(1.5) $$\bar{G}^i_{jk}(x, \dot{x}) = G^i_{jk}(x, \dot{x}) - \sigma_m \dot{\partial}_k \dot{\partial}_j B^{im}(x, \dot{x}),$$

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[1]) The numbers in square brackets refer to the references at the end of the paper.

[2]) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial \dot{x}^i$.

$$(1.6)\qquad \bar{G}^i_{jkh}(x,\dot{x}) = G^i_{jkh}(x,\dot{x}) - \sigma_m \dot{\partial}_h \dot{\partial}_k \dot{\partial}_j B^{im}(x,\dot{x}),$$

$$(1.7)\qquad \bar{l}^i(x,\dot{x}) = \bar{e}^{\sigma} l^i(x,\dot{x}),$$

$$(1.8)\qquad \bar{l}_i(x,\dot{x}) = e^{\sigma} l_i(x,\dot{x}),$$

$$(1.9)\qquad \bar{\dot{x}}^i = \dot{x}^i,$$

$$(1.10)\qquad \bar{A}^i_{jk}(x,\dot{x}) = e^{\sigma} A^i_{jk}(x,\dot{x}),$$

and

$$(1.11)\qquad \bar{C}^i_{jk}(x,\dot{x}) = C^i_{jk}(x,\dot{x}),$$

where

$$\sigma_m \overset{\text{def}}{=} \partial_m \sigma, \quad B^{ij}(x,\dot{x}) \overset{\text{def}}{=} \tfrac{1}{2}F^2 g^{ij} - \dot{x}^i \dot{x}^j,$$

$$l^i(x,\dot{x}) \overset{\text{def}}{=} \dot{x}^i \mid F(x,\dot{x}), \quad l_i(x,\dot{x}) = g_{ik}(x,\dot{x})\, l^k,$$

$$A^i_{mj}(x,\dot{x}) \overset{\text{def}}{=} F C^i_{mj}(x,\dot{x}) \overset{\text{def}}{=} \tfrac{1}{2} F g^{ih} \dot{\partial}_j g_{mh}.$$

The functions $B^{ij}(x,\dot{x})$ are homogeneous of degree two in $\dot{x}$'s.

If we denote the Berwald's covariant derivative with respect to $\bar{g}_{ij}(x,\dot{x})$ in conformal Finsler space by putting a horizontal bar over the same notation of the covariant derivative, then we obtain [3], [7]

$$(1.12)\qquad \bar{F}(\bar{k}) = e^{\sigma}[F\sigma_k + (\dot{\partial}_m F)(\dot{\partial}_k B^{mn})\,\sigma_n],^{3)}$$

$$(1.13)\qquad \bar{l}^i_{(\bar{k})} = -\bar{e}^{\sigma}[l^i \sigma_k - (\dot{\partial}_m l^i)(\dot{\partial}_k B^{mn})\,\sigma_n + \sigma_n l^h \dot{\partial}_h \dot{\partial}_k B^{in}],$$

$$(1.14)\qquad \bar{l}_{i(\bar{k})} = e^{\sigma}[l_{i(k)} + l_i \sigma_k + (\dot{\partial}_m l_i)(\dot{\partial}_k B^{mn})\,\sigma_n + l_m \sigma_n \dot{\partial}_k \dot{\partial}_i B^{mn}],$$

$$(1.15)\qquad \bar{g}_{ij(\bar{k})} = e^{2\sigma}[g_{ij(k)} + 2g_{ij}\sigma_k + \sigma_n\{(\dot{\partial}_m g_{ij})(\dot{\partial}_k B^{mn}) + 2g_{l(i}\dot{\partial}_{j)}\, \dot{\partial}_k B^{ln}\}],$$

and

$$(1.16)\qquad \bar{G}^i_{(\bar{k})} = G^i_{(k)} - (\partial_k \sigma_n)\, B^{in} - \sigma_n[(\partial_k B^{in}) + (\dot{\partial}_k B^{mn})\, G^i_m - (\dot{\partial}_m B^{in})\, G^m_k - G^m \dot{\partial}_k \dot{\partial}_m B^{in} + B^{mn} G^i_{mk} + \sigma_\gamma\{(\dot{\partial}_m B^{in})(\dot{\partial}_k B^{m\gamma}) - B^{m\gamma} \dot{\partial}_k \dot{\partial}_m B^{in}\}].$$

2. **The projective covariant derivative.** Let us consider the following projective change [1]

$$(2.1)\qquad \hat{G}^i(x,\dot{x}) \overset{\text{def}}{=} G^i(x,\dot{x}) - P(x,\dot{x})\,\dot{x}^i,$$

where $P(x,\dot{x})$ is an arbitrary scalar function positively homogeneous of the first degree in the $\dot{x}^i$. Using the homogeneity properties of the function $P(x,\dot{x})$, the suc-

3) The covariant derivative of $\bar{X}^i(x,\dot{x})$ in conformal Finsler space, in the sense of Berwald, is given by $\bar{X}^i_{(k)} \overset{\text{def}}{=} (\partial_k \bar{X}^i) - (\dot{\partial}_m \bar{X}^i)\, \bar{G}^m_k + \bar{X}^m \bar{G}^i_{mk}$.

cessive derivatives of (2.1) with respect to directional arguments are given by

$$\hat{G}^i_i(x, \dot{x}) = G^i_i(x, \dot{x}) - (n + 1) P , \tag{2.2}$$

$$\hat{G}^i_{ik}(x, \dot{x}) = G^i_{ik}(x, \dot{x}) - (n + 1) \dot{\partial}_k P , \tag{2.3}$$

and

$$\hat{G}^i_{ikh}(x, \dot{x}) = G^i_{ikh}(x, \dot{x}) - (n + 1) \dot{\partial}_k \dot{\partial}_h P , \tag{2.4}$$

where

$$\hat{G}^i_{ikh} \stackrel{\text{def}}{=} \dot{\partial}_h \hat{G}^i_{ik} \stackrel{\text{def}}{=} \dot{\partial}_h \dot{\partial}_k \hat{G}^i_i \stackrel{\text{def}}{=} \dot{\partial}_h \dot{\partial}_k \dot{\partial}_i \hat{G}^i . \tag{2.5}$$

Differentiating (2.1) twice with respect to $\dot{x}^j$ and $\dot{x}^k$ and eliminating $\dot{\partial}_k P$ and $\dot{\partial}_j \dot{\partial}_k P$ with the help of equations (2.3) and (2.4), DOUGLAS [5] deduced the projective invariants

$$\Pi^i_{jk}(x, \dot{x}) \stackrel{\text{def}}{=} G^i_{jk} - \frac{1}{n + 1} \{2\delta^i_{(j} G^\gamma_{k)\gamma} + \dot{x}^i G^\gamma_{\gamma jk}\} . \tag{2.6}$$

These entities are known as the projective connection coefficients. They are symmetric in their lower indices and are homogeneous of degree zero in their directional arguments. We define the projective covariant derivative of a vector field $X^i(x, \dot{x})$ with respect to x^k for the projective connection coefficients $\Pi^i_{jk}(x, \dot{x})$ in the following way:

$$X^i_{((k))}(x, \dot{x}) = (\partial_k X^i) - (\dot{\partial}_m X^i) \Pi^m_{pk} \dot{x}^p + X^m \Pi^i_{mk} . \tag{2.7}$$

Substituting (2.6) in (2.7) and using the formula (1.2), we get

(2.8)

$$X^i_{((k))}(x,\dot{x}) = X^i_{(k)} + \frac{1}{n + 1} [(\dot{\partial}_m X^i) \{\delta^m_k G^\gamma_\gamma + \dot{x}^m G^\gamma_{\gamma k}\} - X^m \{2\delta^i_{(k} G^\gamma_{m)\gamma} + \dot{x}^i G^\gamma_{\gamma km}\}] .$$

We may further generalise the formula (2.8) for a tensor of arbitrary rank with respect to x^k for these connection coefficients [6]. Like the Berwald's covariant derivative the projective covariant derivative vanishes for $\dot{x}^i$. The projective covariant derivatives of $l^i(x, \dot{x})$, $F(x, \dot{x})$ and $g_{ij}(x, \dot{x})$ are given by, [6]:

$$l^i_{((k))} = - \frac{1}{n + 1} l^i \left(\frac{1}{F} l_k G^\gamma_\gamma + G^\gamma_{\gamma k} \right), \tag{2.9}$$

$$F_{((k))} = \frac{1}{n + 1} F \left(\frac{1}{F} l_k G^\gamma_\gamma + G^\gamma_{\gamma k} \right), \tag{2.10}$$

and

$$g_{ij((k))} = g_{ij(k)} + \frac{2}{n + 1} \{C_{ijk} G^\gamma_\gamma + g_{ij} G^\gamma_{\gamma k} + g_{k(i} G^\gamma_{j)\gamma} + \dot{x}^m g_{m(i} G^\gamma_{j)\gamma k}\} . \tag{2.11}$$

Using the homogeneity properties of $l^i(x, \dot{x})$, $F(x, \dot{x})$ and $G^i(x, \dot{x})$ and the relation

$$\dot{x}^i_{(k)} = l^i_{(k)} = F_{(k)} = 0\,, \tag{2.12}$$

we obtain the projective covariant derivative of $l_i(x, \dot{x})$ and $G^i(x, \dot{x})$ as follows:

$$l_{i((k))}(x, \dot{x}) = \frac{2}{n+1}\{C_{ijk}G^\gamma_\gamma + g_{ij}G^\gamma_{\gamma k} + g_{k(i}G^\gamma_{j)\gamma} + \dot{x}^m g_{m(i}G^\gamma_{j)\gamma k}\}l^j - - \frac{1}{n+1}\left\{\frac{1}{F}g_{ij}l_k G^\gamma_\gamma - g_{ij}G^\gamma_{\gamma k}\right\} l^j\,, \tag{2.13}$$

and

$$G^i_{((k))}(x, \dot{x}) = G^i_{(k)} + \frac{1}{n+1}\{G^i_k G^\gamma_\gamma + G^i G^\gamma_{\gamma k} - \delta^i_k G^m G^\gamma_{m\gamma} - G^m G^\gamma_{\gamma km}\dot{x}^i\}\,. \tag{2.14}$$

Multiplying (2.13) by $\dot{x}^i$ throughout and using relations

$$C_{ijk}(x, \dot{x})\,\dot{x}^i = C_{ijk}(x, \dot{x})\,\dot{x}^j = C_{ijk}(x, \dot{x})\,\dot{x}^k = 0\,, \tag{2.15a}$$

$$g_{ij}(x, \dot{x})\,\dot{x}^i\dot{x}^j = F^2(x, \dot{x})\,, \tag{2.15b}$$

$$g_{ij(k)}\dot{x}^i = 0\,, \tag{2.15c}$$

and

$$G^i_{hjk}(x, \dot{x})\,\dot{x}^h = 0\,, \tag{2.16}$$

we obtain

$$l_{i((k))}\dot{x}^i = \frac{1}{n+1}[FG^\gamma_{\gamma k} + l_k G^\gamma_\gamma]\,. \tag{2.17}$$

Again, we multiply (2.11) by $\dot{x}^i$ throughout and use the homogeneity property of $G^i(x, \dot{x})$ together with relations (2.15a), (2.15b) (2.15c) and (2.16), we get

$$g_{ij((k))}\dot{x}^i = \frac{1}{n+1}[\dot{x}^i\{2g_{ij}G^\gamma_{\gamma k} + g_{ki}G^\gamma_{\gamma j}\} + g_{kj}G^\gamma_\gamma + F^2 G^\gamma_{j\gamma k}]\,. \tag{2.18}$$

Multiplying (2.18) by $\dot{x}^j$, we easily obtain

$$g_{ij((k))}\dot{x}^i\dot{x}^j = \frac{2}{n+1}\{F^2 G^\gamma_{\gamma k} + g_{kj}\dot{x}^j G^\gamma_\gamma\}\,. \tag{2.19}$$

3. Projective invariant entities in the conformal Finsler space. Since the connection coefficients $\Pi^j_{kl}(x, \dot{x})$ are invariants under the projective change (2.1) so are all the entities deduced in the second section of the paper. For any given geometric object we can find the projective invariant from the formula (2.8) provided the geometric object is taken to be invariant under the projective change. We will now discuss the effect of the conformal transformation over these projective invariant entities:

With the help of equations (1.5), (1.6) and (2.6), the conformal counterpart of the projective connection coefficients $\Pi^i_{jk}(x, \dot{x})$ is given by

(3.1) $$\bar{\Pi}^i_{jk}(x, \dot{x}) = \Pi^i_{jk}(x, \dot{x}) - \sigma_m \left[\dot{\partial}_k \dot{\partial}_j B^{im} - \frac{1}{n+1} \{ 2\delta^i_{(j} \dot{\partial}_{k)} \dot{\partial}_\gamma B^{\gamma m} + \dot{x}^i \dot{\partial}_k \dot{\partial}_j \dot{\partial}_\gamma B^{\gamma m} \} \right].$$

Contracting (3.1) with respect to the indices i and j and using the homogeneity property of $B^{ij}(x, \dot{x})$, we obtain

(3.2) $$\bar{\Pi}^l_{lk}(x, \dot{x}) = \Pi^l_{lk}(x, \dot{x}),$$

i.e. functions $\Pi^l_{lk}(x, \dot{x})$ are invariant under the conformal change (1.1). We have the following theorems:

Theorem 3.1. *When $F_n(x, \dot{x})$ and $\bar{F}_n(x, \dot{x})$ are in conformal correspondence, we have*

(3.3a) $$\bar{F}^2 \bar{l}^i_{((\bar{k}))} + \dot{x}^i \bar{F}_{((\bar{k}))} = 0,$$

and

(3.3b) $$\bar{l}_{i((\bar{k}))} \dot{x}^i = \frac{e^\sigma}{n+1} [l_{i((k))} \dot{x}^i - \sigma_m \{ F \dot{\partial}_k \dot{\partial}_\gamma B^{\gamma m} + l_k \dot{\partial}_\gamma B^{\gamma m} \}],$$

where the notation $((\bar{k}))$ denotes the projective covariant derivative for the connection coefficients $\bar{\Pi}^i_{jk}(x, \dot{x})$ in conformal Finsler space.[4])

Proof. Using equations (1.1c), (1.4), (1.5), (1.7) and (1.8) in (2.10) and (2.11), we obtain

(3.4) $$\bar{l}^i_{((\bar{k}))} = \frac{\bar{e}^\sigma}{n+1} l^i \sigma_m \left\{ \frac{1}{F} l_k \dot{\partial}_\gamma B^{\gamma m} + \dot{\partial}_k \dot{\partial}_\gamma B^{\gamma m} \right\},$$

and

(3.5) $$\bar{F}_{((\bar{k}))} = - \frac{e^\sigma}{n+1} F \sigma_m \left\{ \frac{1}{F} l_k \dot{\partial}_\gamma B^{\gamma m} + \dot{\partial}_k \dot{\partial}_\gamma B^{\gamma m} \right\}.$$

Multiplying (3.4) by $\bar{F}^2(x, \dot{x})$ and (3.5) by $\dot{x}^i$ and adding, we easily get the result (3.3a).

Again, using equations (1.1c), (1.4), (1.5), (1.8) and (1.9) in (2.17), we obtain (3.3b).

Theorem 3.2. *When $F_n(x, \dot{x})$ and $\bar{F}_n(x, \dot{x})$ are in conformal correspondence, we have*

(3.6a) $$\bar{g}_{ij((\bar{k}))} \dot{x}^i = \frac{e^{2\sigma}}{n+1} [g_{ij((k))} \dot{x}^i - \sigma_n \{ \dot{x}^i (g_{ij} \dot{\partial}_k \dot{\partial}_\gamma B^{\gamma n} + 2 g_{i(j} \dot{\partial}_{k)} \dot{\partial}_\gamma B^{\gamma n}) + g_{kj} \dot{\partial}_\gamma B^{\gamma n} + F^2 \dot{\partial}_k \dot{\partial}_\gamma \dot{\partial}_j B^{\gamma n} \}],$$

[4]) We have

$$X^i_{((\bar{k}))} = (\partial_k X^i) - (\dot{\partial}_m X^i) \, \Pi^m_{pk} \, \dot{x}^p + X^m \Pi^i_{mk}.$$

and

$$\bar{g}_{ij((\bar{k}))}\dot{x}^i\dot{x}^j = \frac{2e^{2\sigma}}{n+1}\left[g_{ij((k))}\dot{x}^i\dot{x}^j - \sigma_n\{F\,\dot{\partial}_k\dot{\partial}_\gamma B^{\gamma n} + g_{kj}\dot{x}^j\,\dot{\partial}_\gamma B^{\gamma n}\}\right]. \tag{3.6b}$$

Proof. Under the conformal transformation (1.1), the equation (2.18) reads as follows:

$$\bar{g}_{ij((\bar{k}))}\dot{x}^i = \frac{1}{n+1}\left[\dot{x}^i\{2\bar{g}_{ij}\bar{G}^\gamma_{\gamma k} + \bar{g}_{ki}\bar{G}^\gamma_{\gamma j}\} + \bar{g}_{kj}\bar{G}^\gamma_\gamma + \bar{F}^2\bar{G}^\gamma_{j\gamma k}\right]. \tag{3.7}$$

Using equations (1.1a), (1.1c), (1.4), (1.5), (1.6), (1.9) and the homogeneity property of $G^i(x, \dot{x})$ in (3.7) we get the result (3.6a).

Similarly, we obtain (3.6b) from (2.19).

Theorem 3.3. *We have*

$$\begin{aligned}\bar{G}^i_{((\bar{k}))} = {} & G^i_{((k))} - B^{in}(\dot{\partial}_k\sigma_n) - \sigma_n\Big[\dot{\partial}_k B^{in} + G^i_m\,\dot{\partial}_k B^{mn} - G^m_k\,\dot{\partial}_m B^{in} - G^m\,\dot{\partial}_k\dot{\partial}_m B^{in} + \\ & + B^{mn}G^i_{mk} + \frac{1}{n+1}\{G^i_k\,\dot{\partial}_l B^{ln} + G^l_l\,\dot{\partial}_k B^{in} + G^i\,\dot{\partial}_k\dot{\partial}_l B^{ln} + G^l_{lk}B^{in} - \delta^i_k G^m\,\dot{\partial}_l\dot{\partial}_m B^{ln} - \\ & - \delta^i_k B^{mn}G^l_{ml} - \dot{x}^i(G^m\,\dot{\partial}_m\dot{\partial}_k\dot{\partial}_l B^{ln} + B^{mn}G^l_{lkm})\} + \sigma_\gamma\Big\{\dot{\partial}_m B^{in}\,\dot{\partial}_k B^{m\gamma} - B^{m\gamma}\,\dot{\partial}_k\dot{\partial}_m B^{in} - \\ & - \frac{1}{n+1}(\dot{\partial}_k B^{in}\,\dot{\partial}_l B^{l\gamma} + B^{in}\,\dot{\partial}_k\dot{\partial}_l B^{l\gamma} - \delta^i_k B^{mn}\,\dot{\partial}_m\dot{\partial}_l B^{l\gamma} - B^{mn}\,\dot{\partial}_m\dot{\partial}_k\dot{\partial}_l B^{l\gamma}\dot{x}^i)\Big\}\Big].\end{aligned} \tag{3.8}$$

Proof. The proof follows the pattern of the proof of theorem 3.2.

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