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THE REPRESENTATION OF CARATHÉODORY OPERATORS

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The concept of Carathéodory operator was established in [1] and it was shown that the theory of Carathéodory differential equations can be built up in a similar manner to the classical one, if the right-hand side of differential equations contains a Carathéodory operator instead of functions which fulfil Carathéodory's conditions. In connection with this, the problem was posed (see [1]) whether every Carathéodory operator can be expressed by means of a function fulfilling Carathéodory's conditions. This problem was solved in the affirmative for linear Carathéodory operators in [1]. We shall deal with this problem for general Carathéodory operators in this article. Using the concept of Nemyckij operator [2] we can put the problem in another wording: Is every Carathéodory operator equivalent to some Nemyckij operator?

Definition and basic concepts. Let E_n denote n-dimensional Euclidean space with the norm $|x| = \max_i |x_i|$. Let G be a region in E_n. Let μ be a regular nonnegative measure defined on $\mathscr{F}$, where $\mathscr{F}$ is a σ-field of subsets of the closed interval $I = \langle 0, 1\rangle$ which contains all Borel subsets of I. In the following we shall use only measures fulfilling these conditions. Let $\mathscr{F}^*$ be the completion of $\mathscr{F}$ for μ. We say that a set or a function are μ-measurable it they are $\mathscr{F}^*$-measurable.

Let A be a closed subset of I. Denote by C_A the space of all continuous n-dimensional vector functions $f(t) \equiv [f_1(t), \ldots, f_n(t)]$ defined on A such that $f(t) \in G$ for all $t \in A$. We shall use the norm $\|f\|_A^C = \sup_{t\in A} |f(t)|$.

Let S_A be the set of all m-dimensional vector functions $f(t)$ which are μ-measurable on A. We shall use the quasi-norm $\|f\|_A^S = \int_A \min(1, |f(t)|)\, d\mu$. Denote by $[f]$ the class of all equivalent functions with respect to this quasi-norm which contains f. The symbol $[S]_A$ will be used for the space of the classes of equivalent functions from S_A. $[S]_A$ is an F-space. For $A = I$ we denote $C_A = C$ and $S_A = S$ ($[S_A] = [S]$), $\|\cdot\|_A^C = \|\cdot\|^C$, $\|\cdot\|_A^S = \|\cdot\|^S$).

Let A, B be closed subsets of I such that $B \subset A$ and let a function $f(t)$ be defined on A. The restriction f_B of f to B is the following function: $f_B(t) = f(t)$ for $t \in B$.

Similarly we define the restrictions of vector functions and of classes of equivalent functions.

Definition 1. A transformation T of the space C to the space $[S]$ is called Carathéodory operator, if the domain of T is the whole space C, if T is a continuous transformation and if for every closed interval $J \subset I$ and for every couple of n-dimensional functions $f^{(1)}(t), f^{(2)}(t), f^{(i)} \in C$ for which $f_J^{(1)}(t) = f_J^{(2)}(t)$ we have $(Tf^{(1)})_J = (Tf^{(2)})_J$.

Definition 2. An m-dimensional vector function $h(t, x) \equiv [h_1(t, x_1, \ldots, x_n), \ldots \ldots, h_m(t, x_1, \ldots, x_n)]$ fulfils Carathéodory's conditions, if it is defined on $L \times G$, where $L \in \mathscr{F}^*$, $\mu(L) = \mu(I)$ and

i) $h(t, x)$ is a μ-measurable function of t on L for every n-dimensional vector $x \in G$.

ii) $h(t, x)$ is a continuous function of x for every $t \in L$.

Carathéodory's conditions which are usually used require some kind of boundedness of $h(t, x)$, but it is not necessary for the purpose of this article.

Definition 3. Let T be a Carathéodory operator and $h(t, x)$ be an m-dimensional vector function fulfilling Carathéodory's conditions. We say that the m-dimensional vector function $h(t, x)$ represents the operator T, if $Tf = [h(t, f(t))]$ for every n-dimensional vector function f from C.

The representation is called unique, if $\mu(\{t : \exists_{x \in G}\, h(t, x) \neq g(t, x)\}) = 0$ for every representation $g(t, x)$ of T.

Remark 1. For our purpose we can define Nemyckij operator as follows: A transformation $O : C \to [S]$ is called Nemyckij operator, if it is defined by $Of = [h(t, f(t))]$, where $h(t, x)$ fulfils Carathéodory's conditions [2]. It means that the question about the representation of Carathéodory operators is the question about the equivalence of Carathéodory and Nemyckij operators.

First we shall formulate without proof a well-known theorem. If it is not necessary to point out the dimension of a vector function, we shall speak about functions only.

Theorem 1. *Let a function $h(t, x)$ fulfil Carathéodory's conditions. Then the operator T defined by $Tf = [h(t, f(t))]$ for every $f \in C$ is a Carathéodory operator.*

The purpose of this article is to prove the converse theorem, but first we shall deal only with the most important case when the measure μ is the Lebesgue measure.

Theorem 2. *Let T be a Carathéodory operator (where μ is Lebesgue measure), then a function $h(t, x)$ exists which fulfils Carathéodory's conditions and which represents the operator T. The representation is unique.*

In the proof of Theorem 2 and 3 the notion μ-measurable means Lebesgue measurable. In the proof of Theorem 2 we shall need some auxiliary statement about implicit functions which we shall formulate as a theorem.

Theorem 3. *Let A be an analytic set in the Euclidean space E_{n+m} of points $[x_1, \ldots, x_{n+m}]$. Let B be the orthogonal projection of A on the subspace E_n of points $[x_1, \ldots, x_n]$. A measurable m-dimensional vector function $h(x_1, \ldots, x_n)$ exists such that $[x_1, \ldots, x_n, h_1(x_1, \ldots, x_n), \ldots, h_m(x_1, \ldots, x_n)] \in A$ for all $[x_1, \ldots, x_n] \in B$.*

The connection of this Theorem with the theory of implicit functions is very close. If we define $g(z) = 0$ for $z \in A$ and $g(z) \neq 0$ elsewhere, we search for a measurable solution of $g(x, h(x)) = 0$.

We shall use the following properties of analytic sets:

i) every Borel set is an analytic set,
ii) every analytic set is Lebesgue measurable,
iii) the orthogonal projection of an analytic set is an analytic set,
iv) the union of countable many analytic sets is an analytic set,
v) the intersection of countable many analytic sets is an analytic set,
vi) every analytic set is a range of a continous function $\varphi(t)$ which is defined on $\tilde{I}$, where $\tilde{I}$ is the set of the irrational numbers from I. If J is a subinterval of I, then the range of $\varphi_J(t)$ is an analytic set.

The analytic sets are defined and the mentioned properties are presented in § 35 Chap. III [3].

Now we pass to the proof of Theorem 3. The points in E_{n+m} will be denoted by $z = [x, y]$ where $x_i = z_i$ for $i = 1, \ldots, n$, $y_i = z_{n+i}$ for $i = 1, \ldots, m$. By vi) there exists a parametric expression of A, i.e. there exists a continuous function $z(t) = [x(t), y(t)]$ on $\tilde{I}$ such that $A = \{z : z = z(t) \text{ for some } t \in \tilde{I}\}$.

Let $I_1, \ldots, I_n, \ldots$ be the Baire intervals of the first order, i.e. I_n is the set of numbers which are expressed by $t = 1/(n + \eta)$, $0 \leqq \eta < 1$. Let $I_{n,1}, \ldots, I_{n,s}, \ldots$ be the Baire intervals of the second order, i.e. $I_{n,s}$ is the set of numbers which are expressed by

$$t = \cfrac{1}{n + \cfrac{1}{s + \eta}}\,, \quad 0 \leqq \eta < 1\,.$$

Generally, $I_{n_1,\ldots,n_{k-1},1}, \ldots I_{n_1,\ldots,n_{k-1},s}, \ldots$ denote the Baire intervals of the k-th order. If $x \in B$, we denote by $Q_x = \{t \in \tilde{I} : x(t) = x\}$. Q_x is obviously nonvoid and closed for every $x \in B$ and $Q_x \cap Q_{\bar{x}} = \emptyset$ for $x \neq \bar{x}$. We choose arbitrary numbers $a_{n_1,\ldots,n_k}$ in $\tilde{I}_{n_1,\ldots,n_k}$. Now it is all prepared to define the approximations of $h(x)$.

Definition of $h^{(1)}(x)$: let $x \in B$, then take the least index k such that $\tilde{I}_k \cap Q_x \neq \emptyset$ and put $h^{(1)}(x) = y(a_k)$. The m-dimensional vector function $h^{(1)}(x)$ has at most countable many different values. Obviously $h^{(1)}(x) = y(a_1)$ holds on the set, which is the orthogonal projection of $\{z : z = z(t),\ t \in \tilde{I}_1\}$ on E_n. Generally, $h^{(1)}(x) = y(a_s)$ holds on the set

$$P\{z : z = z(t),\ t \in \tilde{I}_s\} - \bigcup_{l<s} P\{z : z = z(t),\ t \in \tilde{I}_l\},$$

where P denotes the orthogonal projection on E_n.

By ii), iii) and vi) the function $h^{(1)}(x)$ is measurable.

Definition of $h^{(k)}(x)$. Let $x \in B$. We choose the least index n_k such that $I_{n_1,\dots,n_k} \cap Q_x \neq \emptyset$ and put $h^{(k)}(x) = y(a_{n_1,\dots,n_k})$. (The index n_{k-1} was chosen by construction of $h^{(k-1)}(x)$.) The function $h^{(k)}(x)$ is also measurable.

Now it remains to prove that $h^{(k)}(x)$ converge to some $h(x)$, $[x, h(x)] \in A$. Choose a point $x \in B$. According to the construction described above, we take the least index i_1 such that $\tilde{I}_{i_1} \cap Q_x \neq \emptyset$, the least index i_2 such that $I_{i_1,i_2} \cap Q_x \neq \emptyset$ etc. Obviously $h^{(k)}(x) = y(a_{i_1,\dots,i_k})$. The points $a_{i_1,\dots,i_k}$ form a sequence in $\tilde{I}$ such that $a_{i_1,\dots,i_l} \in \tilde{I}_{i_1,\dots,i_k}$ for $l \geqq k$. As $a_{i_1,\dots,i_k} \in I_{i_1,\dots,i_k} \subset \bar{I}_{i_1,\dots,i_k}$ ($\bar{I}_l$ denotes the closure of I_l), there exists $a = \lim_{k\to\infty} a_{i_1,\dots,i_k}$. Since $a \in \bar{I}_{i_1,\dots,i_k}$ for all k, the number a is irrational and $a \in \tilde{I}$. Since $Q_x \cap \tilde{I}_{i_1,\dots,i_k} \neq \emptyset$, there exist numbers $t_k \in Q_x \cap \tilde{I}_{i_1,\dots,i_k}$, and since the length of $\tilde{I}_{i_1,\dots,i_k}$ converges to zero with $k \to \infty$, numbers t_k converge to a, too. Since $t_k \in Q_x$ and Q_x is closed, we obtain $a \in Q_x$. It means $x(a) = x$. On the other hand $h^{(k)}(x) = y(a_{i_1,\dots,i_k}) \to y(a)$ and since $z(a) = [x(a), y(a)] \in A$ we have proved that $[x, h^{(k)}(x)]$ converge to a certain point from A. We can put $h(x) = \lim_{k\to\infty} h^{(k)}(x)$ for all $x \in B$. The function $h(x)$ is obviously measurable and $[x, h(x)] \in A$. Theorem 3 is proved.

Now we return to Theorem 2. First, we shall prove the uniqueness of the representation. Let $h(t, x)$ and $g(t, x)$ be two representations of T, then for every $x \in G \subset E_n$, $h(t, x) = g(t, x)$ for almost all t. Let x_n be a sequence of all n-dimensional vectors from G which have all coordinates rational. Denote by Q_n the set of t for which there is $h(t, x_n) \neq g(t, x_n)$ and by Q_{-1} the set where $h(t, x)$ or $g(t, x)$ is not continuous in x. The sets Q_n, and also the set $\bigcup_n Q_n$ have measure zero. For $t \in \langle 0, 1\rangle - \bigcup_n Q_n$ there holds $h(t, x) = \lim_{n\to\infty} h(t, x_n) = \lim_{n\to\infty} g(t, x_n) = g(t, x)$.

Our proof of Theorem 2 will be constructive, but before we pass to it we must prove several auxiliary lemmas.

Lemma 1. *Let T be a Carathéodory operator and A be a closed subset of $\langle 0, 1\rangle$. Then a continuous operator T_A exists which maps C_A into $[S]_A$ and $(Tf)_A = T_A f_A$ holds, where $f \in C$ and f_A is the restriction of f to A.*

Proof. We shall prove that $(Tf^{(1)})_A = (Tf^{(2)})_A$, if $f^{(1)}, f^{(2)} \in C$, $f_A^{(1)} = f_A^{(2)}$. Choose a sequence of positive numbers ε_s, $\varepsilon_s \to 0$. Put $A_s = O(\varepsilon_s, A) \cap \langle 0, 1\rangle$, where $O(\varepsilon, A)$ is the closed ε-neighbourhood of A. Each A_s consists of a finite number of closed intervals I_s^p. Define $h^{(s)}(t)$ by: $h^{(s)}(t) = f^{(2)}(t)$ for $t \in A_s$. The function $h^{(s)}(t)$ can be extended onto the closed interval $\langle 0, 1 \rangle$ so that $h^{(s)} \in C$ and $\|h^{(s)}(t) - f^{(1)}(t)\|^C = \|h^{(s)}(t) - f^{(1)}(t)\|_{A_s}^C$. Since $f^{(1)}, f^{(2)}$ are uniformly continuous and $f_A^{(1)} = f_A^{(2)}$, we have $\lim_{s \to \infty} \|h^{(s)}(t) - f^{(1)}(t)\|^C = 0$. As the Carathéodory operator T is continuous, we obtain $\lim_{s \to \infty} Th^{(s)} = Tf^{(1)}$. If we recall the definition of $h^{(s)}(t)$, we conclude $(Th^{(s)})_{IP_s} = (Tf^{(2)})_{IP_s}$ for all p and it yields $(Th^{(s)})_{A_s} = (Tf^{(2)})_{A_s}$. As $A \subset A_s$, it is $(Th^{(s)})_A = (Tf^{(2)})_A$. This relation together with $\lim_{s \to \infty} Th^{(s)} = Tf^{(1)}$ implies the desired equality $(Tf^{(1)})_A = (Tf^{(2)})_A$.

Let $f \in C_A$. Obviously a function $g \in C$ exists such that $g_A \equiv f$. We define $T_A f = (Tg)_A$. According to the proved statement, this definition does not depend on the extension of f.

Now, we prove that T_A is a continuous operator. Let $f_A^{(n)} \to f_A$, $f_A^{(n)}, f_A \in C_A$. We extend f_A in an arbitrary way onto the interval $\langle 0, 1 \rangle$ so that this extension belongs to C. Denote such function by f. Further we extend the functions $f_A^{(n)}$ so that $f^{(n)} \in C$ and $\|f^{(n)} - f\|^C = \|f^{(n)} - f\|_A^C$. As T is a Carathéodory operator, we have $\lim_{n \to \infty} T_A f_A^{(n)} = \lim_{n \to \infty} (Tf^{(n)})_A = (Tf)_A = T_A f_A$.

Remark 2. The operators T_A are Carathéodory operators again. Particularly $(T_A f_A)_B = T_B f_B$ for closed B, A, $B \subset A \subset I$ and $f \in C$.

In the next lemma we shall prove that the domain of Carathéodory operator can be extended. In this proof, Lemma 1 will be very useful. Denote by $S_A(G)$ the set of all n-dimensional vector functions $f(t)$ which are defined and measurable on A and for which $f(t) \in G$ for $t \in A$ is fulfilled.

Lemma 2. *Let T be Carathéodory operator, then the operators T_A^* exist for every closed subset $A \subset \langle 0, 1 \rangle$ such that $T_A^* f = T_A f$ for every $f \in C_A$, T_A^* maps the space $S_A(G)$ to $[S]_A$.*

Further, $\lim_{n \to \infty} T_A^* f^{(n)} = T_A^* f$, if $f^{(n)}$ converge almost everywhere to f on A.

Proof. Let an n-dimensional measurable function f on A be given. With respect to Luzin's theorem there exist open sets $B_n \subset A$, $\mu(B_n) < 2^{-n}$ such that $f(t)$ is continuous on $A - B_n$. Since the sets $A - B_n$ are closed, there exist operators T_{A-B_n} (Lemma 1). We define $g_n = T_{A-B_n} f_{A-B_n}$ for $t \in A - B_n$ and $g = 0$ for $t \in B_n$. The classes g_n form a fundamental sequence in $[S]_A$ (cf. Remark 2). Since the space $[S]_A$ is an F-space, there exists a limit $g \in [S]_A$. We put $T_A^* f = g$. If f is continuous on A, then the sets B_n can be chosen empty and this implies $T_A^* f = T_A f$. We pass to the continuity of T_A^*. Let $f^{(n)}$ be a sequence of measurable n-dimensional vector functions which converges a. e. to f on A. Let a number $\varepsilon > 0$ be given. By Egorov's theorem there exists an open set B_{-1}, $B_{-1} \subset A$, $\mu(B_{-1}) < \varepsilon/4$ such that $f^{(n)}$ converge uniformly on $A - B_{-1}$. By Luzin's theorem there exists an open set B_0, $B_0 \subset A$, $\mu(B_0) < \varepsilon/4$ such that f is continuous on $A - B_0$ and there are open sets B_n, $B_n \subset A$, $\mu(B_n) < \varepsilon/2^{n+2}$ such that $f^{(n)}$ is continuous on $A - B_n$. Denote $B = \bigcup_{n=-1}^{\infty} B_n$. Since $f^{(n)}$ are continuous on $A - B$ and $f^{(n)}$ converge to f uniformly on $A - B$, we obtain $\lim_{n\to\infty} T^*_{A-B} f^n_{A-B} = \lim_{n\to\infty} T_{A-B} f^{(n)}_{A-B} = T_{A-B} f_{A-B} = T^*_{A-B} f_{A-B}$. By definition of the quasi-norm on $[S]_A$ we obtain $\|T^*_A f^{(n)} - T^*_A f\|^S_A \leqq \|T^*_{A-B} f^{(n)}_{A-B} - T^*_{A-B} f_{A-B}\|^S_{A-B} + 2\varepsilon$. We obtain easily $\limsup_{n\to\infty} \|T^*_A f^{(n)} - T^*_A f\|^S_A \leqq 2\varepsilon$ and since ε is an arbitrary positive number, Lemma 2 is proved.

Definition 4. Elements of the class Tf will be denoted by $(Tf)(t)$. Carathéodory operator T is called bounded, if a constant $M > 0$ exists such that all elements $(Tf)(t)$, for all $f \in C$, fulfil: $|(Tf)(t)| \leqq M$ for almost all $t \in I$.

In the three following lemmas, we shall deal only with bounded Carathéodory operators. Before we introduce the representative function we must define several auxiliary functions. To every point $[t, x]$ from $\langle 0, 1\rangle \times G$ we define

$$H(t, x) = \int_0^t (Tx^*)(\tau)\,d\tau, \tag{1}$$

where x^* is defined by $x^*(t) = x$ for all $t \in \langle 0, 1\rangle$, $(Tx^*)(t)$ is an arbitrary element from the class Tx^*.

$$\lambda(t, x) = \limsup_{\nu\to 0_+} \frac{H(\tau, y) - H(t, y)}{\tau - t}, \tag{2a}$$

where sup is taken over

$$0 < |\tau - t| \leqq \nu, \quad \|x - y\| \leqq \nu;$$

$$\gamma(t, x) = \liminf_{\nu\to 0_+} \frac{H(\tau, y) - H(t, y)}{\tau - t}, \tag{2b}$$

where inf is taken over

$$0 < |\tau - t| \leqq \nu, \quad \|x - y\| \leqq \nu;$$

$$h(t, x) = \frac{\partial H(t, x)}{\partial t} \tag{3}$$

at all points where the derivative exists.

Lemma 3. *Let T be a bounded Carathéodory operator, then the function $H(t, x)$ is defined on $\langle 0, 1\rangle \times G$, Lipschitz continuous in t, and continuous in both variables. The sets $\{[t, x] : \lambda_i(t, x) \geqq r\}$, $\{[t, x] : \gamma_i(t, x) \leqq r\}$ are analytic sets for every $i = 1, \ldots, n$ and every real number r.*

Proof. Since the operator T is bounded, the integral in (1) exists and is independent of the choice of $(Tx^*)(t)$ from Tx^*. Since the operator T is continuous, the function $H(t, x)$ is continuous in x for t fixed. On the other hand, $H(t, x)$ is Lipschitz continuous in t. From that follows that $H(t, x)$ is continuous in both variables.

Denote $\theta(t, x, y) = H(t, x + y)$. Function $\theta(t, x, y)$ is defined in the space E_{2n+1}. Let P denote the orthogonal projection of a set in E_{2n+1} on the subspace E_{n+1} which consists from the points $[t, x]$. Obviously

(4a)
$$A_1^i(r) = \{[t, x] : \lambda_i(t, x) \geqq r\} =$$
$$= \bigcap_k P\left\{[t, x, y] : \sup_{0<|\tau-t|\leqq 1/k} \frac{\theta_i(\tau, x, y) - \theta_i(t, x, y)}{\tau - t} \geqq r - \frac{1}{k},\ |y| \leqq \frac{1}{k}\right\},$$

(4b)
$$A_2^i(r) = \{[t, x] : \gamma_i(t, x) \leqq r\} =$$
$$= \bigcap_k P\left\{[t, x, y] : \inf_{0<|\tau-t|\leqq 1/k} \frac{\theta_i(\tau, x, y) - \theta_i(t, x, y)}{\tau - t} \leqq r - \frac{1}{k},\ |y| \leqq \frac{1}{k}\right\}.$$

Since the sets in the brackets of P are Borel sets, $A_1^i(r)$ and $A_2^i(r)$ are analytic sets (cf. iii) and v)).

Remark 3. Let A be a closed subset of $\langle 0, 1\rangle$ which consists of the set of disjoint closed nonsingular intervals I_k and from a set B, $\mu(B) = 0$. Let $f(t)$ be defined on A so that it is constant on I_k. Recalling the definition of $h(t, x)$ (see (3)), we have $\int_{I_k} h(t, f(t))\, dt = \int_{I_k} (T_{I_k} f_{I_k})(t)\, dt$ and obviously

(5)
$$\int_A h(t, f(t))\, dt = \int_A (T_A^* f_A)(t)\, dt = \int_A (T^* g)_A (t)\, dt,$$

where g is an arbitrary extension of f.

Now we shall prove a lemma from which it will immediately follow that $h(t, x)$ is the desired representation of the bounded operator T.

Lemma 4. *Let T be a bounded Carathéodory operator, then* $\mu(\{t : \underset{x\in G}{\exists}\ \lambda(t, x) \neq \mu(t, x)\}) = 0$.

Proof. Denote by A the set of points $[t, x]$ for which $\lambda(t, x) > \mu(t, x)$. By (4a) and (4b) $A = \bigcup_i \bigcup_{r_1 > r_2} (A_1^i(r_1) \cap A_2^i(r_2))$, where r_1, r_2 range over the set of all rational numbers $r_1 > r_2$. The set A is analytic and its projection B on the axis t is also analytic. Assume $\mu(B) > 0$; then there exists an index i and rational numbers $r > q$ such that the orthogonal projection B^i of the set $A^i = A_1^i(r) \bigcap A_2^i(q)$ on the axis t has a positive measure: $\mu(B^i) > 0$. By Theorem 3 there exists a measurable n-dimensional vector function $f(t)$ such that $[t, f(t)] \in A^i$ for $t \in B^i$. Accordingly to Luzin's theorem the set $D \subset B^i$ exists, $\mu(D) > 0$ and $f(t)$ is continuous on D. As D is measurable, a closed set $L \subset D$ exists, $\mu(L) > 0$. Obviously $f(t)$ is also continuous on L. For $t \in L$ it holds

(6a)
$$\limsup_{v \to 0_+} \frac{H_i(\tau, y) - H_i(t, y)}{\tau - t} \geqq r,$$

where sup is taken over

$$0 < |\tau - t| \leqq v, \quad |f(t) - y| \leqq v,$$

(6b)
$$\liminf_{v \to 0_+} \frac{H_i(\tau, y) - H_i(t, y)}{\tau - t} \leqq q,$$

where inf is taken over

$$0 < |\tau - t| \leqq v, \quad |f(t) - y| \leqq v.$$

Let $K \subset L$ be the set of all points of density of the set L. Evidently $\mu(K) = \mu(L) \neq 0$. Since the function $H_i(t, y)$ is Lipschitz continuous in t, the relation (6a) and (6b) hold with an additional assumption that numbers τ are from L. It means that to every $t \in K$ there exists a sequence of intervals with end points t and $\bar{\tau}_j$ and a sequence of points $\bar{y}_j$ such that $|\bar{y}_j - f(t)| \to 0$, $|\bar{\tau}_j - t| \to 0$, $\bar{\tau}_j \in L$ and

(7a)
$$\frac{H_i(\bar{\tau}_j, \bar{y}_j) - H_i(t, \bar{y}_j)}{(\bar{\tau}_j - t)} \geqq \tfrac{1}{3}(2r + q).$$

For every $t \in K$ there exists also a sequence of intervals with end points t, $\bar{\bar{\tau}}_j$ and a sequence of points $\bar{\bar{y}}_j$ such that $|\bar{\bar{y}}_j - f(t)| \to 0$, $|\bar{\bar{\tau}}_j - t| \to 0$, $\bar{\bar{\tau}}_j \in L$ and

(7b)
$$\frac{H_i(\bar{\bar{\tau}}_j, \bar{\bar{y}}_j) - H_i(t, \bar{\bar{y}}_j)}{(\bar{\bar{\tau}}_j - t)} \leqq \tfrac{1}{3}(r + 2q).$$

The intervals fulfilling (7a) will be called the intervals of the first type, the intervals fulfilling (7b) will be called the intervals of the second type. The intervals of the first type as well as those of the second type cover the set K. By Vitali's theorem there exists a set of disjoint intervals of the first or of the second type respectively which covers K. Denote these sets by $\bar{Q}_s$ and $\bar{\bar{Q}}_s$. We can assume that for intervals from $\bar{Q}_s$ and $\bar{\bar{Q}}_s$, $|\bar{\tau}_j - t| \leqq 1/s$, $|\bar{\bar{\tau}}_j - t| \leqq 1/s$, $|\bar{y}_j - f(t)| \leqq 1/s$, $|\bar{\bar{y}}_j - f(t)| \leqq 1/s$ holds.

Now we shall define n-dimensional vector functions $\bar{f}^{(s)}(t)$ and $\bar{\bar{f}}^{(s)}(t)$. Let $\tau \in I$, $I \in \bar{Q}_s$, then $\bar{f}^{(s)}(t)$ is constant on I and $\bar{f}^{(s)}(\tau) = \bar{y}_j$ where $\bar{y}_j$ corresponds to I by (7a). For τ which does not belong to any $I, I \in \bar{Q}_s$, we choose $\bar{f}^{(s)}(\tau)$ within $|\bar{f}^{(s)}(\tau) - f(\tau)| \leqq$ $\leqq 1/s$. The definition of $\bar{\bar{f}}^{(s)}(t)$ is the same as the former definition, with the only exception that the set $\bar{Q}_s$ is replaced by $\bar{\bar{Q}}_s$ and the points $\bar{y}_s$ by $\bar{\bar{y}}_s$. By Remark 3 (5) and (7a)

$$\int_L (T_L^* \bar{f}^{(s)})(t)\,\mathrm{d}t = \int_L h(t, \bar{f}^{(s)}(t))\,\mathrm{d}t = \sum_{I \in \bar{Q}_s} \int_I h(t, \bar{f}^{(s)}(t))\,\mathrm{d}t \geqq \frac{2r + q}{3}\,\mu(L)$$

and by (7b)

$$\int_L (T_L^* \bar{f}^{(s)})(t)\,\mathrm{d}t \leqq \frac{r + 2q}{3}\,\mu(L)\,.$$

On the other hand, $\bar{f}^{(s)}(t)$ and $\bar{\bar{f}}^{(s)}(t)$ converge to $f(t)$ and by Lemma 2 it should be

$$\lim_{s\to\infty} \int_L (T_L^* \bar{f}^{(s)})(t)\,\mathrm{d}t = \lim_{s\to\infty} \int_L (T^* \bar{\bar{f}}^{(s)})(t)\,\mathrm{d}t = \int_L (T^* f)(t)\,\mathrm{d}t\,.$$

This relation with the previous inequalities implies that $\mu(L) = 0$ and this contradiction proves Lemma 4.

We have still to complete the solution of the problem for bounded operators.

Lemma 5. *Let T be a bounded Carathéodory operator, then the function $h(t, x)$ defined by (3) fulfils Carathéodory's conditions and it is the representation of T. Function $h(t, x)$ is bounded for almost all t.*

Proof. Since

$$\gamma_i(t, x) \leqq \liminf_{\Delta t\to 0} \frac{H_i(t + \Delta t, x) - H_i(t, x)}{\Delta t} \leqq \limsup_{\Delta t\to 0} \frac{H_i(t + \Delta t, x) - H_i(t, x)}{\Delta t} \leqq$$
$$\leqq \lambda_i(t, x)$$

(see (2a) and (2b)) and by Lemma 4, we obtain that the derivatives $\partial H/\partial t$ exist for almost all t (i.e. the set of t for which this derivatives exist for all $x \in G$ has measure one). As $H(t, x)$ is Lipschitz continuous in t, $h(t, x)$ must be bounded almost everywhere. If we take t from the set for which the derivatives exist, we have

$$\gamma_i(t, x) \leqq \liminf_{y\to x} \frac{\partial H_i(t, y)}{\partial t} \leqq \limsup_{y\to x} \frac{\partial H_i(t, y)}{\partial t} \leqq \lambda_i(t, x)$$

and by Lemma 4 we obtain that the derivative is a continuous function in x for almost all t. We have proved that $h(t, x)$ fulfils Carathéodory's conditions.

We shall prove that $h(t, x)$ is the representation of T. Let us define (as before) $x^*(t) = x$ for every $x \in G$. By (1) and (3) $Tx^* = [h(t, x^*(t))]$. Let I_k be disjoint, semi-opened intervals which cover $\langle 0, 1\rangle$. Let $f(t)$ be constant on every I_k, then $T_{I_k} f_{I_k} = [h(t, f(t))]_{I_k}$ (Lemma 1). By Lemma 2 we have $T_{I_k} f_{I_k} = T_{I_k}^* f_{I_k} = (T^* f)_{I_k}$. Let $g \in C$ and $f^{(s)}$ be a sequence of vector functions of the type just described converging to g. By Lemma 2 we have

$$[h(t, g(t))] = \lim_{s\to\infty} [h(t, f^{(s)}(t))] = \lim_{s\to\infty} T^* f^{(s)} = T^* g = Tg\,.$$

Lemma 5 is proved.

Now we pass to the case of unbounded operators. To every class $[f]$ from $[S]$ we shall define a contracted class arctg $[f]$. The class arctg $[f]$ is the class of m-dimensional vector functions $[\mathrm{arctg} f_1(t), \ldots, \mathrm{arctg} f_m(t)]$, where m-dimensional vector functions $f(t) \equiv [f_1(t), \ldots, f_m(t)]$ belong to the class $[f]$.

Definition 5. Let T be a given operator. The contracted operator arctg T is defined by $(\mathrm{arctg}\ T) f = \mathrm{arctg}\,(Tf)$ for every $f \in C$.

Every contracted operator is bounded and arctg T is Carathéodory operator, if T is Carathéodory operator.

Since arctg T is bounded Carathéodory operator, there exists a function $h^*(t, x)$ fulfilling Carathéodory's conditions, representing arctg T and bounded: $|h^*(t, x)| \leqq \pi/2$. We shall prove that the projection B of the set $A = \{[t, x] : |h^*(t, x)| = \pi/2\}$ on the axis t has measure zero. By Lemma 3 and 4 the sets $\{[t, x] : |h_i^*(t, x)| \geqq \pi/2$ or $h_i^*(t, x)$ does not exists$\}$ are analytic. It means that a measurable function $f(t)$ exists such that $[t, f(t)] \in A$ for $t \in B$, i.e. every function $((\mathrm{arctg}\ T^*) f)(t)$ from the class $(\mathrm{arctg}\ T^*) f$ fulfils $|((\mathrm{arctg}\ T^*) f)(t)| = \pi/2$ almost everywhere on B. On the other hand T^*f is an almost everywhere finite function. It means $\mu(B) = 0$. Now the representative function of T can be defined as $h(t, x) = \mathrm{tg}\ h^*(t, x)$. As $h^*(t, x)$ has the desired properties and for almost all t it is in the region of definition of tg, $h(t, x)$ has also all the desired properties.

Now we shall generalize Theorem 2 for a wider class of Carathéodory operators.

Theorem 4. *Let T be a Carathéodory operator. Providing that the measure from the definition of $[S]$ is a regular nonnegative measure defined on $\mathscr{F}$, where $\mathscr{F}$ is a σ-field of subsets of the closed interval I which contains all Borel subsets of I, then the statement of Theorem 2 is also valid.*

Proof. Let the measure be nontrivial, i.e. $\mu(I) > 0$. Put $\alpha = \mu(I)$ and $F(t) = \mu(\langle 0, t\rangle)/\alpha$. The function $F(t)$ is nondecreasing and continuous from the right, $F(1) = 1$. Denote λ_k the points of discontinuity of $F(t)$. Put $\mu_k = \lim_{t \to \lambda_k -} F(t)$. The index k is of the first type if there exists $t < \lambda_k$ such that $\mu_k = F(t)$. The indices which are not of the first type we shall call of the second type. Denote N the set of all points of discontinuity of $F(t)$. The function $F(t)$ generates the following transformation. To every point $t \in I - N$ there corresponds the point $F(t) \in I$. To a point $t \in N$ there corresponds a semi-open interval $(\mu_k, F(t)\rangle$ if k is of the first type and the closed interval $\langle \mu_k, F(t)\rangle$ if k is of the second type. The set of these intervals will be denoted by M. This point-transformation generates a transformation of subsets of I. Let A be a subset of I. Then the transformation of A is the union of transformations of points from A. Let $\mathscr{L}$ be a σ-field of Lebesgue measurable subsets of I and $\mathscr{L}^+$ be a σ-field of Lebesgue measurable subsets of I which have the following property: if the set A contains a point t which belongs to an interval $J \in M$, then A contains the whole interval J. Since the measure μ is regular and nonnegative, we obtain that to every μ-measurable set there corresponds an $\mathscr{L}^+$-measurable set. The relation between a μ-measure of a set A and the Lebesgue measure of the image $U(A)$ is simply $\mu(A) =$

$= \alpha l(U(A))$, where $l(B)$ is the Lebesgue measure of B. To every μ-measurable function $f(t)$ there corresponds an $\mathscr{L}^+$-measurable function $U(f)(t)$ defined by $U(f)(F(t)) = f(t)$ for $t \in I$, $U(f)(t)$ being constant on intervals from M. (It may happen that this definition of $U(f)(t)$ is not unique but only for a countable number of t). This transformation of functions maps the class of μ-measurable functions on the class of $\mathscr{L}^+$-measurable functions. Finally to every Carathéodory operator T we shall construct an operator $U(T)$ in the following manner: $U(T)f = U(Tf)$. The operator $U(T)$ is Carathéodory operator again but the corresponding measure is the Lebesgue measure. By Theorem 2 there exists the representation $h_0(t, x)$ of $U(T)$. We must still prove that $h_0(t, x)$ is $\mathscr{L}^+$-measurable. However, it follows immediately from the fact that $(U(T)f)_J$ are constant functions for every $J \in M$. It means that $h_0(t, x)$ is a constant function in t (for fixed x) for $t \in J$, $J \in M$. The properties of the transformation U mentioned above yield that there exists a function $h(t, x)$ fulfilling Carathéodory conditions such that $h_0(t, x) = U(h(t, x))$. Obviously the function $h(t, x)$ is the representation of T. Theorem 4 is proved.

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