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LAPLACE L_2-TRANSFORM OF DISTRIBUTIONS

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In our paper [11] Hilbert spaces L_2^k, where k ranged in the set of all integers, were defined. It was shown that $\mathscr{S} = \bigcap L_2^k \subset \ldots \subset L_2^1 \subset L_2^0 \subset L_2^{-1} \subset \ldots \subset \bigcup L_2^k = \mathscr{S}'$, where $\mathscr{S}$ is the space of all rapidly decreasing functions together with their derivatives and the dual $\mathscr{S}'$ of $\mathscr{S}$ is the space of tempered distributions (see [1]). Then Fourier transform $\mathscr{F}$, based on the classical definition with the kernel $\exp(-2\pi i\xi, x)$, is a unitary automorphism on every L_2^k. The purpose of this paper is to transfer these results on Laplace transform.

We make use of the following notation. $\mathscr{D}$ is the linear space, over the field C of complex numbers, of all functions $f : R^n \to C$ with compact support $\operatorname{supp} f$ which possess continuous partial derivatives of all orders. The space $\mathscr{D}'$ of distributions is the dual of $\mathscr{D}$, where $\mathscr{D}$ is provided by usual topology, cf. [1]. Finally, we denote $\mathscr{D}'_+ = \{f \in \mathscr{D}'; \operatorname{supp} f \subset \langle 0, \infty)^n\}$.

By α we denote a multiindex $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$, where all α_j are non-negative integers. We write $|\alpha| = \sum \alpha_j$, $x^\alpha = \prod x_j^{\alpha_j}$, for $x \in R^n$, $D^\alpha = \partial^{|\alpha|}/(\partial x)^\alpha$. If there are for example variables $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and D^α acts only on variables $(x_1, \ldots, x_n)$, then we indicate this by D_x^α. If for a given multiindex α and a function $f : R^n \to C$ there is a function $g : R^n \to C$ such that $\int_{R^n} g\varphi \, dx = (-1)^{|\alpha|} \int_{R^n} f D^\alpha \varphi \, dx$ for all $\varphi \in \mathscr{D}$, then we call g the generalized derivative of f of order α and denote it by $D^\alpha f$.

Inequalities $x \leqq y$ and $x < y$, where $x, y \in R^n$, mean that $x_j \leqq y_j$ and, respectively, $x_j < y_j$, $j = 1, \ldots, n$. As the function $\exp(\sigma, x)$, with σ fixed, will frequently appear throughout our paper, we denote it briefly by e_σ.

Let us remind of definition 1 from [11]: For $k \geqq 0$, integer, let $L_2^k = \{f : R^n \to C;$ there exists $D^\beta f$ for every β, $|\beta| \leqq k$, and $\sum_{|\alpha|+|\beta| \leqq k} \int_{R^n} x^{2\alpha} |D^\beta f|^2 \, dx < +\infty\}$. There is defined an inner product in L_2^k by $(f, g)_k = \int_{R^n} [D_k f, D_k g] \, dx$, where

$$D_k = \left(1 + \sum_{j=1}^n (2\pi i x_j + \partial/\partial x_j)\right)^k = \sum_{|\alpha|+|\beta| \leqq k} a_{\alpha\beta} (2\pi i x)^\alpha D^\beta$$

and

$$[D_k f, D_k g] = \sum_{|\alpha|+|\beta| \leqq k} |a_{\alpha\beta}|^2 (2\pi i x)^\alpha D^\beta f \overline{(2\pi i x)^\alpha D^\beta g}.$$

The space L_2^{-k}, $k \geqq 0$, integer, is the dual of L_2^k. If elements $f, g \in L_2^{-k}$ are, according to Fréchet-Riesz theorem, represented by $\varphi, \psi \in L_2^k$, then $(f, g)_{-k} = (\psi, \varphi)_k$ defines an inner product in L_2^{-k}. All L_2^{-k}, k integer, are Hilbert spaces.

Definition 1. For each integer k and each $\gamma \in R^n$ we define the Hilbert space $L_{2,\gamma}^k = \{f \in \mathscr{D}'_+;\ e_{-\gamma} f \in L_2^k\}$, with an inner product $(f, g)_{k,\gamma} = (e_{-\gamma} f, e_{-\gamma} g)_k$, $k = 0, 1, -1, 2, -2, \ldots$

From definition 1 and from [11] it follows immediately that $\ldots \subset L_{2,\gamma}^2 \subset L_{2,\gamma}^1 \subset L_{2,\gamma}^0 \subset L_{2,\gamma}^{-1} \subset L_{2,\gamma}^{-2} \subset \ldots$, and that the identity-operator $\mathscr{I} : L_{2,\gamma}^k \to L_{2,\gamma}^l$, $k \geqq l$, is continuous. If $\sigma \geqq \gamma$, $\sigma, \gamma \in R^n$, then $L_{2,\gamma}^k \subset L_{2,\sigma}^k$, $k = 0, 1, -1, 2, -2, \ldots$, and again the identity-operator $\mathscr{I} : L_{2,\gamma}^k \to L_{2,\sigma}^k$ is continuous. Let us show that also the operator $\partial/\partial x_1 : L_{2,\gamma}^k \to L_{2,\gamma}^{k-1}$ is continuous.

First we prove the inclusion $(\partial/\partial x_1)(L_{2,\gamma}^k) \subset L_{2,\gamma}^{k-1}$. Actually, for $f \in L_{2,\gamma}^k$ we have $(\partial/\partial x_1) f = (\partial/\partial x_1)(e_\gamma e_{-\gamma} f) = ((\partial/\partial x_1) e_\gamma) e_{-\gamma} f + e_\gamma (\partial/\partial x_1)(e_{-\gamma} f) = \gamma_1 f + e_\gamma (\partial/\partial x_1)(e_{-\gamma} f)$. Evidently $\gamma_1 f \in L_{2,\gamma}^k$ and $(\partial/\partial x_1)(e_{-\gamma} f) \in L_2^{k-1}$. Moreover, $\operatorname{supp} (\partial/\partial x_1)(e_{-\gamma} f) \subset \langle 0, \infty)^n$. This implies that $\operatorname{supp} e_\gamma (\partial/\partial x_1)(e_{-\gamma} f) \subset \langle 0, \infty)^n$ which proves $e_\gamma (\partial/\partial x_1)(e_{-\gamma} f) \in L_{2,\gamma}^{k-1}$ and hence $(\partial/\partial x_1) f \in L_{2,\gamma}^{k-1}$. The continuity of $\partial/\partial x_1 : L_{2,\gamma}^k \to L_{2,\gamma}^{k-1}$ follows then from the continuity of $\partial/\partial x_1 : L_2^k \to L_2^{k-1}$.

In [11], $\mathscr{O}_{p,q}$, $p \geqq q \geqq 0$, signified the normed space of all such functions $\varphi : R^n \to C$ that the mapping $\psi \to \varphi\psi$ maps continuously L_2^{p-k} into L_2^{q-k} for $k = 0, 1, \ldots, q$. The norm was given by $\|\varphi\|_{p,q} = \max_{k=0,1,\ldots,q} \sup_{\|\psi\|_{p-k}=1} \|\varphi\psi\|_{q-k}$.

Then it was shown that $L_2^{q+r} \subset \mathscr{O}_{p,q} \subset L_2^{q-p-r}$, where $r = 1 + [\frac{1}{2}n]$, and that the identity-operators $\mathscr{I} : L_2^{q+r} \to \mathscr{O}_{p,q}$, $\mathscr{I} : \mathscr{O}_{p,q} \to L_2^{q-p-r}$ are continuous. Every polynomial of degree k is an element of $\mathscr{O}_{q+k,q}$, $q \geqq 0$. For $\varphi \in \mathscr{O}_{p,q}$, $f \in L_2^{-q}$, $p \geqq q \geqq 0$, we have defined the product $\varphi f \in L_2^{-p}$ by $(\varphi f)\,\psi = f(\varphi\psi)$, $\psi \in L_2^p$. Then evidently $\|\varphi f\|_{-p} \leqq \|\varphi\|_{p,q} \|f\|_{-q}$. Hence the mapping $(\varphi, f) \to \varphi f$ of $\mathscr{O}_{p,q} \times L_2^{-q}$ into L_2^{-p} is hypocontinuous (i.e. continuous in each variable locally uniformly with respect to the other one). This definition is in accordance with the classical Schwartz's definition of multiplication. Now, we are able to define the product φf, where $\varphi \in \mathscr{O}_{p,q}$, $f \in L_{2,\gamma}^{-q}$, as an element of $L_{2,\gamma}^{-p}$ by $\varphi f = e_\gamma(\varphi e_{-\gamma} f)$. The mapping $(\varphi, f) \to \varphi f$ is then hypocontinuous. In particular, the mapping $f \to (-x)^\alpha f$ of $L_{2,\gamma}^{-q}$ into $L_{2,\gamma}^{-q-|\alpha|}$ is continuous.

For each real number $c > 0$ we define the homogeneity operator $\mathscr{H}_c : L_2^k \to L_2^k$ as follows: Let $f \in L_2^k$. If $k \geqq 0$, then $(\mathscr{H}_c f)(x) = f(x/c)$, $x \in R^n$. If $k \leqq 0$, then $(\mathscr{H}_c f)(\varphi) = c^n f(\mathscr{H}_{1/c}\varphi)$, where $\varphi \in L_2^{-k}$.

Proposition. *Given $c > 0$, a multiindex α, and integers k, p, q, $p \geqq q \geqq 0$. Then*

1) *$\mathscr{H}_c$ is a homeomorphic automorphism of L_2^k for which $\|\mathscr{H}_c\| \leqq (c + 1/c)\, c^{\frac{1}{2}n}$, $\mathscr{H}_c^{-1} = \mathscr{H}_{1/c}$.*

2) *$\mathscr{H}_c \mathscr{F} = c^n \mathscr{F} \mathscr{H}_{1/c}$, where $\mathscr{F}$ is Fourier operator.*

3) *$\mathscr{H}_c D^\alpha = c^{|\alpha|} D^\alpha \mathscr{H}_c$.*

4) *For $\varphi \in \mathscr{O}_{p,q}$, $f \in L_2^{-q}$ we have $\mathscr{H}_c(\varphi f) = (\mathscr{H}_c \varphi)(\mathscr{H}_c f)$.*

Definition 2. Given $\gamma \in R^n$ and an integer k. Then we define Laplace transform $\mathscr{L}$ as a mapping of $L_{2,\gamma}^k$ into the space of all mappings of the set $\{\sigma \in R^n;\ \sigma \geqq \gamma\}$ into L_2^k by the formula

$$(1) \qquad (\mathscr{L}f)_\sigma = \mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{-\sigma}f)\,, \quad f \in L_{2,\gamma}^k\,, \quad \sigma \geqq \gamma\,.$$

Lemma 1. *Given $\gamma, \lambda \in R^n$, an integer k, a multiindex α and $f \in L_{2,\gamma}^k$. Then the following formulae are valid*:

$$(2) \qquad \mathscr{L}((-ix)^\alpha f)_\sigma = \mathrm{D}^\alpha(\mathscr{L}f)_\sigma\,, \qquad \textit{where} \quad \sigma \geqq \gamma\,,$$

$$(3) \qquad \mathscr{L}(\mathrm{e}_\lambda \mathrm{D}^\alpha f)_\sigma = (\sigma - \lambda + i\tau)^\alpha \mathscr{L}(\mathrm{e}_\lambda f)_\sigma\,, \quad \textit{where} \quad \sigma \geqq \gamma + \lambda\,.$$

In particular, for $\lambda = 0$ we get

$$(3a) \qquad \mathscr{L}(\mathrm{D}^\alpha f)_\sigma = (\sigma + i\tau)^\alpha (\mathscr{L}f)_\sigma\,.$$

Proof. 1) $\mathscr{L}((-ix)^\alpha f)_\sigma = \mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{-\sigma}(-ix)^\alpha f) = (2\pi)^n \mathscr{F}\mathscr{H}_{1/2\pi}((-ix)^\alpha \mathrm{e}_{-\sigma}f) = (2\pi)^n \mathscr{F}((-2\pi ix)^\alpha \mathscr{H}_{1/2\pi}(\mathrm{e}_{-\sigma}f)) = (2\pi)^n \mathrm{D}^\alpha \mathscr{F}\mathscr{H}_{1/2\pi}(\mathrm{e}_{-\sigma}f) = \mathrm{D}^\alpha \mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{-\sigma}f) = \mathrm{D}^\alpha(\mathscr{L}f)_\sigma$.

2) $\mathscr{L}(\mathrm{e}_\lambda(\partial f/\partial x_1))_\sigma - (\sigma_1 - \lambda_1)\mathscr{L}(\mathrm{e}_\lambda f)_\sigma = \mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{\lambda-\sigma}(\partial f/\partial x_1) - (\sigma_1 - \lambda_1)\mathrm{e}_{\lambda-\sigma}f) = \mathscr{H}_{2\pi}\mathscr{F}(\partial/\partial x_1)(\mathrm{e}_{\lambda-\sigma}f) = \mathscr{H}_{2\pi}(2\pi i\tau_1)\mathscr{F}(\mathrm{e}_{\lambda-\sigma}f) = i\tau_1 \mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{\lambda-\sigma}f) = i\tau_1 \mathscr{L}(\mathrm{e}_\lambda f)_\sigma$. The mathematical induction completes the proof.

Theorem 1. *Given $\gamma \in R^n$, an integer k and $f \in L_{2,\gamma}^k$. Then for every $\sigma > \gamma$ Laplace image $(\mathscr{L}f)_\sigma$ is a function. Moreover, if we denote the variable of $(\mathscr{L}f)_\sigma$ by τ, then $(\mathscr{L}f)_\sigma(\tau)$ is a holomorphic function of variable $\sigma + i\tau$ on the set $\{\sigma + i\tau \in C^n;\ \sigma > \gamma\}$. Therefore we will further write $(\mathscr{L}f)(\sigma + i\tau)$ instead of $(\mathscr{L}f)_\sigma(\tau)$.*

If $k \leqq 0$, then for each pair of multiindices α, β, $|\alpha| + |\beta| \leqq -k$, there exists a polynomial $P_{\alpha\beta}$ of degree $\leqq |\beta|$ and $g_\beta \in L_{2,\gamma}^0$ such that

$$(4) \qquad (\mathscr{L}f)(u) = \sum_{|\alpha|+|\beta| \leqq -k} P_{\alpha\beta}(u)\,\mathrm{D}_u^\alpha(\mathscr{L}g_\beta)(u)\,, \quad \operatorname{Re} u > \gamma\,.$$

Remark. We shall see in the proof that if we differentiate with respect to $(i \operatorname{Im} u)$ instead of u on the right-hand side of (4), then (4) is valid on the set $\{u \in C^n;\ \operatorname{Re} u \geqq \gamma\}$.

Having already known that $\mathscr{L}f$, $f \in L_{2,\gamma}^k$, is holomorphic for $\operatorname{Re} u > \gamma$ we might replace the operator D_τ by D_u in (2). In this way we would get

$$(2a) \qquad \mathscr{L}((-x)^\alpha f)(u) = (\mathrm{D}_u^\alpha \mathscr{L}f)(u)\,, \quad \operatorname{Re} u > \gamma\,.$$

Proof of Theorem 1. We first prove the second part of Theorem 1, then the first one. Put $m = -k$. According to Theorem 2 of [11] for each multiindex β, $|\beta| \leqq m$, there are $h_\beta \in L_2$ and a polynomial Q_β of degree $\leqq m - |\beta|$ such that $\mathrm{e}_{-\gamma}f = \sum_{|\beta| \leqq m} Q_\beta(-ix)\,\mathrm{D}^\beta h_\beta$. As $\operatorname{supp} \mathrm{e}_{-\gamma}f \subset \langle 0, \infty)^n$, we may suppose that $\operatorname{supp} h_\beta \subset$

$\subset \langle 0, \infty)^n$ for every β, $|\beta| \leqq m$. Hence, using (2), (3), $u = \sigma + i\tau$, we get for $\sigma \geqq \gamma$,

$$(5)\qquad (\mathscr{L}f)_\sigma = \mathscr{L}(\mathrm{e}_\gamma \mathrm{e}_{-\gamma} f)_\sigma = \mathscr{L}(\mathrm{e}_\gamma \sum_{|\beta| \leqq m} Q_\beta(-ix)\, \mathrm{D}^\beta h_\beta)_\sigma =$$

$$= \sum_{|\beta| \leqq m} \mathscr{L}(Q_\beta(-ix)\, \mathrm{e}_\gamma\, \mathrm{D}_x^\beta h_\beta)_\sigma = \sum_{|\beta| \leqq m} Q_\beta(\mathrm{D}_\tau)\, \mathscr{L}(\mathrm{e}_\gamma\, \mathrm{D}_x^\beta h_\beta)_\sigma =$$

$$= \sum_{|\beta| \leqq m} Q_\beta(\mathrm{D}_\tau)\, ((\sigma - \gamma + i\tau)^\beta\, \mathscr{L}(\mathrm{e}_\gamma h_\beta)_\sigma)\,.$$

If we carry out the indicated differentiation we can write

$$(\mathscr{L}f)_\sigma = \sum_{|\alpha| + |\beta| \leqq m} P_{\alpha\beta}(u)\, \mathrm{D}_u^\alpha \mathscr{L}(\mathrm{e}_\gamma h_\beta)_\sigma,$$

where $P_{\alpha\beta}$ is a polynomial of degree $\leqq |\beta| - (m - |\beta| - |\alpha|) \leqq |\beta|$. It remains to put $g_\beta = \mathrm{e}_\gamma h_\beta$ which is evidently an element of $\overset{0}{L}_{2,\gamma}$ for every β, $|\beta| \leqq m$.

To prove the first part of Theorem 1 it suffices to show that $(\mathscr{L}g_\beta)_\sigma$ is holomorphic on $\{u;\ \mathrm{Re}\, u > \gamma\}$. In fact, $(\mathscr{L}g_\beta)_\sigma = \mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{-\sigma} g_\beta) = \int_{R^n} \exp(-(\sigma + i\tau), x)\, g_\beta(x)\, \mathrm{d}x = \int_{R^n} \exp(\gamma - u, x)\, h_\beta(x)\, \mathrm{d}x$.

Let us take $A > 0$, then evidently $H_A(u) = \int_{\langle 0, A\rangle^n} \exp(\gamma - u, x)\, h_\beta(x)\, \mathrm{d}x$ is an entire function of u. Put for brevity $M = \langle 0, \infty)^n - \langle 0, A\rangle^n$. Then for each $\varepsilon \in (0, \infty)^n$, $\mathrm{Re}\, u \geqq \gamma + \varepsilon$, we have

$$\left(\int_M |\exp(\gamma - u, x)\, h_\beta(x)|\, \mathrm{d}x\right)^2 \leqq$$

$$\leqq \int_M |\exp(\gamma - u, x)|^2\, \mathrm{d}x \int_M |h_\beta|^2\, \mathrm{d}x \leqq \|h\|_{L_2}^2 \prod_{s=1}^n \frac{1}{2\varepsilon_s} \exp(-2A\varepsilon_s)\,.$$

Thus $\lim_{A \to \infty} H_A(u) = 0$ uniformly on $\{u;\ \mathrm{Re}\, u \geqq \gamma + \varepsilon\}$. According to well-known Weierstrass theorem the limit-function $(\mathscr{L}g_\beta)_\sigma$ is holomorphic on $\{u;\ \mathrm{Re}\, u > \gamma\}$. The proof is complete.

Corollary. *It follows from the proof of Theorem* 1 *that there exists a function* $\varphi(u)$, *holomorphic on* $\{u;\ \mathrm{Re}\, u > \gamma\}$ *and bounded on* $\{u;\ \mathrm{Re}\, u \geqq \gamma + \varepsilon\}$ *for every* $\varepsilon \in (0, \infty)^n$, *such that*

$$(6)\qquad (\mathscr{L}f)(u) = \varphi(u) \prod_{j=1}^n (u_j - \gamma_j)^m\,, \quad \mathrm{Re}\, u > \gamma\,.$$

Hence if we write $\mathscr{L}f = F$, then the integral

$$g(x) = \int_{R^n} \frac{F(\sigma + i\tau)}{\prod_{j=1}^n (\sigma_j + i\tau_j - \gamma_j)^{m+2}} \exp(\sigma + i\tau, x)\, \mathrm{d}\tau\,,$$

is absolutely convergent for every $\sigma > \gamma$, does not depend on σ and is equal to zero for $x \notin \langle 0, \infty)^n$, see [10], Lemma 1.

Evidently g is a distribution from $\bigcap_{\sigma>\gamma} L^0_{2,\sigma}$ and can be differentiated. Let us write

$$\prod_{j=1}^{n}\left(\frac{\partial}{\partial x_j} - \gamma_j\right)^{m+2} (2\pi)^{-n} g =$$

$$= \prod_{j=1}^{n}\left(\frac{\partial}{\partial x_j} - \gamma_j\right)^{m+2} \mathrm{e}_\sigma \mathscr{F}^{-1}(F(\sigma + 2\pi i\tau) \prod_{j=1}^{n} (\sigma_j + 2\pi i\tau_j - \gamma_j)^{-m-2}) =$$

$$= \mathrm{e}_\sigma \mathscr{F}^{-1}(F(\sigma + 2\pi i\tau)) = \mathrm{e}_\sigma \mathscr{F}^{-1} \mathscr{H}_{1/2\pi} F(\sigma + i\tau) =$$

$$= \mathrm{e}_\sigma \mathscr{F}^{-1} \mathscr{F}(\mathrm{e}_{-\sigma} f) = \mathrm{e}_\sigma \mathrm{e}_{-\sigma} f = f .$$

We have got an inversion formula

Theorem 2. *Given $\gamma \in R^n$, a non-negative integer m and $f \in L^{-m}_{2,\gamma}$. Let $F = \mathscr{L}f$. Then for every $\sigma > \gamma$ we have*

$$(7) \quad f = (2\pi i)^{-n} \prod_{j=1}^{n}\left(\frac{\partial}{\partial x_j} - \gamma_j\right)^{m+2} \int_{\sigma + iR^n} F(u) \prod_{j=1}^{n} (u_j - \gamma_j)^{-m-2} \exp(u, x) \, \mathrm{d}u ,$$

where the indicated differentiation is in the sense of distribution theory.

Definition 3. Given $\gamma \in R^n$ and an integer $k \geqq 0$. Then we put $H^k_{2,\gamma} = \{F;\ F$ is holomorphic for $\operatorname{Re} u > \gamma$ and $\sup_{\sigma>\gamma} \sum_{|\alpha|+|\beta|\leqq k} \int_{R^n} |(\sigma + i\tau)^\alpha|^2 \,.\, |\mathrm{D}^\beta F(\sigma + i\tau)|^2 \, \mathrm{d}\tau < +\infty\}$.

Lemma 2. *Let $\gamma \in R^n$ and $k \geqq 0$ be an integer. Then*

1) *Laplace transform $\mathscr{L}$ is an isomorphism of $L^k_{2,\gamma}$ onto $H^k_{2,\gamma}$.*

2) *For each $F \in H^k_{2,\gamma}$ and each multiindex β, $|\beta| \leqq k$, there exists* $\lim_{\sigma\to\gamma+} \mathrm{D}^\beta_\tau F(\sigma + i\tau)$ *in the topology of L_2 (with respect to the variable τ) and this limit is the generalized derivative of order β of* $\lim_{\sigma\to\gamma+} F(\sigma + i\tau)$. *We denote* $\lim_{\sigma\to\gamma+} F(\sigma + i\tau) = F(\gamma + i\tau)$.

3) *For each α, β, $|\alpha| + |\beta| \leqq k$, the function $\int_{R^n} |(\sigma + i\tau)^\alpha \mathrm{D}^\beta_\tau F(\sigma + i\tau)|^2 \, \mathrm{d}\tau$ is non-increasing in all variables on the set $\{\sigma \in R^n;\ \sigma \geqq \gamma\}$. In particular,*

$$\sup_{\sigma>\gamma} \int_{R^n} |(\sigma + i\tau)^\alpha \mathrm{D}^\beta_\tau F(\sigma + i\tau)|^2 \, \mathrm{d}\tau = \int_{R^n} |(\sigma + i\tau)^\alpha \mathrm{D}^\beta_\tau F(\gamma + i\tau)|^2 \, \mathrm{d}\tau .$$

4) *$H^k_{2,\gamma}$ is Hilbert space with an inner product*

$$(8) \qquad (F, G)_{H^k_{2,\gamma}} = \left(\frac{1}{2\pi}\right)^n \int_{R^n} [\tilde{\mathrm{D}}_k F(\gamma + i\tau), \tilde{\mathrm{D}}_k G(\gamma + i\tau)] \, \mathrm{d}\tau ,$$

where

$$\tilde{\mathrm{D}}_k = \left(1 + \sum_{j=1}^{n} i\tau_j + 2\pi \frac{\partial}{\partial \tau_j}\right)^k = \sum_{|\alpha| + |\beta| \leqq k} a_{\alpha\beta}(i\tau)^\alpha \mathrm{D}_\tau^\beta$$

and

$$[\tilde{\mathrm{D}}_k F(\gamma + i\tau), \tilde{\mathrm{D}}_k G(\gamma + i\tau)] =$$

$$= \sum_{|\alpha| + |\beta| \leqq k} |a_{\alpha\beta}|^2 (i\tau)^\alpha \mathrm{D}_\tau^\beta F(\gamma + i\tau) \overline{(i\tau)^\alpha \mathrm{D}_\tau^\beta G(\gamma + i\tau)} .$$

5) *$\mathscr{L}$ is a unitary mapping of $L_{2,\gamma}^k$ onto $H_{2,\gamma}^k$.*

Proof. Points 1, 2, 3 follow immediately from [10], Lemma 6, point 4 is evident. We prove point 5 with the help of [11], Theorem 1.

Compute at first

$$\mathrm{D}\mathscr{H}_{1/2\pi} = \left(1 + \sum\left(2\pi i\tau_j + \frac{\partial}{\partial \tau_j}\right)\right)\mathscr{H}_{1/2\pi} =$$

$$= \mathscr{H}_{1/2\pi}\left(1 + \sum\left(i\tau_j + 2\pi\frac{\partial}{\partial \tau_j}\right)\right) = \mathscr{H}_{1/2\pi}\tilde{\mathrm{D}} .$$

Now for $f, g \in L_{2,\gamma}^k$, we have

$$(f, g)_{L^k{}_{2,\gamma}} = (\mathrm{e}_{-\gamma}f, \mathrm{e}_{-\gamma}g)_k = (\mathscr{F}(\mathrm{e}_{-\gamma}f), \mathscr{F}(\mathrm{e}_{-\gamma}g))_k =$$

$$= (\mathscr{H}_{1/2\pi}\mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{-\gamma}f), \mathscr{H}_{1/2\pi}\mathscr{H}_{2\pi}\mathscr{F}(\mathrm{e}_{-\gamma}g))_k = (\mathscr{H}_{1/2\pi}\mathscr{L}f, \mathscr{H}_{1/2\pi}\mathscr{L}g)_k =$$

$$= \int_{R^n} [\mathrm{D}_k\mathscr{H}_{1/2\pi}\mathscr{L}f, \mathrm{D}_k\mathscr{H}_{1/2\pi}\mathscr{L}g]\,\mathrm{d}\tau = \int_{R^n} [\mathscr{H}_{1/2\pi}\tilde{\mathrm{D}}_k\mathscr{L}f, \mathscr{H}_{1/2\pi}\tilde{\mathrm{D}}_k\mathscr{L}g]\,\mathrm{d}\tau =$$

$$= (2\pi)^{-n}\int_{R^n} [\tilde{\mathrm{D}}_k\mathscr{L}f, \tilde{\mathrm{D}}_k\mathscr{L}g]\,\mathrm{d}\tau = (\mathscr{L}f, \mathscr{L}g)_{H^k{}_{2,\gamma}} .$$

Definition 4. Given $\gamma \in R^n$ and an integer $k > 0$. Then we denote $H_{2,\gamma}^{-k}$ the linear hull of all functions $u^\alpha \mathrm{D}_u^\beta \Phi$, $|\alpha| + |\beta| \leqq k$, where $\Phi \in H_{2,\gamma}^0$. For $F = u^\alpha \mathrm{D}_u^\beta \Phi$, $G = u^\varkappa \mathrm{D}_u^\lambda \psi$, $\Phi, \psi \in H_{2,\gamma}^0$, $|\alpha| + |\beta| \leqq k$, $|\varkappa| + |\lambda| \leqq k$, we define the inner product

$$(F, G)_{^{-k}H_{2,\gamma}} = (\mathrm{D}^\alpha((-x)^\beta \mathscr{L}^{-1}\Phi), \quad \mathrm{D}^\varkappa((-x)^\lambda \mathscr{L}^{-1}\Psi))_{L^{-k}{}_{2,\gamma}} . \tag{9}$$

Theorem 3. *Laplace transform $\mathscr{L} : L_{2,\gamma}^k \to H_{2,\gamma}^k$ is a unitary isomorphism for each integer k.*

Proof. The case $k \geqq 0$ has been already proved in Lemma 2. Further let $k < 0$. As it follows from Theorem 1 Laplace transform $\mathscr{L}$ is a one-to-one mapping of $L_{2,\gamma}^k$ onto $H_{2,\gamma}^k$. The linearity of $\mathscr{L}$ is evident. The inner product (9) was defined so that $\mathscr{L}$

turns out to be unitary. Indeed, let $F, G \in H^k_{2,\gamma}$, $F = \sum_{|\alpha|+|\beta|\leqq -k} a_{\alpha\beta} u^\alpha D^\beta_u \Phi_{\alpha\beta}$, $G = \sum_{|\varkappa|+|\lambda|\leqq -k} b_{\varkappa\lambda} u^\varkappa D^\lambda_u \psi_{\varkappa\lambda}$, $\Phi_{\alpha\beta}, \psi_{\varkappa\lambda} \in H^0_{2,\gamma}$. Then

$$(F, G)_{H_{2,\gamma}{}^k} =$$
$$= \sum_{|\alpha|+|\beta|\leqq -k} a_{\alpha\beta} \sum_{|\varkappa|+|\lambda|\leqq -k} \bar{b}_{\varkappa\lambda} (D^\alpha((-x)^\beta \mathscr{L}^{-1}\Phi_{\alpha\beta}), D^\varkappa((-x)^\lambda \mathscr{L}^{-1}\psi_{\varkappa\lambda}))_{L^k{}_{2,\gamma}} =$$
$$= \sum_{|\alpha|+|\beta|\leqq -k} a_{\alpha\beta} \sum_{|\varkappa|+|\lambda|\leqq -k} \bar{b}_{\varkappa\lambda} (\mathscr{L}^{-1}(u^\alpha D^\beta \Phi_{\alpha\beta}), \mathscr{L}^{-1}(u^\varkappa D^\lambda \psi_{\varkappa\lambda}))_{L^k{}_{2,\gamma}} =$$
$$= (\mathscr{L}^{-1}F, \mathscr{L}^{-1}G)_{L^k{}_{2,\gamma}}.$$

Corollary. *$H^k_{2,\gamma}$ is the image of a unitary mapping of Hilbert space $L^k_{2,\gamma}$. Hence it is also Hilbert space.*

Definition 5. Given $\gamma \in R^n$ and integers $p, q, p \geqq q \geqq 0$. Then we define a normed space $\mathcal{O}^*_{p,q,\gamma} = \{f \in \mathscr{D}'_+;$ there exists $g \in \mathcal{O}_{p,q}$ such that $e_{-\gamma} f = \mathscr{F} g\}$. For $f \in \mathcal{O}^*_{p,q,\gamma}$ we put $\|f\|^*_{p,q,\gamma} = \|\mathscr{F}^{-1} e_{-\gamma} f\|_{p,q}$.

Remark. It was shown in [11] that $\mathcal{O}^*_{p,q,\gamma} \subset L^{q-p-r}_{2,\gamma}$ and that the identity-operator $\mathscr{I} : \mathcal{O}^*_{p,q,\gamma} \to L^{q-p-r}_{2,\gamma}$ is continuous.

Lemma 3. *Given integers $p, q, p \geqq q \geqq 0$, $f \in \mathscr{D}'_+ \cap \mathscr{F}(\mathcal{O}_{p,q})$ and $g \in \mathscr{D}'_+ \cap L^{-q}_2$. Then $f * g \in \mathscr{D}'_+$. (The convolution $(.*.)$ was defined in [11] as a mapping from $\mathscr{F}(\mathcal{O}_{p,q}) \times L^{-q}_2$ into L^{-p}_2 by $f * g = \mathscr{F}^{-1}(\mathscr{F}f . \mathscr{F}g)$).*

Proof. We show at first that $\operatorname{supp} \mathscr{F}^2 g \subset (-\infty, 0\rangle^n$. Take $\varphi \in L^q_2$ for which $\operatorname{supp} \varphi \cap (-\infty, 0\rangle^n = \emptyset$. Then $\operatorname{supp} \mathscr{F}^2 \varphi \cap \langle 0, \infty)^n = \operatorname{supp} \varphi(-x) \cap \langle 0, \infty)^n = \emptyset$. Hence $(\mathscr{F}^2 g)\, \varphi = g(\mathscr{F}^2 \varphi) = 0$.

As $\mathscr{S}$ is dense in L^p_2, it suffices to prove that $(f * g)\, \varphi = 0$ for every $\varphi \in \mathscr{S}$ fulfilling $\operatorname{supp} \varphi \cap \langle 0, \infty)^n = \emptyset$. We have $(f * g)\, \varphi = \mathscr{F}^2 g(\mathscr{F}^{-2} \varphi * f)$. Hence it suffices to show that $\operatorname{supp} (\mathscr{F}^{-2}\varphi * f) \cap (-\infty, 0\rangle^n = \emptyset$.

Actually, $(\mathscr{F}^{-2}\varphi * f) \in L^{q-p-r}_2$. Take $\psi \in L^{p+r-q}_2$ for which $\operatorname{supp} \psi \subset (-\infty, 0\rangle^n$. Then $(\mathscr{F}^{-2}\varphi * f)\, \psi = (\mathscr{F}^2 f)\,(\mathscr{F}^{-2}\varphi * \mathscr{F}^2 \psi) = 0$. We have received the last equality from the following equality

$$\operatorname{supp} (\mathscr{F}^{-2}\varphi * \mathscr{F}^{-2}\psi) \cap (-\infty, 0\rangle^n = \operatorname{supp} \left(\int_{R^n} \varphi(y - x)\, \psi(-y)\, dy\right) \cap (-\infty, 0\rangle^n = \emptyset.$$

Remark. It follows from Lemma 3 that $f \in \mathcal{O}^*_{p,q,\gamma}$, $g \in L^{-q}_{2,\gamma}$ implies $(e_{-\gamma} f * e_{-\gamma} g) \in \mathscr{D}'_+$. This enables us to state

Definition 6. *Let $\gamma \in R^n$, integers $p, q, p \geqq q \geqq 0$, $f \in \mathcal{O}^*_{p,q,\gamma}$ and $g \in L^{-q}_{2,\gamma}$ be given. Then we define the convolution $f * g$ as an element of $L^{-p}_{2,\gamma}$ by*

$$f * g = e_\gamma(e_{-\gamma} f * e_{-\gamma} g). \tag{10}$$

Remark. For $f \in \mathcal{O}^*_{p,q,\gamma}$, $g \in L^{-q}_{2,\gamma}$ we have $\|f * g\|_{L^{-p}{}_{2,\gamma}} = \|e_{-\gamma}(f * g)\|_{-p} = \|e_{-\gamma}f * e_{-\gamma}g\|_{-p} \leqq \|\mathcal{F}(e_{-\gamma}f)\|_{p,q} \|e_{-\gamma}g\|_{-q} = \|f\|^*_{p,q,\gamma} \|g\|_{L^{-q}{}_{2,\gamma}}$. Thus the mapping $(f, g) \to f * g$ of $\mathcal{O}^*_{p,q,\gamma} \times L^{-q}_{2,\gamma}$ into $L^{-p}_{2,\gamma}$ is hypocontinuous.

Lemma 4. *Let $\gamma \in R^n$, integers p, q, $p \geqq q \geqq 0$, $f \in \mathcal{O}^*_{p,q,\gamma}$ and $g \in L^{-q}_{2,\gamma}$ be given. Then for every $\sigma \geqq \gamma$ the equality*

$$e_\sigma(e_{-\sigma}f * e_{-\sigma}g) = e_\gamma(e_{-\gamma}f * e_{-\gamma}g) \tag{11}$$

holds.

Proof. Put $F = e_{-\gamma}f \in \mathcal{F}(\mathcal{O}_{p,q}) \cap \mathcal{D}'_+$, $G = e_{-\gamma}g \in L^{-q}_2 \cap \mathcal{D}'_+$. We have to show that $(e_{\gamma-\sigma}F) * (e_{\gamma-\sigma}G) = e_{\gamma-\sigma}(F * G)$ holds for every $\sigma \geqq \gamma$. According to the hypocontinuity of convolution and density of $\mathcal{S}$ in L^{-q}_2 we may assume that $G \in \mathcal{S} \cap \mathcal{D}'_+$. Take $\varphi \in \mathcal{D}$. Then

$$(e_{\gamma-\sigma}F * e_{\gamma-\sigma}G)\,\varphi = (e_{\gamma-\sigma}F)\,(\mathcal{F}^2(e_{\gamma-\sigma}G) * \varphi) = F(e_{\gamma-\sigma}(\mathcal{F}^2(e_{\gamma-\sigma}G) * \varphi)),$$
$$e_{\gamma-\sigma}(F * G)\,\varphi = F(\mathcal{F}^2 G * e_{\gamma-\sigma}\varphi).$$

However,

$$e_{\gamma-\sigma}(\mathcal{F}^2(e_{\gamma-\sigma}G) * \varphi)(x) = e_{\gamma-\sigma}(x) \int_{R^n} e_{\gamma-\sigma}(y - x)\, G(y - x)\, \varphi(y)\, dy =$$
$$= \int_{R^n} G(y - x)\, e_{\gamma-\sigma}(y)\, \varphi(y)\, dy = (\mathcal{F}^2 G * e_{\gamma-\sigma}\varphi)(x).$$

As $\mathcal{D}$ is dense in L^p_2, the proof is complete.

Theorem 4. *Given $f \in \mathcal{O}^*_{p,q,\gamma}$ and $g \in L^{-q}_{2,\gamma}$, $p \geqq q \geqq 0$, integers, $\gamma \in R^n$. Then*

$$\mathcal{L}(f * g)(u) = \mathcal{L}f(u) \,.\, \mathcal{L}g(u), \quad \operatorname{Re} u > \gamma. \tag{12}$$

Proof. Take $\sigma > \gamma$. Then according to (11) we may write

$$\mathcal{L}(f * g)(\sigma + 2\pi i\tau) = \mathcal{F}(e_{-\sigma}(f * g))(\tau) = \mathcal{F}(e_{-\sigma}f * e_{-\sigma}g)(\tau) =$$
$$= \mathcal{F}(e_{-\sigma}f)(\tau)\, \mathcal{F}(e_{-\sigma}g)(\tau) = \mathcal{L}f(\sigma + 2\pi i\tau)\, \mathcal{L}g(\sigma + 2\pi i\tau).$$

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