Ivo Vrkoč

The weak exponential stability and periodic solutions of Ito stochastic equations with small stochastic terms

Czechoslovak Mathematical Journal, Vol. 18 (1968), No. 4, 722–752

Persistent URL: `http://dml.cz/dmlcz/100868`

Czechoslovak Mathematical Journal, 18 (93) 1968, Praha

THE WEAK EXPONENTIAL STABILITY AND PERIODIC SOLUTIONS OF ITO STOCHASTIC EQUATIONS WITH SMALL STOCHASTIC TERMS

Ivo Vrkoč, Praha

(Received July 20, 1967)

The problem investigated in this article is very similar to that in [1]. However, now we are going to assume another conditions, particularly conditions concerning the stability of (2). In the previous paper we utilised the uniform exponential stability of all solutions of (2). Some of consequence were, for example, that $a(t, x, \varepsilon)$ cannot be bounded and that we had to replace relations 8), 9) which are usually used in the averaging theory by 5), 6) from [1]. In this article we shall investigate the case that $a(t, x, \varepsilon)$ and $B(t, x, \varepsilon)$ are bounded and, at the same time, we shall have to use a certain weaker concept of stability than in the article [1]. The proof of the theorem on periodic solutions depends substantially on the theorem on total boundedness. To prove the total boundedness we need to have the existence of the expectation of $|x(t)|^{2q}$ where $x(t)$ is a solution of (1) and q is an integer $q > 1$. This would be obvious, if $w_\varepsilon(t)$ were a continuous process, but we do not assume this. The last theorem of the paper deals with this problem. The stability theorem is also formulated in terms of the q-norm.

Since notation and conditions used here differ slightly from those in [1], we shall formulate them again.

Notation and basic conditions. Let E_n denote the n-dimensional Euclidean space, $x \equiv [x_1, x_2, \ldots, x_n]$ an n-dimensional vector, ε a small parameter and t a real variable. We shall use the norms $|x|_q = \sqrt[2q]{(\sum_{i=1}^{n} x_i^{2q})}$ for vectors and $|B|_q = \sqrt[2q]{(\sum_{ij=1}^{n} B_{ij}^{2q})}$ for matrices where q is a positive integer. For $q = 1$, $|x| = |x|_1$.

Let Ω be an abstract space, $\mathscr{F}$ a σ-field of subsets of Ω and P be a probabilistic measure defined on $\mathscr{F}$. We shall suppose that all random variables are $\mathscr{F}$-measurable. The norms of vector random variables and of matrix random variables are defined by $\|z\|_q = \sqrt[2q]{(E|z|_q^{2q})}$, where E is the expectation, $\|z\| = \|z\|_1$. Denode by $P(A|B)$, $A \in \mathscr{F}$, $B \in \mathscr{F}$ a conditional probability.

Let an n-dimensional vector function $a(t, x, \varepsilon)$ be defined on $\langle 0, \infty) \times E_n \times \times \langle 0, \delta)$ and an $n \times n$ matrix function $B(t, x, \varepsilon)$ on $\langle 0, \infty) \times E_n \times (0, \delta)$, where δ is a positive number.

1) Let $a(t, x, \varepsilon)$, $B(t, x, \varepsilon)$ be measurable in t for fixed x, ε and $|a(t, x, \varepsilon) - a(t, y, \varepsilon)|_q \leqq K|x - y|_q$, $|B(t, x, \varepsilon) - B(t, y, \varepsilon)|_q \leqq K|x - y|_q$.

2) Let $w_\varepsilon(t)$ be n-dimensional stochastic processes with independent increments such that $\|w_\varepsilon(t) - w_\varepsilon(0)\|_q$ exist and are continuous functions of t for some q. We shall suppose that $E(w_\varepsilon^{(i)}(t_2) - w_\varepsilon^{(i)}(t_1))^{2s+1} = 0$ for $s = 0, 1, \ldots, q - 1$, where $w_\varepsilon^{(i)}(t)$ are the components of the process $w_\varepsilon(t)$. Under these assumptions it is possible to prove that there are continuous nondecreasing functions $F_\varepsilon(t)$ such that

$$E|w_\varepsilon(t_2) - w_\varepsilon(t_1)|_s^{2s} \leqq F_\varepsilon(t_2) - F_\varepsilon(t_1) \quad \text{whenever} \quad t_1 < t_2\,, \quad s = 1, 2, \ldots, q\,.$$

We suppose that the parameter ε is from an open interval $(0, \delta)$. Let us denote $\mathscr{F}_\varepsilon(t)$ the least σ-field generated by increments $w_\varepsilon(t_2) - w_\varepsilon(t_1)$ for $t_1 < t_2 \leqq t$. Then $\mathscr{F}_\varepsilon(t) \subset \mathscr{F}$.

3) A continuous nondecreasing function $F(t)$ exists such that $F_\varepsilon(t_2) - F_\varepsilon(t_1) \leqq \leqq F(t_2) - F(t_1)$ for all $t_1 < t_2$, $\varepsilon \in (0, \delta)$.

4) Let initial values $x_0(\omega)$ be random variables which are independent of increments of $w_\varepsilon(t)$, i.e. of σ-fields $\mathscr{F}_\varepsilon(t)$ and such that $E|x_0|_q^{2q} < \infty$. If we consider several initial values simultaneously we shall suppose that the entire σ-field generated by all initial values is independent of $\mathscr{F}_\varepsilon(t)$. We assume that the structure of $(\Omega, \mathscr{F}, P)$ enables to construct an initial value $x_0(\omega)$ for every given distribution function.

Under these conditions we can consider a stochastic integral equation

$$(1) \qquad x(t) = x_0 + \int_0^t a(\tau, x(\tau), \varepsilon), \varepsilon)\,\mathrm{d}\tau + \int_0^t B(\tau, x(\tau), \varepsilon)\,\mathrm{d}w_\varepsilon(\tau)\,.$$

This stochastic equation will be compared with

$$(2) \qquad y(t) = y_0 + \int_0^t a(\tau, y(\tau), 0)\,\mathrm{d}\tau$$

where only the initial value can be stochastic.

Now, we shall formulate conditions on the boundedness of $a(t, x, \varepsilon)$ and $B(t, x, \varepsilon)$.

5) There exists a function $h(t)$ which is measurable and such that $|a(t, x, \varepsilon)|_q \leqq h(t)$, $\int_{t_1}^{t_2} h(\tau)\,\mathrm{d}\tau < \infty$ for every $0 \leqq t_1 < t_2 < \infty$ and $\int_{t_1}^{t_2} |B(\tau, 0, \varepsilon)|_q^{2q}\,\mathrm{d}F(\tau) < \infty$.

6) There exists a function $g(t)$ which is measurable and such that $|B(t, x, \varepsilon)|_q \leqq \leqq g(t)$, $\int_{t_1}^{t_2} g^{2q}(\tau)\,\mathrm{d}F(\tau) < \infty$ for every $0 \leqq t_1 < t_2 < \infty$.

7) Let the vector function $a(t, x, \varepsilon)$ and the matrix function $B(t, x, \varepsilon)$ have partial derivatives with respect to x_i which are Lipschitz continuous in the following sense

$$\left|\frac{\partial a}{\partial x}(t, x, \varepsilon) - \frac{\partial a}{\partial x}(t, y, \varepsilon)\right|_q \leqq K|x - y|_q,$$

$$\left[n^{2q-1} \sum_{k=1}^{n} \left|\frac{\partial B}{\partial x_k}(t, x, \varepsilon) - \frac{\partial B}{\partial x_k}(t, y, \varepsilon)\right|_q^{2q}\right]^{1/2q} \leqq K|x - y|_q.$$

We shall use following notation.

8) By $\psi(\tau, d, \varepsilon)$ we denote a function which fulfils $|\int_{\tau_1}^{\tau_2} (a(\tau, x, \zeta) - a(\tau, x, 0)) \, d\tau|_q \leqq \leqq \psi(\delta, d, \varepsilon)$ for $0 \leqq \tau_2 - \tau_1 \leqq \delta$, $|x|_q \geqq d$, $\zeta \leqq \varepsilon$.

9) By $\chi(\tau, d, \varepsilon)$ we denote a function which fulfils $\int_{\tau_1}^{\tau_2} |B(\tau, x, \zeta)|_q^{2q} \, dF_\zeta(\tau) \leqq \leqq \chi^{2q}(\tau, d, \varepsilon)$ for $0 \leqq \tau_2 - \tau_1 \leqq \delta$, $|x|_q \geqq d$, $\zeta \leqq \varepsilon$.

The further assumptions on ψ and χ will be given in Theorem 1 (implicitly) or in Remarks 1,2 and these assumptions are related to the used averaging method.

The difference from the previous paper [1] consists in essence in the different type of stability which we shall use now.

Definition 1. The solution $\bar{z}(t)$ of (3)

$$z(t) = z_0 + \int_{t_0}^{t} a(\tau, z(\tau)) \, d\tau + \int_{t_0}^{t} B(\tau, z(\tau)) \, dw(\tau) \tag{3}$$

is called stable, if to every $\varepsilon > 0$ and $t_0 \geqq 0$ there exists $\delta > 0$ such that $|\bar{z}(t_0) - - z(t_0)|_q < \delta$ implies $|\bar{z}(t) - z(t)|_q < \varepsilon$ for $t \geqq t_0$.

The solutions of (3) are called weakly uniformly exponentially stable, if there exist a function $\beta(d)$, $0 < \beta(d) < 1$ for all $d \geqq 0$, and a constant $S > 0$ such that $\|z_1(t) - - z_2(t)\|_q \leqq \beta(d) \|z_1(t_0) - z_2(t_0)\|_q$ for $t \geqq t_0 + S$ provided $|z_i(t_0)|_q \leqq d$, $i = 1, 2$ with probability 1.

This definition can be applied also for nonstochastic ordinary equation (2). The condition for the weak uniform exponential stability is then $|y_1(t) - y_2(t)|_q \leqq \leqq \beta(d) |y_1(t_0) - y_2(t_0)|_q$ for $t \geqq t_0 + S$, $|y_i(t_0)| \leqq d$, $i = 1, 2$.

Definition 2. The solutions of a nonstochastic ordinary equation (2) are called totally bounded, if there exist numbers $\zeta \geqq 0$, $\alpha > 0$, $S > 0$ such that $|y(t)|_q \leqq \leqq |y(t_0)|_q - \alpha$ for $|y(t_0)|_q \geqq \zeta$, $t \geqq t_0 + S$.

The definitions of a periodic process and of a process with periodic increments are the same as in the previous paper, but they are restated for completeness.

Definition 3. A process $z(t)$ is called periodic with period T if it is defined for all t and if for every positive integer s and for all numbers $t_1, t_2, \ldots, t_s$ and for all Borel sets $A_1, A_2, \ldots, A_s$ we have $P(z(t_1) \in A_1, z(t_2) \in A_2, \ldots z(t_s) \in A_s) = P(z(t_1 + T) \in \in A_1, z(t_2 + T) \in A_2, \ldots z(t_s + T) \in A_s)$.

Definition 4. A process $w(t)$ has periodic increments with period T if for every t, $h > 0$ and for every Borel set A we have $P(w(t + h) - w(t) \in A) = P(w(t + T + h) - w(t + T) \in A)$.

Before we shall formulate and prove Theorem 1 we need introduce Lemma 1 and Lemma 2.

Lemma 1. *Let the conditions* 1) *to* 5) *and* 7) *be fulfilled. Let L be a positive number, then there exists a function* $\varphi_1(d, \varepsilon)$, *which is given by* (1,9), *such that*

$$\sup_{\langle 0,L\rangle} \|x(t) - y(t)\|_q \leqq \varphi_1(d, \varepsilon)$$

where $x(t)$, $y(t)$ *is the solution of the equation* (1) *and* (2), *respectively, with the same initial nonstochastic value* x_0, $|x_0| \geqq d$.

Proof. By (1), (2) we easily obtain

$$\|x(t) - y(t)\|_q \leqq \left\| \int_0^t (a(\tau, x(\tau), \varepsilon) - a(\tau, y(\tau), 0))\, d\tau \right\|_q + \left\| \int_0^t B(\tau, x(\tau), \varepsilon)\, dw_\varepsilon(\tau) \right\|_q . \tag{1.1}$$

First, we estimate the first term

$$\begin{aligned} \left\| \int_0^t (a(\tau, x(\tau), \varepsilon) - a(\tau, y(\tau), 0))\, d\tau \right\|_q &\leqq K \int_0^t \|x(\tau) - y(\tau)\|_q\, d\tau + \\ + \left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau_i), \varepsilon))\, d\tau \right\|_q &+ \left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y(\tau_i), \varepsilon) - \right. \\ \left. - a(\tau, y(\tau_i), 0))\, d\tau \right\|_q &+ \left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y(\tau_i), 0) - a(\tau, y(\tau), 0))\, d\tau \right\|_q \end{aligned} \tag{1.2}$$

where $0 = \tau_0 < \tau_1 < \ldots < \tau_s = t$ and $\max(\tau_{i+1} - \tau_i) \leqq \delta$, $s < (L/\delta) + 1$.

By condition 5) $|y(t_2) - y(t_1)| \leqq \int_{t_1}^{t_2} h(\tau)\, d\tau$ holds for $t_1 < t_2$ such that the third and the last term in (1.2) are estimated by

$$\left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau_i), \varepsilon))\, d\tau \right\|_q \leqq K \sum \int_{\tau_i}^{\tau_{i+1}} \int_{\tau_i}^{\tau} h(\xi)\, d\xi\, d\tau \leqq LK\, \theta(\delta) \tag{1.3}$$

where $\theta(\delta) = \sup_{0 \leqq \tau \leqq L} \int_\tau^{\tau+\delta} h(\xi)\, d\xi$. By 8) we obtain the estimate for the fourth term

$$\begin{aligned} &\left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y(\tau_i), \varepsilon) - a(\tau, y(\tau_i), 0))\, d\tau \right\|_q \leqq \\ &\leqq \sum \sqrt[2q]{\left[E \left| \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y(\tau_i), \varepsilon) - a(\tau, y(\tau_i), 0))\, d\tau \right|_q^{2q} \right]} \leqq \psi(\delta, \alpha(d), \varepsilon)\, s \end{aligned} \tag{1.4}$$

where $\alpha(d) = d - \int_0^L h(\xi)\,d\xi$. Recalling (1,2) to (1,4) and the fact that the last term has the same estimate as (1,3), we get

$$(1.5) \qquad \left\| \int_0^t (a(\tau, x(\tau), \varepsilon) - a(\tau, y(\tau), 0))\,d\tau \right\|_q \leqq$$

$$\leqq K \int_0^t \|x(\tau) - y(\tau)\|_q\,d\tau + 2KL\theta(\delta) + \psi(\delta, \alpha(d), \varepsilon)\,s\,.$$

Now, we turn to the last term of (1,1)

$$(1.6) \quad \left\| \int_0^t B(\tau, x(\tau), \varepsilon)\,d\omega_\varepsilon(\tau) \right\|_q \leqq \left\| \int_0^t (B(\tau, x(\tau), \varepsilon) - B(\tau, y(\tau), \varepsilon))\,dw_\varepsilon(\tau) \right\|_q +$$

$$+ \left\| \int_0^t B(\tau, y(\tau), \varepsilon)\,dw_\varepsilon(\tau) \right\|_q\,.$$

Using (6,1) we obtain

$$(1.7) \qquad \left\| \int_0^t (B(\tau, x(\tau), \varepsilon) - B(\tau, y(\tau), \varepsilon))\,dw_\varepsilon(\tau) \right\|_q \leqq$$

$$\leqq H \sqrt[2q]{\left[\int_0^t \|B(\tau, x(\tau), \varepsilon) - B(\tau, y(\tau), \varepsilon)\|_q^{2q}\,dF(\tau) \right]} \leqq$$

$$\leqq KH \sqrt[2q]{\left[\int_0^t \|x(\tau) - y(\tau)\|_q^{2q}\,dF(\tau) \right]}$$

where $H = 2n \exp\{(1/4q)\,4^q(F(L) - F(0))\}$ and

$$(1.8) \quad \left\| \int_0^t B(\tau, y(\tau), \varepsilon)\,dw_\varepsilon(\tau) \right\|_q \leqq H \sqrt[2q]{\left[\int_0^t |B(\tau, y(\tau), \varepsilon)|_q^{2q}\,dF_\varepsilon(\tau) \right]} \leqq$$

$$\leqq H \sqrt[2q]{\left[\sum \int_{\tau_i}^{\tau_{i+1}} |B(\tau, y(\tau), \varepsilon) - B(\tau, y(\tau_i), \varepsilon)|_q^{2q}\,dF_\varepsilon(\tau) \right]} +$$

$$+ H \sqrt[2q]{\left[\sum \int_{\tau_i}^{\tau_{i+1}} |B(\tau, y(\tau_i), \varepsilon)|_q^{2q}\,dF_\varepsilon(\tau) \right]} \leqq$$

$$\leqq HK \sqrt[2q]{\left[\sum \int_{\tau_i}^{\tau_{i+1}} |y(\tau) - y(\tau_i)|_q^{2q}\,dF(\tau) \right]} + H\chi(\delta, \alpha(d), \varepsilon) \sqrt[2q]{\left[\frac{L}{\delta} + 1 \right]}\,.$$

For deriving these inequalities we made use of conditions 1), 3) and of 9). By (1,1) and (1,5) to (1,8) we obtain

$$\|x(t) - y(t)\|_q \leqq A_1 \left[\int_0^t \|x(\tau) - y(\tau)\|_q \, d\tau + \sqrt[2q]{\int_0^t \|x(\tau) - y(\tau)\|_q^{2q} \, dF(\tau)} + \right.$$

$$\left. + \theta(\delta) + \frac{\psi(\delta, \alpha(d), \varepsilon)}{\delta} + \frac{\chi(\delta, \alpha(d), \varepsilon)}{\sqrt[2q]{\delta}} \right]$$

where all coefficients are substituted by one constant A_1. This integral inequality implies that

$$\|x(t) - y(t)\|_q \leqq A_2 \left[\theta(\delta) + \frac{\psi(\delta, \alpha(d), \varepsilon)}{\delta} + \frac{\chi(\delta, \alpha(d), \varepsilon)}{\sqrt[2q]{\delta}} \right] \tag{1.9}$$

where A_2 is another constant.

Lemma 2. *Let conditions* 1) *to* 5) *and* 7) *be fulfilled; if a positive number L is given, then for every point* $\bar{x} \in E_n$ *and for every number* $\eta > 0$ *a neighbourhood* $S_\mu(\bar{x}) = \{x \in E_n : |x - \bar{x}|_q < \mu\}$ *exists such that*

$$\sup_{\langle 0,L \rangle} \|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\|_q \leqq (\varphi_2(d, \varepsilon) + \eta) \, \|x_0^{(1)} - x_0^{(2)}\|_q$$

where $x^{(1)}(t)$, $x^{(2)}(t)$ *are solutions of* (1), $y^{(1)}(t)$, $y^{(2)}(t)$ *are solutions of* (2) *with the same nonstochastic initial values* $x_0^{(1)}, x_0^{(2)}$ *from* $S_\mu(\bar{x})$. *The function* $\varphi_2(d, \varepsilon)$ *is given by* (2.31).

Proof. Let a point $\bar{x}$ be given. For the sake of brevity the expression on the right-hand side of (1,9) will be denoted by $\varphi_1(d, \varepsilon)$. We can choose $\varphi_1(d, \varepsilon)$ as a non-increasing function of d. For $\bar{x}$ and for a given $\eta > 0$ a number $\mu > 0$ exists such that

$$\begin{aligned} &\varphi_1(|x|_q, \varepsilon) < \varphi_1(|\bar{x}|_q, \varepsilon) + \eta \quad &\text{for} \quad |x|_q \geqq |\bar{x}|_q - \mu \\ &\psi(\delta, |x|_q, \varepsilon) < \psi(\delta, |\bar{x}|_q, \varepsilon) + \eta \quad &\text{for} \quad |x|_q \geqq |\bar{x}|_q - \mu \\ &\chi(\delta, |x|_q, \varepsilon) < \chi(\delta, |\bar{x}|_q, \varepsilon) + \eta \quad &\text{for} \quad |x|_q \geqq |\bar{x}|_q - \mu\,. \end{aligned} \tag{2.1}$$

Let $x_0^{(1)}, x_0^{(2)}$ be initial nonstochastic values from $S_\mu(\bar{x})$ and $x^{(i)}(t)$, $y^{(i)}(t)$, $i = 1, 2$ be defined as in the Lemma. We shall establish an estimate for

$$\begin{aligned} \|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\|_q \leqq & \left\| \int_0^t (a(\tau, x^{(1)}(\tau), \varepsilon) - a(\tau, x^{(2)}(\tau), \varepsilon) - \right. \\ & \left. - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0)) \, d\tau \right\|_q + \\ & + \left\| \int_0^t (B(\tau, x^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon)) \, dw_\varepsilon(\tau) \right\|_q . \end{aligned} \tag{2.2}$$

Let τ_i be a finite sequence of numbers $0 = \tau_0 < \tau_1 < \ldots < \tau_s = t$, $\max(\tau_{i+1} - \tau_i) \leqq \delta$, $s < 1 + L/\delta$. We can estimate the first term on the right-hand side of (2,2) by

$$(2.3)\quad \left\| \int_0^t (a(\tau, x^{(1)}(\tau), \varepsilon) - a(\tau, x^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau \right\|_q \leqq$$

$$(2.4)\quad \leqq \left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y^{(1)}(\tau), \varepsilon) - a(\tau, y^{(2)}(\tau), \varepsilon) - a(\tau, x^{(1)}(\tau), \varepsilon) + a(\tau, x^{(2)}(\tau), \varepsilon))\, d\tau \right\|_q +$$

$$(2.5)\quad + \left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y^{(1)}(\tau), \varepsilon) - a(\tau, y^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau_i), \varepsilon) + a(\tau, y^{(2)}(\tau_i), \varepsilon))\, d\tau \right\|_q +$$

$$(2.6)\quad + \left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y^{(1)}(\tau_i), \varepsilon) - a(\tau, y^{(2)}(\tau_i), \varepsilon) - a(\tau, y^{(1)}(\tau_i), 0) + a(\tau, y^{(2)}(\tau_i), 0))\, d\tau \right\|_q +$$

$$(2.7)\quad + \left\| \sum \int_{\tau_i}^{\tau_{i+1}} (a(\tau, y^{(1)}(\tau_i), 0) - a(\tau, y^{(2)}(\tau_i), 0) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau \right\|_q .$$

Similarly as in [2], the condition 7) implies that

$$(2.8)\quad |a(t, x, \varepsilon) - a(t, y, \varepsilon) - a(t, u, \varepsilon) + a(t, v, \varepsilon)|_q \leqq \leqq K|x - y|_q\, |x - u|_q\,; \quad v = u + y - x\,.$$

Employing this inequality we can estimate (2,4) by

$$(2.9)\quad K \sqrt[2q]{E \left| \int_0^t |y^{(1)}(\tau) - y^{(2)}(\tau)|_q\, |x^{(1)}(\tau) - y^{(1)}(\tau)|_q\, d\tau \right|^{2q}} + K \int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q\, d\tau\,.$$

Since $y^{(i)}(t)$ are solutions of (2) and 1) holds we have

$$(2.10)\quad |y^{(1)}(t) - y^{(2)}(t)|_q \leqq e^{Kt}|y^{(1)}(0) - y^{(2)}(0)|_q = e^{Kt}|x_0^{(1)} - x_0^{(2)}|_q\,.$$

From (2.10) follows that (2.9) cannot exceed

$$(2.11) \qquad K\|x_0^{(1)} - x_0^{(2)}\|_q \sqrt[2q]{E\left|\int_0^t e^{K\tau} \left|x^{(1)}(\tau) - y^{(1)}(\tau)\right|_q d\tau\right|^{2q}} +$$

$$+ K\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q d\tau \leqq$$

$$\leqq (e^{Kt} - 1)(\varphi_1(|\bar{x}|, \varepsilon) + \eta)\|x_0^{(1)} - x_0^{(2)}\|_q +$$

$$+ K\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q d\tau .$$

The last inequality results from Hölder inequality, Lemma 1. and (2.1). Recalling (2.8) we deduce that both (2.5) and (2.7) cannot exceed

$$(2.12) \qquad K\sqrt[2q]{\left|\sum \int_{\tau_i}^{\tau_{i+1}} |y^{(1)}(\tau) - y^{(2)}(\tau)|_q \, |y^{(1)}(\tau) - y^{(1)}(\tau_i)|_q d\tau\right|^{2q}} +$$

$$+ K\left|\sum \int_{\tau_i}^{\tau_{i+1}} (y^{(1)}(\tau) - y^{(2)}(\tau) - y^{(1)}(\tau_i) + y^{(2)}(\tau_i)) d\tau\right|_q .$$

By (2.10) we obtain an estimate for (2.12) as

$$(2.13) \qquad K\sum \left|\int_{\tau_i}^{\tau_{i+1}} e^{Kt} |y^{(1)}(\tau) - y^{(1)}(\tau_i)|_q d\tau\right| \|x_0^{(1)} - x_0^{(2)}\|_q +$$

$$+ K\left|\sum \int_{\tau_i}^{\tau_{i+1}} (y^{(1)}(\tau) - y^{(2)}(\tau) - y^{(1)}(\tau_i) + y^{(2)}(\tau_i)) d\tau\right|_q ,$$

and using condition 5) we find that (2.13) is less than

$$(2.14) \qquad (e^{Kt} - 1)\,\theta(\delta)\,\|x_0^{(1)} - x_0^{(2)}\|_q +$$

$$+ K\sum \int_{\tau_i}^{\tau_{i+1}} |y^{(1)}(\tau) - y^{(2)}(\tau) - y^{(1)}(\tau_i) + y^{(2)}(\tau_i)|_q d\tau .$$

Since

$$|y^{(1)}(t) - y^{(2)}(t) - y^{(1)}(\tau_i) + y^{(2)}(\tau_i)|_q \leqq$$

$$\leqq K\int_{\tau_i}^t |y^{(1)}(\tau) - y^{(2)}(\tau)|_q d\tau \leqq e^{K\tau_i}(e^{K(t-\tau_i)} - 1)\,|x_0^{(1)} - x_0^{(2)}|_q$$

we obtain that (2.14) cannot exceed

$$(2.15) \qquad (e^{KL} - 1)(\theta(\delta) + (e^{K\delta} - 1))\,\|x_0^{(1)} - x_0^{(2)}\|_q .$$

Before turning to the term (2.6), we prove the inequality

$$(2.16) \qquad \left| \int_{t_1}^{t_2} (a(\tau, x, \varepsilon) - a(\tau, y, \varepsilon) - a(\tau, x, 0) + a(\tau, y, 0)) \, d\tau \right|_q \leqq$$
$$\leqq |x - y|_q \sqrt{[8K(\psi(\delta, |\bar{x}|, \varepsilon) + \eta)(t_2 - t_1)]}$$

if $x, y \in S_\mu(\bar{x})$ and $t_1 < t_2$, $t_2 \leqq t_1 + \delta$. By conditions 8), (2,1), 7) we obtain

$$2(\psi(\delta, |\bar{x}|_q, \varepsilon) + \eta) \geqq \left| \int_{t_1}^{t_2} (a(\tau, x, \varepsilon) - a(\tau, y, \varepsilon) - a(\tau, x, 0) + a(\tau, y, 0)) \, d\tau \right|_q =$$
$$= \left| \int_0^1 \int_{t_1}^{t_2} \left(\frac{\partial a}{\partial x} (\tau, y + \lambda(x - y), \varepsilon) - \frac{\partial a}{\partial x} (\tau, y + \lambda(x - y), 0) \right) (x - y) \, d\tau \, d\lambda \right|_q \geqq$$
$$\geqq \left| \int_{t_1}^{t_2} \left(\frac{\partial a}{\partial x} (\tau, y, \varepsilon) - \frac{\partial a}{\partial x} (\tau, y, 0) \right) (x - y) \, d\tau \right|_q - K(t_2 - t_1) |x - y|_q^2$$

for $y \in S_{\mu/2}(\bar{x})$, $|x|_q \geqq |\bar{x}|_q - \mu$. It follows that for arbitrary vectors $y \in S_{\mu/2}(\bar{x})$, u, $|u|_q = 1$ we have

$$\left| \int_{t_1}^{t_2} \left(\frac{\partial a}{\partial x} (\tau, y, \varepsilon) - \frac{\partial a}{\partial x} (\tau, y, 0) \right) u \, d\tau \right|_q \leqq$$
$$\leqq \inf_{\varrho > 0} \left[\frac{2(\psi + \eta)}{\varrho} + K(t_2 - t_1) \varrho \right] \leqq \sqrt{[8(\psi + \eta)(t_2 - t_1) K]} \, .$$

This last inequality yields

$$\left| \int_{t_1}^{t_2} (a(\tau, x, \varepsilon) - a(\tau, y, \varepsilon) - a(\tau, x, 0) + a(\tau, y, 0)) \, d\tau \right|_q \leqq$$
$$\leqq \int_0^1 \left| \int_{t_1}^{t_2} \left(\frac{\partial a}{\partial x} (\tau, y + \lambda(x - y), \varepsilon) - \frac{\partial a}{\partial x} (\tau, y + \lambda(x - y), 0) \right) (x - y) \, d\tau \right|_q d\lambda \leqq$$
$$\leqq |x - y|_q \sqrt{[8K(\psi + \eta)(t_2 - t_1)]} \, ,$$

which proves inequality (2.16).

By (2.16) we obtain an estimate for (2.6),

$$(2.17) \qquad \sum |y^{(1)}(\tau_i) - y^{(2)}(\tau_i)|_q \sqrt{[8K(\psi + \eta)(\tau_{i+1} - \tau_i)]} \leqq$$
$$\leqq \frac{4}{K} (e^{K(L+\delta)} - 1) \, \|x_0^{(1)} - x_0^{(2)}\|_q \sqrt{\left[\frac{(\psi + \eta)K}{\delta} \right]} .$$

By (2.3) to (2.7), (2.11) to (2.15) and (2.17) we have

$$(2.18)\quad \left\|\int_0^t (a(\tau, x^{(1)}(\tau), \varepsilon) - a(\tau, x^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau\right\|_q \leqq$$

$$\leqq A_3\left(\varphi_1(|\bar{x}|_q, \varepsilon) + \eta + \theta(\delta) + \delta + \sqrt{\frac{\psi(\delta, \alpha(|\bar{x}|_q), \varepsilon) + \eta}{\delta}}\right)\|x_0^{(1)} - x_0^{(2)}\|_q +$$

$$+ K\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q\, d\tau\,.$$

Further we shall deal with the last term of (2.2). By formula (6.1) and condition 3) we have

$$(2.19)\quad \left\|\int_0^t (B(\tau, x^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon))\, dw_\varepsilon(\tau)\right\|_q \leqq$$

$$\leqq H \sqrt[2q]{\int_0^t E|B(\tau, x^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon)|_q^{2q}\, dF_\varepsilon(\tau)} \leqq$$

$$(2.20)\quad \leqq H \sqrt[2q]{\left[\sum \int_{\tau_i}^{\tau_{i+1}} E|B(\tau, y^{(1)}(\tau), \varepsilon) - B(\tau, y^{(2)}(\tau), \varepsilon) - B(\tau, x^{(1)}(\tau), \varepsilon) + B(\tau, x^{(2)}(\tau), \varepsilon)|_q^{2q}\, dF(\tau)\right]} +$$

$$(2.21)\quad + H \sqrt[2q]{\left[\sum \int_{\tau_i}^{\tau_{i+1}} E|B(\tau, y^{(1)}(\tau), \varepsilon) - B(\tau, y^{(2)}(\tau), \varepsilon) - B(\tau, y^{(1)}(\tau_i), \varepsilon) + B(\tau, y^{(2)}(\tau_i), \varepsilon)|_q^{2q}\, dF(\tau)\right]} +$$

$$(2.22)\quad + H \sqrt[2q]{\left[\sum \int_{\tau_i}^{\tau_{i+1}} E|B(\tau, y^{(1)}(\tau_i), \varepsilon) - B(\tau, y^{(2)}(\tau_i), \varepsilon)|_q^{2q}\, dF_\varepsilon(\tau)\right]}$$

where $H = 2n \exp\{(4^{q-1}/q)(F(L) - F(0))\}$. The inequality (2.8) holds for the matrix B, too, only the norm used is the norm of matrices now. By such an inequality we proceed analogously as in the case of (2.4). For (2.20) we obtain an estimate

$$(2.23)\quad A_4(\varphi_1(|\bar{x}|_q, \varepsilon) + \eta)\,\|x_0^{(2)} - x_0^{(1)}\|_q +$$

$$+ HK \sqrt[2q]{\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q^{2q}\, dF(\tau)}$$

and for (2.21)

$$(2.24)\quad A_5(\theta(\delta) + \delta)\|\, x_0^{(1)} - x_0^{(2)}\|_q\,.$$

Analogously as we needed (2.16) in the previous case we must prove the following inequality now

$$(2.25) \qquad \sqrt[2q]{\int_{t_1}^{t_2} |B(\tau, x, \varepsilon) - B(\tau, y, \varepsilon)|_q^{2q} \, dF(\tau)} \leqq$$
$$\leqq 2 \sqrt{[K(\chi(t_2 - t_1, |\bar{x}|_q, \varepsilon) + \eta) \sqrt[2q]{(F(t_2) - F(t_1))}]} \, |x - y|_q \, .$$

Similarly as in the first part of the proof of (2.16) we obtain

$$2(\chi(t_2 - t_1, |\bar{x}|_q, \varepsilon) + \eta) \geqq \sqrt[2q]{\int_{t_1}^{t_2} |B(\tau, x, \varepsilon) - B(\tau, y, \varepsilon)|_q^{2q} \, dF_\varepsilon(\tau)} \geqq$$
$$\geqq \sqrt[2q]{\int_{t_1}^{t_2} \left| \sum_i \frac{\partial B_i}{\partial x} (\tau, y, \varepsilon)(x - y) \right|_q^{2q} dF_\varepsilon(\tau)} - \tfrac{1}{2} K |x - y|_q^2 \sqrt[2q]{[F(t_2) - F(t_1)]} \, ,$$

where B_i is the i-th row of B, such that

$$\sqrt[2q]{\int_{t_1}^{t_2} \left| \sum_i \frac{\partial B_i}{\partial x} (\tau, y, \varepsilon) \, u \right|_q^{2q} dF_\varepsilon(\tau)} \leqq$$
$$\leqq 2 \sqrt{[K(\chi(t_2 - t_1, |\bar{x}|_q, \varepsilon) + \eta) \sqrt[2q]{(F(t_2) - F(t_1))}]}$$

for arbitrary vectors $y \in S_{\mu/2}(\bar{x})$, u, $|u|_q = 1$; this last inequality yields (2.25).

By (2.25) we obtain easily an estimate for (2.22),

$$(2.26) \qquad 2He^{K(L+\delta)} \sqrt{[K(\chi(\delta, \alpha(|\bar{x}|_q), \varepsilon) + \eta) \sqrt[2q]{\bar{\theta}(\delta)}]} \sqrt[2q]{\left[\frac{1}{\delta K q}\right]} \, \|x_0^{(1)} - x_0^{(2)}\|_q$$

where $\bar{\theta}(\delta) = \max\limits_{\langle 0, L \rangle} (F(t + \delta) - F(t))$.

From (2.19) to (2.24) and (2.26) we have

$$(2.27) \qquad \left\| \int_0^t (B(\tau, x^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon)) \, dw_\varepsilon(\tau) \right\|_q \leqq$$
$$\leqq A_6 \left(\varphi_1(|\bar{x}|_q, \varepsilon) + \eta + \theta(\delta) + \delta + \frac{\sqrt{[(\chi(\delta, \alpha(|\bar{x}|_q), \varepsilon) + \eta) \sqrt[2q]{\bar{\theta}(\delta)}]}}{\sqrt[2q]{\delta}} \right) \|x_0^{(1)} -$$
$$- x_0^{(2)}\|_q + HK \sqrt[2q]{\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q^{2q} \, dF(\tau)} \, .$$

By (2.2), (2.18) and (2.27) we obtain

$$(2.28) \qquad \|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\|_q \leqq (A_7 \eta + Q) \, \|x_0^{(1)} - x_0^{(2)}\|_q +$$
$$+ K \int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q \, d\tau +$$
$$+ HK \sqrt[2q]{\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|_q^{2q} \, dF(\tau)}$$

where A_7 is a constant and

(2.29)

$$Q = A_8\left(\varphi_1(|\bar{x}|_q, \varepsilon) + \theta(\delta) + \delta + \sqrt{\frac{\psi(\delta, \alpha(|\bar{x}|_q), \varepsilon)}{\delta}} + \sqrt{\frac{\chi(\delta, \alpha(|\bar{x}|_q), \varepsilon)\sqrt[2q]{\theta(\delta)}}{\sqrt[q]{\delta}}}\right).$$

Analogously as in Lemma 2 in [3] we obtain

$$\|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\|_q \leqq A_9(A_7\eta + Q))\,\|x_0^{(1)} - x_0^{(2)}\|_q\,. \tag{2.30}$$

If we put

$$\varphi_2(d, \varepsilon) = A_9 Q \tag{2.31}$$

we see that the inequality in the statement of Lemma 2 is a consequence of inequality (2.30)

We now pass to Theorem 1. First we express the function Q defined by (2.29) more specifically by using (1.9).

$$Q = A_{10}\left[\theta(\delta_1(d)) + \frac{\psi(\delta_1(d), \alpha(d), \varepsilon)}{\delta_1(d)} + \frac{\chi(\delta_1(d), \alpha(d), \varepsilon)}{\sqrt[2q]{\delta_1(d)}} + \right. \tag{2.32}$$

$$\left. + \theta(\delta_2(d)) + \delta_2(d) + \sqrt{\frac{\psi(\delta_2(d), \alpha(d), \varepsilon)}{\delta_2(d)}} + \sqrt{\frac{\chi(\delta_2(d), \alpha(d), \varepsilon)\sqrt[2q]{\theta(\delta_2(d))}}{\sqrt[q]{\delta_2(d)}}}\right]$$

where $L = 2S$ now and $\delta_1(d) > 0$, $\delta_2(d) > 0$ are arbitrary functions with $\delta_1(d) \to 0$, $\delta_2(d) \to 0$ as $d \to \infty$.

Theorem 1. *Let the conditions* 1) *to* 5) *and* 7) *be fulfilled, let* $a(t, x, \varepsilon)$ *and* $B(t, x, \varepsilon)$ *be periodic with period* T *and let* $w_\varepsilon(t)$ *have periodic increments with period* T. *If the solutions of* (2) *are weakly uniformly exponentially stable and if exist functions* $\delta_1(d) > 0$, $\delta_2(d) > 0$ *such that* $\delta_1(d) \to 0$, $\delta_2(d) \to 0$ *for* $d \to \infty$ *and*

$$\varphi_2(d, \varepsilon) + \beta(d) < 1 \quad \textit{for all nonnegative } d \textit{ and } \quad 0 < \varepsilon \leqq \varepsilon_0 \tag{2.33}$$

then the solutions of (1) *are weakly uniformly exponentially stable for* $0 \leqq \varepsilon \leqq \varepsilon_0$.

The function $\varphi_2(d, \varepsilon)$ is given by (2.31) (Q by (2.32)), $\beta(d)$ corresponds to the weak exponential stability of (2) (Definition 1).

Proof. We put $\bar{\beta}_\varepsilon(d) = \sup\,(\varphi_2(v, \varepsilon) + \beta(v))$ for $0 \leqq v \leqq d$ and by (2,33) we obtain $0 \leqq \bar{\beta}_\varepsilon(d) < 1$.

Let u, v be two points from E_n with $|u|_q \leqq d$, $|v|_q \leqq d$. Take an abscissa $u + \lambda(v - u)$, $0 \leqq \lambda \leqq 1$. According to Lemma 2 we can choose a finite number of points $x_0^{(k)}$ lying on this abscissa such that $\|x^{(i+1)}(t) - x^{(i)}(t) - y^{(i+1)}(t) + y^{(i)}(t)\|_q \leqq \leqq \max_{j=i,i+1}\,(\varphi_2(|x_0^{(j)}|_q, \varepsilon) + \eta)\,\|x_0^{(i)} - x_0^{(i+1)}\|_q$ for $t \in \langle 0, 2S\rangle$ where $x^{(i)}(t)$ are solutions of (1) with the initial values $x_0^{(i)}$ and $y^{(i)}(t)$ are solutions of (2) with the initial values $x_0^{(i)}$. Since the solutions of (2) are weakly uniformly exponentially stable, we have $\|x^{(i+1)}(t) - x^{(i)}(t)\|_q \leqq \max_{j=i,i+1} (\varphi_2(|x_0^{(j)}|_q, \varepsilon) + \beta(|x_0^{(j)}|_q) + \eta) \|x_0^{(i)} - x_0^{(i+1)}\|_q$ for $t \in \langle S, 2S \rangle$. By these inequalities it follows that

$$\|u(t) - v(t)\|_q \leqq \sum_i \|x^{(i+1)}(t) - x^{(i)}(t)\|_q \leqq \sup_{|x|_q \leqq d} (\varphi_2(|x|_q, \varepsilon) + \beta(|x|_q) + \eta) \cdot$$

$$\cdot \sum_i \|x_0^{(i+1)} - x_0^{(i)}\|_q = (\bar{\beta}_\varepsilon(d) + \eta) \|u - v\|_q \quad \text{for} \quad t \in \langle S, 2S \rangle$$

where $u(t)$, $v(t)$ are solutions of (1) with initial values $u(0) = u$ and $v(0) = v$, respectively. The last inequality means that the solutions of (1) are weakly uniformly exponentially stable, since η is an arbitrary positive number.

The proof of the stability of solutions of (1) is the same as that in the end of the proof of Theorem 1 in [1].

Remark 1. At the first glance it seems very difficult to find $\varepsilon_0 > 0$ fulfilling (2.33); however, if the functions $\psi(\delta, d, \varepsilon)$, $\chi(\delta, d, \varepsilon)$ are continuous in ε, $\chi(\delta, d, 0) = 0$ and if we do not need and explicite expression for ε_0 but only an existence statement is sufficient, then it suffices to find $\varepsilon_1 > 0$ such that (2.33) holds for d large enough, i.e. for $d \geqq d_0 \geqq 0$, where d_0 is a number. Obviously, we can choose $\varepsilon_2 > 0$ such that (2.33) holds on the compact set $0 \leqq d \leqq d_0$. Since $\psi(\delta, d, \varepsilon)$ and $\chi(\delta, d, \varepsilon)$ are nonincreasing functions of ε, the inequality (2.33) holds for all d, and $0 \leqq \varepsilon \leqq \leqq \varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. From these considerations the sufficient conditions follow, i.e.,

$$\frac{\theta(\delta_1(d))}{1 - \beta(d)} \to 0\,, \quad \frac{\psi(\delta_1(d), \alpha(d), \varepsilon)}{(1 - \beta(d))\, \delta_1(d)} \to 0\,, \quad \frac{\chi(\delta_1(d), \alpha(d), \varepsilon)}{(1 - \beta(d)) \sqrt[2q]{\delta_1(d)}} \to 0 \quad \text{as} \quad d \to \infty\,,$$

$$\frac{\theta(\delta_2(d))}{1 - \beta(d)} \to 0\,, \quad \frac{\delta_2(d)}{1 - \beta(d)} \to 0\,, \quad \frac{\psi(\delta_2(d), \alpha(d), \varepsilon)}{(1 - \beta(d))^2\, \delta_2(d)} \to 0\,,$$

$$\frac{\chi(\delta_2(d), \alpha(d), \varepsilon)}{(1 - \beta(d))^2} \sqrt[2q]{\frac{\theta(\delta_2(d))}{\delta_2^2(d)}} \to 0 \quad \text{as} \quad d \to \infty\,.$$

If $F(t)$ is linear and $h(t)$ is bounded, then it is obvious that we can simplify this system of conditions. Thus, the following sufficient condition for (2.33) is obtained.

Let a number $\varepsilon > 0$ and a function $\delta(d)$, $\delta(d) > 0$ exist such that

$$(2.34) \quad \frac{\delta(d)}{1 - \beta(d)} \to 0\,, \quad \frac{\psi(\delta(d), \alpha(d), \varepsilon)}{(1 - \beta(d))^2\, \delta(d)} \to 0\,, \quad \frac{\chi(\delta(d), \alpha(d), \varepsilon)}{(1 - \beta(d))^2 \sqrt[2q]{\delta(d)}} \to 0 \quad \text{as} \quad d \to \infty$$

then a number $\varepsilon_0 > 0$ exists such that (2.33) is satisfied.

Further we shall consider a more special case. We shall assume that the functions

ψ, χ can be estimated by the product of functions which depend either on δ, ε or on d, ε only, i.e.,

$$(2.35)\qquad \psi(\delta, \alpha(d), \varepsilon) \leqq \psi_1(\delta, \varepsilon)\, \psi_2(d, \varepsilon)\,, \quad \chi(\delta, \alpha(d), \varepsilon) \leqq \chi_1(\delta, \varepsilon)\, \chi_2(d, \varepsilon)\,,$$

ψ_1, χ_1 are continuous in δ, ε.

Remark 2. Let (2.35) be fulfilled and let a number $\varepsilon > 0$ exist such that one of the functions $\psi_1(1/d, \varepsilon)\, d$, $\psi_2(d, \varepsilon)/(1 - \beta(d))^3$ tends to zero as $d \to \infty$ and one of the functions $\chi_1(1/d, \varepsilon) \sqrt[2q]{d}$, $\chi_2(d, \varepsilon)/(1 - \beta(d))^{2+1/2q}$ tends to zero as $d \to \infty$; then a number $\varepsilon_0 > 0$ exists such that (2.33) holds.

For proving this statement we have to use the fact that deriving Lemma 1 we estimated (1.2) and (1.8) separately so that we could choose different $\delta^{(i)}$ in (1.5) and in (1.8); this means that in (2.34) we can choose different functions $\delta^{(1)}(d)$, $\delta^{(2)}(d)$ in conditions for $\psi(\delta(d), \alpha(d), \varepsilon)\,(1 - \beta(d))^{-2}\, \delta(d)^{-1} \to 0$ and for $\chi(\delta(d), \alpha(d), \varepsilon) \times (1 - \beta(d))^{-2}\, \delta(d)^{-1/2q} \to 0$.

Finally we shall show how to estimate the function $\beta(d)$ for a wide class of differential equations whose solutions are weakly uniformly exponentially stable.

Remark 3. Let a bounded function $f(t, r)$ defined for $r \geqq 0$ exist such that $f(t, r) \leqq 0$, $(\partial^2 f/\partial r^2)(t, r) \geqq 0$ and

$$(2.36)\qquad \sum_i (y_i^{(1)} - y_i^{(2)})^{2q-1}\,(a_i(t, y^{(1)}, 0) - a_i(t, y^{(2)}, 0)) \leqq$$
$$\leqq |y^{(2)} - y^{(1)}|_q^{2q-1}\, f(t, |y^{(2)} - y^{(1)}|_q)$$

holds. Condition (2.36) means that an inequality $|y^{(1)}(t) - y^{(2)}(t)|_q \leqq |r(t)|$ holds where $y^{(i)}(t)$ are solutions of (2) and $r(t)$ is the solution of

$$(2.37)\qquad \dot{r} = f(t, r)$$

with initial condition $r(t_0) = |y^{(2)}(t_0) - y^{(1)}(t_0)|_q$.

Obviously the function $\beta(d)$ for (2) can be estimated by the function $\tilde{\beta}(d)$ for equation (2.37). The variational equation of (2.37) is $\delta\dot{r} = (\partial f(t, r(t))/\partial r)\, \delta r$ and the solution of the variational equation fulfils $\delta r(t)/\delta r(0) = \exp\{\int_0^L (\partial f(t, r(t))/\partial r)\, dt\}$, where $r(t)$ is a solution of (2.37). Since $f(t, r) \leqq 0$ and $(\partial f/\partial r)(t, r)$ is nondecreasing, we have $\delta r(L)/\delta r(0) \leqq \exp\{\int_0^L (\partial f(t, r_0)/\partial r)\, dt\}$ and by definition of $\beta(d)$ we obtain $\beta(d) \leqq \exp\{\int_0^L (\partial f(t, d)/\partial r)\, dt\}$. Since $\int_0^L (\partial f(t, d)/\partial r)\, dt \to 0$ as $d \to \infty$, the function $(1 - \beta(d))^{-1}$ can be estimated by $(\int_0^L (\partial f(t, d)/\partial r)\, dt)^{-1}$ as $d \to \infty$, and we can replace $(1 - \beta(d))^{-1}$ by $(\int_0^L (\partial f(t, d)/\partial r)\, dt)^{-1}$ in (2.34) and in Remark 1.

Now, we shall turn to the investigation of the boundedness of solutions of (1). We shall show that if we take the total boundedness of solutions of (2) into account then we obtain an explicite estimate for the solutions of (1). In the sequel the index ε in (1) can be omitted.

Theorem 2. *Let conditions* 1), 2) *be fulfilled, let the solutions of* (2) *be totally bounded and let constants* $\mu > 0$, $\mu < (1/\sqrt[2q]{2})\,\alpha/2$, $L \geqq S$ *exists such that* $\|x(t_0 + L) - y(t_0 + L)\|_q < \mu$ *for some* $q > 1$, *where* $x(t)$ *and* $y(t)$ *are solutions of* (1) *and* (2), *respectively, which have the same nonstochastic initial value; then a distribution function* $F^+(\theta)$ *depending on one variable* θ *only exists such that* $\int \theta^2\, dF^+(\theta) < \infty$, $\int_d^\infty \theta^{2i}\, dF^+(\theta) \leqq e_i/d^{2q-2i-1}$, $i = 1, \ldots, q-1$, *where* e_i *are some constants and* $\int_{|\lambda|_q \leqq \Theta} dF(t_0, \lambda_1, \ldots, \lambda_n) \geqq F^+(\Theta)$ *for every* $\Theta \geqq 0$ *implies that* $\int_{|\lambda|_q \leqq \Theta} dF(t_0 + L, \lambda_1, \ldots, \lambda_n) \geqq F^+(\Theta)$, *where* $F(t, \lambda_1, \ldots, \lambda_n)$ *is the distribution function of an arbitrary solution of* (1).

Proof. We shall seek the distribution function $F^+(\theta)$ as a solution of the following inequality

$$(3.1)\qquad \int_0^{\zeta+\alpha} \frac{\mu^{2q}}{(\theta - \zeta)^{2q}}\, dF^+(\theta') + \int_{\zeta+\alpha}^{\theta} \frac{\mu^{2q}}{(\theta - \theta' + \alpha)^{2q}}\, dF^+(\theta') \leqq$$

$$\leqq \left(F^+\left(\theta + \frac{\alpha}{2}\right) - F^+(\theta)\right)\left(1 - \frac{\mu^{2q}4^q}{\alpha^{2q}}\right) \quad \text{for} \quad \theta > \zeta + \alpha\,.$$

This inequality will be satisfied, if

$$(3.2)\qquad p_{k+1} = \frac{4^q \mu^{2q}}{\alpha^{2q} - 4^q \mu^{2q}} \left(\frac{p_0}{(k+2)^{2q}} + \sum_{l=1}^{k} \frac{p_l}{(k - l + 2)^{2q}}\right)$$

where $F^+(\theta) = \sum p_l$ for such l that $\zeta + (l+2)\,\alpha/2 \leqq \theta$.

Denote $\lambda = 4^q\mu^{2q}/(\alpha^{2q} - 4^q\mu^{2q})$; then for $p_0 = 0$ we obtain the solution of (3.2),

$$p_k = p_1 \left(\frac{\lambda}{k^{2q}} + \lambda^2 \sum_{\substack{l_1 + l_2 = k+1 \\ l_i \geqq 2}} \frac{1}{l_1^{2q}} \frac{1}{l_2^{2q}} + \ldots + \lambda^s \sum_{\substack{\sum_1^s l_i = k+s-1 \\ l_i \geqq 2}} \frac{1}{l_1^{2q}} \cdots \frac{1}{l_s^{2q}} + \right.$$

$$\left. + \ldots + \lambda^{k-1} \frac{1}{4^{(k-1)q}}\right) \quad \text{for} \quad k > 1$$

where p_1 must be choosen such that $\sum p_k = 1$. The convergence of this series follows from further estimates. We have

$$\int_{N(\alpha/2)+\xi} \theta^{2i}\, dF^+(\theta) \leqq (\zeta + \tfrac{3}{2}\alpha)^{2i} \sum_{N-2}^{\infty} k^{2i} p_k \leqq (\zeta + \tfrac{3}{2}\alpha)^{2i}\, p_1 \left(\sum_{N-2}^{\infty} \frac{\lambda}{l_1^{2(q-i)}} + \right.$$

$$\left. + \lambda^2 \sum \frac{1}{l_1^{2(q-i)}} \frac{1}{l_2^{2(q-i)}} + \ldots + \lambda^s \sum \frac{1}{l_1^{2(q-i)}} \cdots \frac{1}{l_s^{2(q-i)}} + \ldots\right),$$

where in the s-th term, the summation is extended over l_i's satisfying $\sum_{i=1}^{s} l_i \geqq N + s - 3$, $l_i \geqq 2$.

Since

$$\sum_{s=1}^{\infty} \lambda^s \sum_{l_1+\ldots+l_s \geqq s+N-3, l_i \geqq 2} \frac{1}{l_1^\nu} \ldots \frac{1}{l_s^\nu} \leqq \sum \frac{s^\nu}{(\nu-1)^s} \frac{\lambda^s}{(N-3)^{\nu-1}} \leqq \frac{c_\nu}{(N-3)^{\nu-1}} \frac{\lambda}{\nu-1}$$

for $\nu > 1$ where c_ν are some constant which are independent of N, we obtain

(3.3)

$$\int_d^\infty \theta^{2i} \, \mathrm{d}F^+(\theta) \leqq \frac{(\frac{3}{2}\alpha + \zeta)^{2i}}{2^{2q-2i-1}} \alpha^{2q-2i-1} \frac{\lambda}{2q-2i-1} c_{2q-2i} \frac{1}{(d-\zeta-2\alpha)^{2q-2i-1}} \leqq$$

$$\leqq \frac{4^{1+i}(\frac{3}{2}\alpha + \zeta)^{2i} \alpha^{2q-2i-1}}{(\alpha^{2q} - 4^q\mu^{2q})(2q-2i-1)} c_{2q-2i}\mu^{2q} \frac{1}{(d-\zeta-2\alpha)^{2q-2i-1}} \quad \text{for} \quad d > \zeta + 2\alpha \, .$$

Let $F(t_0, \lambda_1, \ldots, \lambda_n)$ be a given distribution function fulfilling the assumptions of Theorem 2. Let $x(t)$ be a solution of (1) with this initial distribution function. Put $\bar{F}(t, \theta) = P(|x(t)|_q \leqq \theta)$. With respect to the assumptions we have $\bar{F}(t_0, \theta) \geqq F^+(\theta)$. Choose $\bar{\theta} > \zeta + \alpha$ and define $\hat{F}(t_0, \theta) = F^+(\theta)$ for $\theta < \bar{\theta}$ and $\hat{F}(t_0, \theta) = \bar{F}(t_0, \theta)$ for $\theta \geqq \bar{\theta}$. For the sake of brevity we put $\bar{F}(\theta) = \bar{F}(t_0, \theta)$.

Since the solutions of (2) are totally bounded and the estimate $\|x(t_0 + L) - - y(t_0 + L)\|_q < \mu$ holds, we obtain

$$P(|x(t_0+L)|_q \leqq \bar{\theta}) \geqq \int_0^{\bar{\theta}+0} P(|x(t_0+L)|_q \leqq \bar{\theta} \,\|\, |x(t_0)|_q = \theta) \, \mathrm{d}\bar{F}(\theta) +$$

$$+ \int_{\bar{\theta}+0}^{\bar{\theta}+\alpha/2} P(|x(t_0+L)|_q \leqq \bar{\theta} \,\|\, |x(t_0)|_q = \theta) \, \mathrm{d}\bar{F}(\theta) \geqq \bar{F}(\bar{\theta}) -$$

$$- \int_0^{\bar{\theta}+0} P(|x(t_0+L)|_q > \bar{\theta} \,\|\, |x(t_0)|_q = \theta) \, \mathrm{d}\bar{F}(\theta) + \left(1 - \frac{\mu^{2q}4^q}{\alpha^{2q}}\right) (\bar{F}(\bar{\theta} + \tfrac{1}{2}\alpha) - \bar{F}(\bar{\theta})) \geqq$$

$$\geqq \bar{F}(\bar{\theta}) - \int_0^{\zeta+0} \frac{\mu^{2q}}{(\bar{\theta}-\zeta)^{2q}} \, \mathrm{d}\bar{F}(\theta) - \int_{\zeta+0}^{\bar{\theta}+0} \frac{\mu^{2q}}{(\bar{\theta}-\theta+\alpha)^{2q}} \, \mathrm{d}\bar{F}(\theta) +$$

$$+ \left(1 - \frac{\mu^{2q}4^q}{\alpha^{2q}}\right)\left(\bar{F}\left(\bar{\theta} + \frac{\alpha}{2}\right) - \bar{F}(\bar{\theta})\right) \geqq \bar{F}(\bar{\theta}) - \int_0^{\bar{\theta}+0} \frac{\mu^{2q}}{(\bar{\theta}-\theta+\alpha(\theta))^{2q}} \, \mathrm{d}\bar{F}(\theta) +$$

$$+ \left(1 - \frac{\mu^{2q}4^q}{\alpha^{2q}}\right)\left(\bar{F}\left(\bar{\theta} + \frac{\alpha}{2}\right) - \bar{F}(\bar{\theta})\right) = \bar{F}(\bar{\theta}) - \frac{\mu^{2q}}{\alpha^{2q}} \bar{F}(\bar{\theta}) +$$

$$+ 2q \int_0^{\bar{\theta}+0} \frac{\mu^{2q}(1-\alpha'(\theta))}{(\bar{\theta}-\theta+\alpha(\theta))^{2q+1}} \bar{F}(\theta) \, \mathrm{d}\theta + \left(1 - \frac{\mu^{2q}4^q}{\alpha^{2q}}\right)\left(\bar{F}\left(\bar{\theta} + \frac{\alpha}{2}\right) - \bar{F}(\bar{\theta})\right) \geqq$$

$$\geqq \hat{F}(\bar{\theta}) - \frac{\mu^{2q}}{\alpha^{2q}} \hat{F}(\bar{\theta}) + 2q \int_0^{\bar{\theta}+0} \frac{\mu^{2q}(1-\alpha'(\theta))}{(\bar{\theta}-\theta+\alpha(\theta))^{2q+1}} \hat{F}(\theta) \, \mathrm{d}\theta +$$

$$+\left(1-\frac{\mu^{2q}4^q}{\alpha^{2q}}\right)\left(\hat{F}\left(\bar{\theta}+\frac{\alpha}{2}\right)-\hat{F}(\bar{\theta})\right)=\hat{F}(\bar{\theta})-\int_0^{\bar{\theta}+0}\frac{\mu^{2q}}{(\bar{\theta}-\theta+\alpha(\theta))^{2q}}\,\mathrm{d}\hat{F}(\theta)+$$

$$+\left(1-\frac{\mu^{2q}4^q}{\alpha^{2q}}\right)\left(\hat{F}\left(\bar{\theta}+\frac{\alpha}{2}\right)-\hat{F}(\bar{\theta})\right)\geqq\hat{F}(\bar{\theta}-0)-\int_0^{\bar{\theta}-0}\frac{\mu^{2q}}{(\bar{\theta}-\theta+\alpha(\bar{\theta}))^{2q}}\,\mathrm{d}\hat{F}(\theta)+$$

$$+\left(1-\frac{\mu^{2q}4^q}{\alpha^{2q}}\right)\left(\hat{F}\left(\bar{\theta}+\frac{\alpha}{2}\right)-\hat{F}(\bar{\theta}-0)\right)\geqq F^+(\bar{\theta}-0)-$$

$$-\int_0^{\bar{\theta}-0}\frac{\mu^{2q}}{(\bar{\theta}-\theta+\alpha(\theta))^{2q}}\,\mathrm{d}F^+(\theta)+\left(1-\frac{\mu^{2q}4^q}{\alpha^{2q}}\right)\left(F^+\left(\bar{\theta}+\frac{\alpha}{2}\right)-F^+(\bar{\theta}-0)\right)\geqq$$

$$\geqq F^+(\bar{\theta}-0)\,.$$

With respect to the definition of $\hat{F}$ we have $\hat{F}(\theta) = F^+(\theta)$ for $\theta < \bar{\theta}$ and $\hat{F}(\bar{\theta} + \alpha/2) - \hat{F}(\bar{\theta} - 0) \geqq F^+(\bar{\theta} + \alpha/2) - F^+(\bar{\theta} - 0)$. The last inequality follows from (3.1) and the function $\alpha(\theta)$ is defined by $\alpha(\theta) = \theta - \zeta$ for $0 < \theta \leqq \zeta + \alpha$ and $\alpha(\theta) = \alpha$ for $\theta > \zeta + \alpha$. These inequalities mean that $\int_{|\lambda|_q \leqq \theta} \mathrm{d}F(t_0 + L, \lambda_1, \ldots, \lambda_n) \geqq F^+(\theta)$ for $\theta > \zeta + \alpha$. (The integral as a function of θ is continuous from the right). If we have $\theta \leqq \zeta + \alpha$, then $F^+(\theta) = 0$ and the proved inequality is satisfied too. This completes the proof of Theorem 2.

Analogously as in paper [1] we shall find periodic solutions by using the weak uniform exponencial stability and the total boundedness of solutions. These two properties allow us to derive explicite estimates for periodic solutions.

Theorem 3. *Let the assumptions of Theorem* 1 *with* 6) *be fulfilled for* $q > 1$ *and let the solutions of* (2) *be totally bounded; then an* $\varepsilon_0 > 0$ *exists such that equations* (1) *have periodic solutions* $\bar{x}_\varepsilon(t)$ *with period* T *for* $0 < \varepsilon \leqq \varepsilon_0$, *equation* (2) *has a periodic solution* $\bar{y}(t)$ *with the same period and* $\lim_{\varepsilon\to 0}\sup_t \|\bar{x}_\varepsilon(t) - \bar{y}(t)\|_i = 0$, $i = 1, \ldots, q - 1$,

Remark 4. Since the solutions of (1) are weakly uniformly exponentially stable, the periodic solutions are determined uniquely in the sense that their distribution functions are determined uniquely.

Proof. First, let a positive number d be given. We shall decompose the process $w_\varepsilon(t)$ into two $w_\varepsilon(t) = w_\varepsilon^d(t) + \bar{w}_\varepsilon^d(t)$ such that $w_\varepsilon^d(t)$, $\bar{w}_\varepsilon^d(t)$ are mutually independent and are processes with independent increments. The process $w_\varepsilon^d(t)$ is that centered at its expectation part of $w_\varepsilon(t)$ for which $|w_\varepsilon^d(t + 0) - w_\varepsilon^d(t)| \leqq d$ almost everywhere. Obviously $\|\bar{w}_\varepsilon^d(t_2) - \bar{w}_\varepsilon^d(t_1)\| \to 0$ as $d \to \infty$ for every t_1, t_2. This means, that for the process $\bar{w}_\varepsilon^d$ a function F_ε^d exists such that $E|\bar{w}_\varepsilon^d(t_2) - \bar{w}_\varepsilon^d(t_1)|^2 \leqq F_\varepsilon^d(t_2) - F_\varepsilon^d(t_1)$ and $\lim_{d\to\infty} (F_\varepsilon^d(t_2) - F_\varepsilon^d(t_1)) = 0$. For the process w_ε^d we have $E|w_\varepsilon^d(t_2) - w_\varepsilon^d(t_1)|_q^{2q} \leqq F_\varepsilon(t_2) - F_\varepsilon(t_1)$.

In the following Lemma we shall use a function $\chi_d(x)$ defined by

$$\begin{aligned}
\chi_d(x) &= 0 && \text{for} \quad |x| = |x|_1 \geqq d\,, \\
\chi_d(x) &= 1 && \text{for} \quad |x| < \max(d-1, 0)\,, \\
\chi_d(x) &= d - |x| && \text{for} \quad \max(d-1, 0) \leqq |x| < d\,.
\end{aligned}$$

Lemma 3. *Let the assumptions of Theorem 3. be fulfilled and L be a positive number. Denote $x(t, d)$ the solution of*

$$(4.1)\quad x(t, d) = x_0 + \int_0^t a(\tau, x(\tau, d), \varepsilon)\,\mathrm{d}\tau + \int_0^t \chi_d(x(\tau, d))\,B(\tau, x(\tau, d), \varepsilon)\,\mathrm{d}w_\varepsilon^d(\tau)$$

and $x(t)$ the solution of (1) which has the same initial value as $x(t, d)$. Then to every $\eta > 0$ a number d_0 exists such that $E|x(t) - x(t, d)|^2 < \eta$ for $d \geqq d_0$, $t \in \langle 0, L\rangle$.

Proof of Lemma 3. Denote $\Delta(t) = x(t, d) - x(t)$. We have

$$\Delta(t) = \int_0^t (a(\tau, x(\tau) + \Delta(\tau), \varepsilon) - a(\tau, x(\tau), \varepsilon))\,\mathrm{d}\tau + \int_0^t (\chi_d(x(\tau) + \Delta(\tau)) - \chi_d(x(\tau))) \times$$

$$\times\, B(\tau, x(\tau) + \Delta(\tau), \varepsilon)\,\mathrm{d}w_\varepsilon^d(\tau) + \int_0^t (\chi_d(x(\tau)) - 1)\,B(\tau, x(\tau) + \Delta(\tau), \varepsilon)\,\mathrm{d}w_\varepsilon^d(\tau) +$$

$$+ \int_0^t (B(\tau, x(\tau) + \Delta(\tau), \varepsilon) - B(\tau, x(\tau), \varepsilon))\,\mathrm{d}w_\varepsilon^d(\tau) - \int_0^t B(\tau, x(\tau), \varepsilon)\,\mathrm{d}\bar{w}_\varepsilon^d(\tau)\,.$$

From this we obtain by 1) and 6),

$$\|\Delta(t)\| \leqq K \int_0^t \|\Delta(\tau)\|\,\mathrm{d}\tau + n^{1/2}\Bigg(\sqrt{\int_0^t g^2(\tau)\,\|\Delta(\tau)\|^2\,\mathrm{d}F(\tau)} +$$

$$+ \sqrt{\int_0^t g^2(\tau)\,\|\chi_d(x(\tau)) - 1\|^2\,\mathrm{d}F(\tau)} + K\sqrt{\int_0^t \|\Delta(\tau)\|^2\,\mathrm{d}F(\tau)} + \sqrt{\int_0^t g^2(\tau)\,\mathrm{d}F_\varepsilon^d(\tau)}\Bigg),$$

and from the last inequality there follows an estimate

$$\|\Delta(t)\| \leqq 2\Bigg(\sqrt{\int_0^t g^2(\tau)\,\|\chi_d(x(\tau)) - 1\|^2\,\mathrm{d}F(\tau)} +$$

$$+ \sqrt{\int_0^t g^2(\tau)\,\mathrm{d}F_\varepsilon^d(\tau)}\Bigg)\,n^{\frac{1}{2}} \exp\left\{2K^2t^2 + 2K^2(F(t) - F(0))n + 2n\int_0^t g^2(\tau)\,\mathrm{d}F(\tau)\right\}.$$

It suffices to prove that $\|\chi_d(x(t)) - 1\| \to 0$ and $\int_0^t g^2(\tau)\,\mathrm{d}F_\varepsilon^d(\tau) \to 0$ as $d \to \infty$. For the first term we have $E|\chi_d(x(t)) - 1|^2 \leqq \int_{|\lambda| \geqq d-1} \mathrm{d}\tilde{F}(\lambda)$, where $\tilde{F}(\lambda)$ is a distribution function of $x(t)$. Thus, the first term tends to zero. Since the increments of $F_\varepsilon^d(t)$

tend to zero as $d \to \infty$ and $\int_0^t g^2(\tau)\, dF(\tau) < \infty$, it follows that the second term tends also to zero.

Lemma 4. *Let the assumptions of Theorem 2 be fulfilled; then for every $d > \zeta$ (for ζ, α see Definition 2) equation (4,1) has a periodic solution.*

Proof. Since $\chi_d(x)\, B(t, x, \varepsilon) = 0$ for $|x| \geqq d$ and $P(|w_\varepsilon^d(t + 0) - w_\varepsilon^d(t)| > d) = 0$, we have $P(|x(t, d)| > 2d) = 0$ if $P(|x(0, d)| > 2d) = 0$. Recalling Theorem 1 we obtain

$$\|x^{(1)}(t_0 + t, d) - x^{(2)}(t_0 + t, d)\| \leqq \bar{\beta}(2d) \| x^{(1)}(t_0, d) - x^{(2)}(t_0, d)\| \quad \text{for} \quad t \geqq S$$

where $x^{(i)}(t, d)$ are solutions of (4.1) for which $x^{(i)}(t_0, d)$ are nonstochastic and $|x^{(i)}(t_0, d)| \leqq 2d$. By Theorem 2. from [1] it follows that the uniquely determined periodic solution $\bar{x}_\varepsilon(t, d)$ of (4.1) exists. Obviously, $P(|\bar{x}_\varepsilon(t, d)| > 2d) = 0$ and by Theorem 3 we have $P(|\bar{x}_\varepsilon(0, d)| \leqq \theta) \geqq F^+(\theta)$.

For the proof of Theorem 3 we shall still need another lemma.

Lemma 5. *Let a sequence of distribution functions $F_m(\lambda_1, \ldots, \lambda_n)$ be given such that $F_m(\lambda_1, \ldots, \lambda_n) \to F_0(\lambda_1, \ldots, \lambda_n)$ in all points where $F_0(\lambda_1, \ldots, \lambda_n)$ is continuous and let a distribution funkcion $F^+(\theta)$ (depending only on one variable) exist such that $F^+(\theta) = 0$ for $\theta < 0$, $\int \theta^2\, dF^+(\theta) < \infty$ and $\int_{|\lambda| \leqq \theta} dF_m(\lambda_1, \ldots, \lambda_n) \geqq F^+(\theta)$ for all θ; then a sequence of random variables z_m (which are n-dimensional) exists whose distribution functions are $F_m(\lambda_1, \ldots, \lambda_n)$ and for which $E|z_m - z_0|^2 \to 0$ as $m \to \infty$.*

Proof. First, consider the case that $n = 1$. In this case we choose $\bar{\Omega} = \langle 0, 1\rangle$ as the space of definition of z_m, where the field of Lebesgue measurable sets $\bar{\mathscr{F}}$ is defined and $\bar{P}$ is Lebesgue measure. Define $z_m(\omega) = F_m^{-1}(\omega)$, where F_m^{-1} is the inverse function to $F_m(\lambda)$. If the inverse function is not defined i.e. if F_m is constant on an interval $\langle a, b)$, $F_m(a) = v$, we take $F_m^{-1}(v) = a$ and if $F_m(\lambda)$ is discontinuous in a, $F_m(a - 0) = \alpha$, $F_m(a) = \beta > \alpha$, we take $F_m^{-1}(v) = a$ for $v \in \langle \alpha, \beta\rangle$. Obviously $F_m(\lambda)$ are distribution functions of z_m. Let K_m be a sequence of nonnegative numbers; then

$$E|z_m(\omega) - z_0(\omega)|^2 = \int_{\substack{|z_m| \leqq K_m \\ |z_0| \leqq K_m}} |z_m(\omega) - z_0(\omega)|^2\, d\bar{P} +$$

$$+ \int_{\max(|z_m|, |z_0|) > K_m} |z_m(\omega) - z_0(\omega)|^2\, d\bar{P}\,.$$

Using the assumption of Lemma 5 we can estimate the last term,

$$E|z_m(\omega) - z_0(\omega)|^2 \leqq \int_{\substack{|z_m| \leqq K_m \\ |z_0| \leqq K_m}} |z_m(\omega) - z_0(\omega)|^2\, d\bar{P} + 12 \int_{K_m}^{\infty} \theta^2\, dF^+(\theta)\,.$$

Since

$$\int_{\substack{|z_m| \leqq K_m \\ |z_0| \leqq K_m}} |z_m(\omega) - z_0(\omega)| \, \mathrm{d}\bar{P} =$$

$$= \int_{\max(F_m(-K_m), F_0(-K_m))}^{\min(F_m(K_m), F_0(K_m))} |F_m^{-1}(\omega) - F_0^{-1}(\omega)| \, \mathrm{d}\bar{P} \leqq \int_{-K_m}^{K_m} |F_m(\lambda) - F_0(\lambda)| \, \mathrm{d}\lambda$$

we can estimate the other term so that

$$E|z_m(\omega) - z_0(\omega)|^2 \leqq 2K_m \int_{-K_m}^{K_m} |F_m(\lambda) - F_0(\lambda)| \, \mathrm{d}\lambda + 12 \int_{K_m}^{\infty} \theta^2 \, \mathrm{d}F^+(\Theta)$$

holds. Evidently, a sequence of K_m can be chosen that the estimate converges to zero.

Now we shall prove Lemma 5 for $n = 2$. First we construct random variables $x_m^{(1)}, x_m^{(2)}$ on $\langle 0, 1 \rangle \times \langle 0, 1 \rangle$ where the probabilistic measure is Lebesgue measure. The first components $x_m^{(1)}$ we construct as in the case $n = 1$: $x_m^{(1)}(\omega_1, \omega_2) = F_m^{-1}(\omega_1, \infty)$. Since $F_m(\lambda, \mu)$ is absolutely continuous with respect to $F_m(\lambda, \infty)$ for every μ, there exists a function $F_m(\mu \,|\, \gamma)$ such that $F_m(\lambda, \mu) = \int_{-\infty}^{\lambda} F_m(\mu \,|\, \gamma) \, \mathrm{d}F_m(\gamma, \infty)$. We put $F_m(\mu \,|\, \mathscr{F})(\omega_1) = F_m(\mu \,|\, x_m^{(1)}(\omega_1))$. $F_m(\mu \,|\, \mathscr{F})$ is a random variable for every fixed μ and it is a distribution function for every fixed ω_1. We can define $F_m^{-1}(\omega_2 \,|\, \mathscr{F})$ as in the case $n = 1$ and put $x_m^{(2)}(\omega_1, \omega_2) = F_m^{-1}(\omega_2 \,|\, \mathscr{F})(\omega_1)$. Certainly, the distribution function of $x_m^{(1)}, x_m^{(2)}$ is $F_m(\lambda, \mu)$.

Now we choose positive numbers K, η and a sequence $\ldots < \lambda_k < \lambda_{k+1} < \ldots$, $|\lambda_{k+1} - \lambda_k| < \eta$, $|\lambda_{k+1} - \lambda_k| > \eta/2$ such that almost all points on lines $\lambda = \lambda_k$ are the points of continuity of $F_0(\lambda, \mu)$. Further we denote $\eta_k = F_0(\lambda_k, \infty)$, $\eta_{k,m} = F_m(\lambda_k, \infty)$ and we remove the points η_{k+l} for which $\eta_{k+l} = \eta_k$, $l > 0$. Changing the indices we obtain again the sequence $\ldots < \eta_k < \eta_{k+1} < \ldots$ with $\eta_k = F_0(\lambda_k, \infty)$, $|\lambda_k - \lambda_{k+1}| > \eta/2$.

For every integer k we put

$$x_m^{(i)k}(\bar{\omega}_1, \omega_2) = x_m^{(i)}(\eta_k + \bar{\omega}_1(\eta_{k+1} - \eta_k), \omega_2) \quad \text{for} \quad 0 \leqq \bar{\omega}_1 \leqq 1 \,, \quad 0 \leqq \omega_2 \leqq 1 \,.$$

Random values $x_m^{(i)k}(\bar{\omega}_1, \omega_2)$ have the following distribution functions

(5.1)
$$F_m^{(i)k}(\mu) = \bar{P}(x_m^{(i)k}(\bar{\omega}_1, \omega_2) \leqq \mu) = \frac{1}{\eta_{k+1} - \eta_k} \, \bar{P}(x_m^{(i)}(\omega_1, \omega_2) \leqq \mu, \eta_k \leqq \omega_1 \leqq \eta_{k+1}) \,.$$

We can also determine conditional distribution functions

$$F_m^{(1)k}(\lambda \,|\, \mathscr{F}_m^k) = \bar{P}(x_m^{(1)k}(\bar{\omega}_1, \omega_2) \leqq \lambda \,|\, \mathscr{F}_m^k) \tag{5.2}$$

where $\mathscr{F}_m^k$ are the least σ-fields generated by $x_m^{(2)k}(\varpi_1, \omega_2)$. An estimate for $|F_0^{(2)k}(\mu) - F_m^{(2)k}(\mu)|$ will be needed.

(5.3) $$|F_0^{(2)k}(\mu) - F_m^{(2)k}(\mu)| =$$

$$= \frac{1}{\eta_{k+1} - \eta_k} |\bar{P}(x_0^{(2)} \leqq \mu, \eta_k \leqq \omega_1 \leqq \eta_{k+1}) - \bar{P}(x_m^{(2)} \leqq \mu, \eta_k \leqq \omega_1 \leqq \eta_{k+1})| \leqq$$

$$\leqq \frac{1}{\eta_{k+1} - \eta_k} |\bar{P}(x_0^{(2)} \leqq \mu, \eta_k \leqq \omega_1 \leqq \eta_{k+1}) - \bar{P}(x_m^{(2)} \leqq \mu, \eta_{k,m} \leqq \omega_1 \leqq \eta_{k+1,m})| +$$

$$+ \frac{1}{\eta_{k+1} - \eta_k} \bar{P}(x_m^{(2)} \leqq \mu, \min(\eta_k, \eta_{k,m}) \leqq \omega_1 \leqq \max(\eta_k, \eta_{k,m})) +$$

$$+ \frac{1}{\eta_{k+1} - \eta_k} \bar{P}(x_m^{(2)} \leqq \mu, \min(\eta_{k+1}, \eta_{k+1,m}) \leqq \omega_1 \leqq \max(\eta_{k+1}, \eta_{k+1,m})) \leqq$$

$$\leqq \frac{1}{\eta_{k+1} - \eta_k} |F_0(\lambda_{k+1}, \mu) - F_m(\lambda_{k+1}, \mu)| + \frac{1}{\eta_{k+1} - \eta_k} |F_0(\lambda_k, \mu) - F_m(\lambda_k, \mu)| +$$

$$+ \frac{|\eta_k - \eta_{k,m}|}{\eta_{k+1} - \eta_k} + \frac{|\eta_{k+1} - \eta_{k+1,m}|}{\eta_{k+1} - \eta_k}.$$

Let y_m^k be auxiliary random variables defined on $\langle 0, 1 \rangle$ by $y_m^k(\omega) = (F_m^{(2),k}(\mu))^{-1}$ (as in case $n = 1$). y_m^k have the same distribution functions as $x_m^{(2)k}(\varpi_1, \omega_2)$ have. There exist measure preserving maps which transfer $y_0^k(\omega)$ into $x_0^{(2)k}(\varpi_1, \omega_2)$. By means of these transformation the random variables $y_m^k(\omega)$ are transferred into random variables which we denote by $z_m^{(2)k}(\varpi_1, \omega_2)$. Then

(5.4) $$\iint |z_m^{(2)k} - x_0^{(2)k}| \chi(\varpi_1, \omega_2) \, d\varpi_1 \, d\omega_2 = \int |y_m^k(\omega) - y_0^k(\omega)| \chi(\omega) \, d\omega \leqq$$

$$\leqq \int_{-K}^{K} |F_m^{(2)k}(\mu) - F_0^{(2)k}(\mu)| \, d\mu$$

where $\chi(\varpi_1, \omega_2)$ is the characteristic function of $|z_m^{(2)k}| \leqq K$, $|x_0^{(2)k}| \leqq K$ and $\chi(\omega)$ is the characteristic function of $|y_m^k| \leqq K$, $|y_0^k| \leqq K$.

To define $z_m^{(1)k}$ $(m \geqq 1)$ we extend the domain of definition of $z_m^{(2)k}$ onto $\langle 0, 1 \rangle \times \langle 0, 1 \rangle \times \langle 0, 1 \rangle$ by

$$z_m^{(2)k}(\varpi_1, \omega_2, \omega_3) = z_m^{(2)k}(\varpi_1, \omega_2).$$

Random variables $z_m^{(1)k}(\varpi_1, \omega_2, \omega_3)$ $(m \geqq 1)$ can be defined by means of conditional distribution (5.2) such that the distribution function of $z_m^{(1)k}$, $z_m^{(2)k}$ is the same as the distribution function of $x_m^{(1)k}$, $x_m^{(2)k}$. With respect to the construction of $z_m^{(1)k}$ it follows that

(5.5) $$|z_m^{(1)k} - x_m^{(1)k}| > \eta$$

on a set whose Lebesgue measure is less than $(\eta_{k+1} - \eta_k)^{-1} (|\eta_k - \eta_{k,m}| + |\eta_{k+1} - \eta_{k+1,m}|)$. Now, we put $z_m^{(i)}(\omega_1, \omega_2, \omega_3) = z_m^{(i)k}((\omega_1 - \eta_k)/(\eta_{k+1} - \eta_k), \omega_2, \omega_3)$ for $\eta_k \leqq \omega_1 \leqq \eta_{k+1}$. Since the distribution function of $z_m^{(1)}, z_m^{(2)}$ is a linear combination of the distribution functions of $z_m^{(1)k}, z_m^{2(k)}$ we obtain that the distribution function of $z_m^{(1)}, z_m^{(2)}$ is the same as that for $x_m^{(1)}, x_m^{(2)}$, i.e. $F_m(\lambda, \mu)$.

If the number m is large enough, we have $|\eta_{k,m} - \eta_k| < |\eta_{k+1} - \eta_k|$, $|\eta_{k,m} - \eta_k| < |\eta_k - \eta_{k-1}|$ for all k with $|\lambda_k| \leqq K$. Let $\tilde{\chi}$ be the characteristic function of $|z_m^{(2)}| \leqq K$, $|x_0^{(2)}| \leqq K$, $|x_0^{(1)}| \leqq K$, $\tilde{\chi}^k$ the characteristic function of $|z_m^{(2)k}| \leqq K$, $|x_0^{(2)k}| \leqq K$, $|x_0^{(1)k}| \leqq K$, $\chi^\dagger$ be the characteristic function of $|z_m^{(1)}| \leqq K$, $|x_m^{(1)}| \leqq K$, and χ^* be the characteristic function of $|x_m^{(1)}| \leqq K$, $|x_0^{(1)}| \leqq K$; then for the norm $\|x_0 - z_m\|$ we obtain an estimate

$$E|z_m - x_0|^2 \leqq 2 \iiint |z_m^{(1)} - x_m^{(1)}|^2 \, d\omega_1 \, d\omega_2 \, d\omega_3 + 2 \int |x_m^{(1)} - x_0^{(1)}|^2 \, d\omega_1 + \\ + \iint |z_m^{(2)} - x_0^{(2)}|^2 \, d\omega_1 \, d\omega_2 \leqq$$

$$\leqq 4K \iiint |z_m^{(1)} - x_m^{(1)}| \, \chi^\dagger \, d\omega_1 \, d\omega_2 \, d\omega_3 + 4K \int |x_m^{(1)} - x_0^{(1)}| \, \chi^* \, d\omega_1 + \\ + 2K \int |z_m^{(2)} - x_0^{(2)}| \, \tilde{\chi} \, d\omega_1 \, d\omega_2 + 52 \int_K^\infty \theta^2 \, dF^+(\theta) \, ;$$

by (5.5) this is less than

$$16K^2 \sum_{|\lambda_{k+1}| \leqq K + 2\eta} |\eta_k - \eta_{k,m}| + 4K\eta + 4K \int_0^1 |x_m^{(1)} - x_0^{(1)}| \, \chi^* \, d\omega_1 + \\ + 2K \sum_k (\eta_{k+1} - \eta_k) \int_0^1 \int_0^1 |z_m^{(2)k} - x_0^{(2)k}| \, \tilde{\chi}^k \, d\bar{\omega}_1 \, d\omega_2 + 52 \int_K^\infty \theta^2 \, dF^+(\theta) \leqq$$

and applying (5.1), (5.3), (5.4), this is less than

$$16K^2 \sum_{|\lambda_{k+1}| \leqq K + 2\eta} |\eta_k - \eta_{k,m}| + 4K\eta + 4K \int_0^1 |x_m^{(1)} - x_0^{(1)}| \, \chi^* \, d\omega_1 + \\ + 2K \sum_{|\lambda_{k+1}| \leqq K + 2\eta} \int_{-K}^K |F_0(\lambda_{k+1}, \mu) - F_m(\lambda_{k+1}, \mu)| \, d\mu + \\ + 2K \sum_{|\lambda_{k+1}| \leqq K + 2\eta} \int_{-K}^K |F_0(\lambda_k, \mu) - F_m(\lambda_k, \mu)| \, d\mu + \\ + 2K \sum_{|\lambda_{k+1}| \leqq K + 2\eta} |\eta_k - \eta_{k,m}| + 2K \sum_{|\lambda_{k+1}| \leqq K + 2\eta} |\eta_{k+1} - \eta_{k+1,m}| + 52 \int_K^\infty \theta^2 \, dF^+(\theta) \, .$$

With respect to the continuity assumption of $F_0(\lambda, \mu)$ on $\lambda = \lambda_k$ we obtain $F_m(\lambda_k, \mu) \to \to F_0(\lambda_k, \mu)$ for almost all μ so that the 4-th and 5-th term tend to 0 as $m \to \infty$. Since $|\eta_k - \eta_{k,m}| \to 0$ as $m \to \infty$, we obtain that the 1-st, 6-th and 7-th term tend to 0. Similarly as in the case $n = 1$ we obtain

$$4K \int_0^1 |x_m^{(1)} - x_0^{(1)}| \chi^* \, d\omega_1 = 4K \int_{|x_m(1)| \leqq K, |x_0(1)| \leqq K} |x_m^{(1)} - x_0^{(1)}| \, d\omega_1 \leqq$$

$$\leqq 4K \int_{-K}^{K} |F_m(\lambda, \infty) - F_0(\lambda, \infty)| \, d\lambda$$

and the term on the right-hand side tends to 0. We have conluded that

$$\limsup_{m \to \infty} E|z_m - x_0|^2 \leqq 4K\eta + 52 \int_K^\infty \theta^2 \, dF^+(\theta) .$$

For a given $\varepsilon > 0$ we can find $K > 0$, $\eta > 0$ such that $4K\eta + 52 \int_K^\infty \theta^2 \, dF^+(\theta) < < \varepsilon/2$ and for these K, η we can find m_0 such that

$$E|z_m - x_0|^2 \leqq 4K\eta + 52 \int_K^\infty \theta^2 \, dF^+(\theta) + \frac{\varepsilon}{2} < \varepsilon \quad \text{for} \quad m > m_0 .$$

We have proved that $\lim_{m \to \infty} E|z_m - x_0|^2 = 0$. If we define $z_0 = x_0$, Lemma 5 is proved for $n = 2$. For $n > 2$ it is possible to proceed by induction.

Now we pass to the proof of Theorem 3. Let d_s be a divergent sequence of numbers. For every $d_s > \zeta$ a periodic solution of (4.1) exists due to Lemma 4. Denote $F_s(\lambda_1, \dots, \lambda_n)$ the initial distribution functions for these periodic solutions. In the previous paper [1] we have constructed these distribution functions as limits of distribution functions $F_s(lT, \lambda_1, \dots, \lambda_n)$ of solutions $x(t, d_s)$ of (4,1). The distribution function $F_s(0, \lambda_1, \dots, \lambda_n)$ of the initial value $x(0, d_s)$ can be chosen arbitraryly. If we take the initial distribution function and ε such that the conditions of Theorem 2 are satisfied we obtain $\int_{|\lambda| \leqq \theta} dF_s(lTk, \lambda_1, \dots, \lambda_n) \geqq F^+(\theta)$ for $\theta \geqq 0$, $kT \geqq S$, and consequently $\int_{|\lambda| \leqq \theta} dF_s(\lambda_1, \dots, \lambda_n) \geqq F^+(\theta)$. This means that it is possible to choose a subsequence of $F_s(\lambda_1, \dots, \lambda_n)$ (we denote this subsequence in the same way as the original sequence) and a distribution function $F_0(\lambda_1, \dots, \lambda_n)$ such that $F_s(\lambda_1, \dots, \lambda_n) \to F_0(\lambda_1, \dots, \lambda_n)$ in all points of continuity of $F_0(\lambda_1, \dots, \lambda_n)$ and such that

$$\int_{|\lambda| \leqq \theta} dF_0(\lambda_1, \dots, \lambda_n) \geqq F^+(\theta) . \tag{5.6}$$

By (5,6), $\int |\lambda|^2 \, dF_0(\lambda_1, \dots, \lambda_n) < \infty$. We shall show that $F_0(\lambda_1, \dots, \lambda_n)$ is the distribution function of the initial value of a periodic solution $x_0(t)$ of (1). By Lemma 5 we take the sequence of random variables $x_s(0)$, whose distribution functions are $F_s(\lambda_1, \dots, \lambda_n)$, $E|x_s(0) - x_0(0)|^2 \to 0$ is fulfilled and condition 4) holds. Let $x_0(t, d_s)$ be the solution of (4.1) with the initial value $x_0(0, d_s) = x_0(0)$, $x_s(t, d_s)$ be periodic solution of (4.1), $\psi(x)$ be an arbitrary bounded and Lipschitz continuous function and $\bar{K}$ the Lipschitz coefficient of ψ; then

$$\begin{aligned}|E(\psi(x_0(T)) - \psi(x_0(0)))| \leqq |E(\psi(x_0(T)) - \psi(x_0(T, d_s)))| + \\ + |E(\psi(x_0(T, d_s)) - \psi(x_s(T, d_s)))| + |E(\psi(x_s(T, d_s)) - \psi(x_s(0, d_s)))| + \\ + |E(\psi(x_s(0, d_s)) - \psi(x_0(0)))| \leqq \bar{K}\sqrt{E}\,|x_0(T) - x_0(T, d_s)|^2 + \\ + \bar{K}\sqrt{E}\,|x_0(T, d_s) - x_s(T, d_s)|^2 + \bar{K}\sqrt{E}\,|x_s(0) - x_0(0)|^2 .\end{aligned}$$

The fourth term equals zero since $x_s(t, d_s)$ is periodic. By Lemma 3, $\|x_0(T) - x_0(T, d_s)\| \to 0$ and, by (7,3) and Lemma 5, the other terms tend to zero. We proved that $E(\psi(x_0(T)) - \psi(x_0(0))) = 0$ for an arbitrary bounded and Lipschitz continuous function $\psi(x)$ and this means that $x_0(t)$ is periodic.

We pass to the uniqueness of the periodic solution. Let two periodic solutions $x^{(i)}(t, \omega)$, $E|x^{(i)}(t, \omega)|^2 < \infty$, $i = 1, 2$ exist. We can choose the random values $x^{(i)}(0, \omega)$ as functions of ω such that $E|x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2$ attains its minimal value provided their distribution functions $F^{(i)}(\lambda_1, \ldots, \lambda_n) = P(x_1^{(i)}(0) \leqq \lambda_1, \ldots \ldots, x_n^{(i)}(0) \leqq \lambda_n)$ are given. Since $x^{(i)}(t, \omega)$ are periodic it follows that

$$E|x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2 \leqq E|x^{(1)}(kT, \omega) - x^{(2)}(kT, \omega)|^2 , \quad kT \geqq S .$$

For every $\eta > 0$ a constant A and a measurable set $\Lambda \in \mathscr{F}$ exist such that $|x^{(i)}(0, \omega)| \leqq A$ for $\omega \in \Lambda$ and $\int_{\Omega-\Lambda} |x^{(i)}(0, \omega)|^2 \,\mathrm{d}P < \eta$. Denote $\beta = \bar{\beta}(A)$ (for $\bar{\beta}(d)$ see Theorem 1). By Theorem 1 we have

$$\begin{aligned}\int_\Lambda |x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2 \,\mathrm{d}P + \int_{\Omega-\Lambda} |x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2 \,\mathrm{d}P = \\ = E|x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2 \leqq E|x^{(1)}(kT, \omega) - x^{(2)}(kT, \omega)|^2 = \\ = \int_\Lambda |x^{(1)}(kT, \omega) - x^{(2)}(kT, \omega)|^2 \,\mathrm{d}P + \int_{\Omega-\Lambda} |x^{(1)}(kT, \omega) - x^{(2)}(kT, \omega)|^2 \,\mathrm{d}P \leqq \\ \leqq \beta^2 \int_\Lambda |x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2 \,\mathrm{d}P + \int_{\Omega-\Lambda} |x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2 \,\mathrm{d}P .\end{aligned}$$

Since $\beta < 1$ it follows that $\int_\Lambda |x^{(1)}(0, \omega) - x^{(2)}(0, \omega)|^2 \,\mathrm{d}P = 0$, and since η was arbitrary we obtain that $x^{(1)} = x^{(2)}$ almost everywhere and it yields $F^{(1)} = F^{(2)}$.

It still remains to prove the last relation of Theorem 3. Let $\bar{x}_\varepsilon(t)$ be the periodic solution of (1). By (5.6) we obtain $\int_{|\bar{x}_\varepsilon(kT)| \geqq \zeta + 2\alpha} |\bar{x}_\varepsilon(kT)|^2 \,\mathrm{d}P \leqq \int_{\zeta+2\alpha}^\infty \theta^2 \,\mathrm{d}F^+(\theta)$. Actually, equation (2) has also a periodic solution denoted by $\bar{y}(t)$. Let $\bar{y}^*(t)$ be a solution of (2) having the initial value $\bar{y}^*(t_0) = \bar{x}_\varepsilon(t_0)$, where t_0 is from $\langle 0, kT \rangle$, $kT > S$. By Lemma 1 we have

$$\|\bar{x}_\varepsilon(t_0 + kT) - \bar{y}(t_0 + kT)\|_i \leqq$$

$$\leqq \|\bar{x}_\varepsilon(t_0 + kT) - \bar{y}^*(t_0 + kT)\|_i + \|\bar{y}^*(t_0 + kT) - \bar{y}(t_0 + kT)\|_i \leqq$$

$$\leqq \varphi_1(0, \varepsilon) + \sqrt[2i]{\left[\int_{|\bar{x}_\varepsilon(t_0)|_i < \zeta + 2\alpha} |\bar{y}^*(t_0 + kT) - \bar{y}(t_0 + kT)|_i^{2i} \, dP + \right.}$$

$$\left. + \int_{|\bar{x}_\varepsilon(t_0)|_i \geqq \zeta + 2\alpha} |\bar{y}^*(t_0 + kT) - \bar{y}(t_0 + kT)|_i^{2i} \right] dP \leqq$$

$$\leqq \varphi_1(0, \varepsilon) + \beta(\zeta + 2\alpha) \sqrt[2i]{\int_{|\bar{x}_\varepsilon(t_0)|_i \leqq \zeta + 2\alpha} |\bar{y}^*(t_0) - \bar{y}(t_0)|_i^{2i} \, dP} +$$

$$+ \sqrt[2i]{\int_{|\bar{x}_\varepsilon(t_0)|_i \geqq \zeta + 2\alpha} |\bar{y}^*(t_0) - \bar{y}(t_0)|_i^{2i} \, dP} \leqq$$

$$\leqq \varphi_1(0, \varepsilon) + \beta(\zeta + 2\alpha) \|\bar{x}_\varepsilon(t_0) - \bar{y}(t_0)\|_i + 2 \sqrt[2i]{\int_{\zeta + 2\alpha}^{\infty} \theta^{2i} \, dF_\varepsilon^+(\theta)} .$$

Since $\bar{x}_\varepsilon$, $\bar{y}$ are periodic we obtain

$$\|\bar{x}_\varepsilon(t_0) - \bar{y}(t_0)\|_i \leqq \frac{\varphi_1(0, \varepsilon) + 2 \sqrt[2i]{\int_{\zeta+2\alpha}^{\infty} \theta^{2i} \, dF_\varepsilon^+(\theta)}}{1 - \beta(\zeta + 2\alpha)} .$$

By Lemma 1 and by Theorem 2 (see (3.3) – where μ can be chosen small if ε is small) the right-hand side tends to zero as ε tends to zero. This completes the proof of Theorem 3.

Remark 5. Since we proved the uniqueness of the periodic solution $\bar{x}_\varepsilon(t)$ of (1) we need not choose a subsequence from the sequence $\bar{x}_\varepsilon(0, d_s)$ for constructing the periodic solution $\bar{x}_\varepsilon(t)$. It is obvious now, that the original sequence $\bar{x}_\varepsilon(0, d_s)$ must converge to $\bar{x}_\varepsilon(t)$. As mentioned above it is possible to find estimates for the periodic solution $\bar{x}_\varepsilon(t)$. As the estimates for periodic solutions $\bar{x}_\varepsilon(t, d_s)$ of (4.1) were derived in [1] it suffices to estimate the difference $\bar{x}_\varepsilon(0, d_s) - \bar{x}_\varepsilon(0)$. We shall suppose that $T \geqq S$.

Let $\bar{F}_\varepsilon(\lambda_1, \ldots, \lambda_n)$, $\bar{F}_\varepsilon(\lambda_1, \ldots, \lambda_n, d)$ be the distribution functions of the initial values $\bar{x}_\varepsilon(0)$, $\bar{x}_\varepsilon(0, d)$. Random values $\bar{x}_\varepsilon(0)$, $\bar{x}_\varepsilon(0, d)$ as functions of ω are taken now such that the expression $E|\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2$ attains its minimal value provided $\bar{F}_\varepsilon(\lambda_1, \ldots, \lambda_n)$ and $\bar{F}_\varepsilon(\lambda_1, \ldots, \lambda_n, d)$ are given. We denote

$$\Lambda_s = \{\omega : |\bar{x}_\varepsilon(0)| \leqq s, \; |\bar{x}_\varepsilon(0, d)| \leqq s\} \tag{5.7}$$

where $s \geqq 0$ is a number. Since $\bar{x}_\varepsilon(t)$, $\bar{x}_\varepsilon(t, d)$ are periodic processes, then recalling the choice of $\bar{x}_\varepsilon(0)$, $\bar{x}_\varepsilon(0, d)$ we have

$$\|\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)\| \leqq \|\bar{x}_\varepsilon(T) - \bar{x}_\varepsilon(T, d)\| . \tag{5.8}$$

Let $z(t)$ be a solution of (1) with the initial value $z(0) = \bar{x}_\varepsilon(0, d)$. Obviously,

$$\|\bar{x}_\varepsilon(T) - \bar{x}_\varepsilon(T, d)\| \leqq \|\bar{x}_\varepsilon(T) - z(T)\| + \|z(T) - \bar{x}_\varepsilon(T, d)\| . \tag{5.9}$$

By Theorem 1,

$$\|\bar{x}_\varepsilon(T) - z(T)\|^2 \leqq \beta^2(s) \int_{A_s} |\bar{x}_\varepsilon(0) - z(0)|^2 \, \mathrm{d}P + \int_{\Omega - A_s} |\bar{x}_\varepsilon(0) - z(0)|^2 \, \mathrm{d}P = \tag{5.10}$$

$$= \beta^2(s) \int_{A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P + \int_{\Omega - A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P .$$

Since the term on the left-hand side of (5.8) can be written as

$$E|\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 = \int_{A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P + \int_{\Omega - A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P ,$$

we obtain by (5.8) to (5.10)

$$\sqrt{\left[\int_{A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P + \int_{\Omega - A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P\right]} \leqq$$

$$\leqq \sqrt{\left[\beta^2(s) \int_{A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P + \right.}$$

$$\overline{\left. + \int_{\Omega - A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P\right]} + \|z(T) - \bar{x}_\varepsilon(T, d)\| .$$

From this inequality follows that

$$\sqrt{\int_{A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P} \leqq (1 - \beta^2(s))^{-1} \Big\{(\beta(s) \|z(T) - \bar{x}_\varepsilon(T, d))\| +$$

$$+ \sqrt{\Big[\|z(T) - \bar{x}_\varepsilon(T, d)\|^2 + }$$

$$\overline{+ 2(1 - \beta^2(s)) \|z(T) - \bar{x}_\varepsilon(T, d)\| \sqrt{\int_{\Omega - A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P}\Big]}\Big\} .$$

Since

$$\int_{|\bar{x}_\varepsilon(0)| \geqq s} |\bar{x}_\varepsilon(0)|^2 \, \mathrm{d}P \leqq \int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta) , \quad \int_{|\bar{x}_\varepsilon(0, d)| \geqq s} |\bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P \leqq \int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta)$$

(Theorem (2 and (5.6)).

We obtain

$$\int_{\max(|\bar{x}_\varepsilon(0)|,|\bar{x}_\varepsilon(0,d)|)\geqq s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P \leqq 12 \int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta) ;$$

using this last inequality we obtain an estimate

$$(5.11) \quad \sqrt{\int_{A_s} |\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)|^2 \, \mathrm{d}P} \leqq (1 - \beta^2(s))^{-1} \Big\{ \beta(s) \, \|z(T) - \bar{x}_\varepsilon(T, d)\| + $$
$$+ \sqrt{\Big[\|z(T) - \bar{x}_\varepsilon(T, d)\|^2 + 4(1 - \beta^2(s)) \sqrt{\Big[3 \int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta) \Big]} \, \|z(T) - \bar{x}_\varepsilon(T, d)\| \Big]} \Big\} .$$

If we want to have an estimate for the norm $\|\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)\|$, we obtain easily

$$(5.12) \qquad \|\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)\| \leqq (1 - \beta^2(s))^{-1} \Big\{ \beta(s) \, \|z(T) - \bar{x}_\varepsilon(T, d)\| + $$
$$+ \sqrt{\Big[\|z(T) - \bar{x}_\varepsilon(T, d)\|^2 + 4(1 - \beta^2(s)) \sqrt{\Big[3 \int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta) \Big]} \, \|z(T) - \bar{x}_\varepsilon(T, d)\| \Big]} \Big\} +$$
$$+ 2 \sqrt{\Big[3 \int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta) \Big]} .$$

This estimate is rather complicated; hower, it can be shown that

$$(5.13) \quad \|\bar{x}_\varepsilon(0) - \bar{x}_\varepsilon(0, d)\| \leqq 2 \frac{\|z(T) - \bar{x}_\varepsilon(T, d)\|}{1 - \beta^2(s)} + 2 \sqrt{\Big[6 \int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta) \Big]} .$$

Estimate (5.13) is efficient in the case that

$$\sqrt{\int_s^\infty \theta^2 \, \mathrm{d}F^+(\theta)} \ll \frac{\|z(T) - \bar{x}_\varepsilon(T, d)\|}{(1 - \beta^2(s))} .$$

The expression $\|z(T) - \bar{x}_\varepsilon(T, d)\|$ can be estimated by Lemma 3 provided the initial value of $z(0) = \bar{x}_\varepsilon(0, d)$ fulfils the relation $\|z(0)\| = \|\bar{x}_\varepsilon(0, d)\| \leqq \sqrt{\int_0^\infty \theta^2 \, \mathrm{d}F^+(\theta)}$.

The last section of the article is devoted to an existence theorem. In this section we shall not consider the index ε in equation (1) and shall use a strong q-norm $|||x|||_q$ defined by

$$|||x(t, \omega)|||_q = \sqrt[2q]{E \sup_{t \in \langle 0, L \rangle} |x(t, \omega)|_q^{2q}} .$$

First we state some properties of stochastic integrals. Let $w(t)$ be a stochastic process fulfilling condition 2) and let $f(t, \omega)$ be a matrix function which is $\mathscr{L} \times \mathscr{F}$-measurable in both arguments ($\mathscr{L}$ denotes the Lebesgue measurability), $\mathscr{F}(t)$ measurable for every t and such that $\int_0^L E|f(\tau, \omega)|_q^{2q} \, \mathrm{d}F(\tau) < \infty$. It is possible, in an ordinary way, to construct a stochastic integral $\int_0^L f(t, \omega) \,\mathrm{d}w(t)$ as a limit of $\sum_i f(\tau_i, \omega) (w(\tau_{i+1}) - - w(\tau_i))$ such that the inequality

$$(6.1) \qquad \left\| \int_0^L f(t) \,\mathrm{d}w(t) \right\|_q \leqq 2n \sqrt[2q]{\int_0^L E \max_j \sum_i f_{ij}^{2q} (t, \omega) \, e^{\frac{1}{2}4q(F(L)-F(t))} \,\mathrm{d}F(t)}$$

holds; since $|\int_0^t f(\xi) \,\mathrm{d}w(\xi)|_q$ is a semi-martingale

$$(6.2) \qquad \left|\left|\left| \int_0^t f(\xi) \,\mathrm{d}w(\xi) \right|\right|\right|_q \leqq \frac{2q}{2q-1} \left\| \int_0^L f(\xi) \,\mathrm{d}w(\xi) \right\|_q$$

holds.

Theorem 4. *Let conditions* 1), 2), 4) *be fulfilled for some* $q \geqq 1$ *and let the following condition* 5') *hold.*

$$(5') \qquad |a(t, 0)|_q \quad \text{is integrable on every} \quad \langle 0, L \rangle \quad \text{and}$$

$$\int_0^L |B(t, 0)|_q^{2q} \,\mathrm{d}F(t) < \infty \quad \text{for every} \quad L \geqq 0 \,.$$

Then a solution of (1) *exists, i.e. a process* $x(t, \omega)$ *which is* $\mathscr{L} \times \mathscr{F}$ *measurable in both arguments,* $\mathscr{F}(t)$*-measurable for every* t *and* $E \sup_{\langle 0,L \rangle} |x(t, \omega)|^2 < \infty$. *The process* $x(t, \omega)$ *fulfils equation* 1) *in the sense that*

$$E \sup_{\langle 0,L \rangle} \left| x(t, \omega) - x_0(\omega) - \int_0^t a(\tau, x(\tau, \omega)) \,\mathrm{d}\tau - \int_0^t B(\tau, x(\tau, \omega)) \,\mathrm{d}w(\tau) \right|_q^{2q} = 0$$

holds.

Proof. Denote $\mathscr{M}$ the linear space of all n-dimensional processes $x(t, \omega)$ defined on $\langle 0, L \rangle$ which are $\mathscr{L} \times \mathscr{F}$ measurable in both arguments, $\mathscr{F}(t)$ measurable for all t and such that $|||x(t, \omega)|||_q < \infty$.

Theorem 4 is a consequence of the two following lemmas.

Lemmas 6. *Let* $x(t, \omega) \in \mathscr{M}$; *then an* $\hat{x}(t, \omega)$ *exists such that* $\hat{x}(t, \omega) \in \mathscr{M}$ *and*

$$\hat{x}(t, \omega) = x_0(\omega) + \int_0^t a(\tau, x(\tau, \omega)) \,\mathrm{d}\tau + \int_0^t B(\tau, x(\tau, \omega)) \,\mathrm{d}w(\tau) \,.$$

Proof. Obviously, the integral $\int_0^t a(\tau, x(\tau)) \,\mathrm{d}\tau$ exists and belongs to $\mathscr{M}$. Since $f(t, \omega) = B(t, x(t, \omega))$ fulfils the conditions mentioned above, the stochastic integral $\int_0^t B(\tau, x(\tau)) \,\mathrm{d}w(\tau)$ exists. By 1), 2), (6.1) and (6.2) it follows that $|||\int_0^t B(\tau, x(\tau)) \,. \,. \,\mathrm{d}w(\tau)|||_q < \infty$. Since the integral $\int_0^t B(\tau, x(\tau)) \,\mathrm{d}w(\tau)$ is stochastically continuous an $\mathscr{L} \times \mathscr{F}$-measurable modification of this integral exists. Since $x(t, \omega)$ and $w(t_2) - - w(t_1)$ for $t_1 < t_2 \leqq t$ are $\mathscr{F}(t)$-measurable, the stochastic integral is $\mathscr{F}(t)$-measurable, too. This means that the stochastic integral and also the entire right-hand side belongs to $\mathscr{M}$.

We shall find the solution of (1) by using the method of successive approximations, i.e., let

$$x_{m+1}(t) = x_0 + \int_0^t a(\tau, x_m(\tau))\, d\tau + \int_0^t B(\tau, x_m(\tau))\, dw(\tau)\,.$$

Since $x_0 \in \mathscr{M}$, all successive approximations $x_m \in \mathscr{M}$. Since $a(t, x)$, $B(t, x)$ are Lipschitz continuous we have by (6.1), (6.2)

$$|||x_{m+1}(t) - x_m(t)|||_q \leqq K \int_0^t |||x_m(\tau) - x_{m-1}(\tau)|||_q\, d\tau +$$

$$+ K' \sqrt[2q]{\int_0^t |||x_m(\tau) - x_{m-1}(\tau)|||_q^{2q}\, dF(\tau)}\,.$$

From this inequality we obtain

$$|||x_{m+1}(t) - x_m(t)|||_q \leqq$$

$$\leqq \left(\int_0^L \|a(\tau, x_0)\|_q\, d\tau + \frac{2qH}{2q-1} \sqrt[2q]{\int_0^L \|B(\tau, x_0)\|_q^{2q}\, dF(\tau)}\right) \sqrt[2q]{\left[\frac{c^m}{m!}(L + F(L) - F(0))^m\right]}$$

where c is a constant.

Since the series $\sum \sqrt[2q]{[(c^m/m!)\,(t + F(t) - F(0))^m]}$ converges the sequence $x_m(t)$ is fundamental in the strong q-norm.

Lemma 7. *The linear space $\mathscr{M}$ is complete.*

Proof. Let $x_m(t)$ be an arbitrary fundamental sequence in $\mathscr{M}$. Take a sequence of positive numbers ε_m such that $\sum_{k=1}^{\infty} \varepsilon_k^{2q/(2q+1)} < \infty$. We select a subsequence $x_{n_k}(t)$ from x_m in the following way. The index n_1 is chosen such that $|||x_m(t) - x_{n_1}(t)|||_q < \varepsilon_1$ for $m > n_1$, etc. Denoting this subsequence in the same way as the original sequence, we shall have the inequalities $|||x_k(t) - x_m(t)|||_q < \varepsilon_k$ for $m > k$. Obviously,

$$P\left(\sup_{t \in \langle 0, L\rangle} |x_k(t) - x_m(t)|_q > |||x_k(t) - x_m(t)|||_q^{2q/(2q+1)}\right) \leqq |||x_k(t) - x_m(t)|||_q^{2q/(2q+1)}\,.$$

Denote

$$\Lambda = \bigcap_k \bigcup_{m \geqq k} \{\omega : \sup_t |x_m(t, \omega) - x_{m+1}(t, \omega)|_q > |||x_m - x_{m+1}|||_q^{2q/(2q+1)}\}\,.$$

For $\omega \notin \Lambda$ there exists an $n_0(\omega)$ such that

$$\sup_t |x_m(t, \omega) - x_{m+1}(t, \omega)|_q \leqq |||x_m - x_{m+1}|||_q^{2q/(2q+1)} \quad \text{for} \quad m \geqq n_0(\omega)\,.$$

For $m \geqq n \geqq n_0(\omega)$ we have

$$\sup_t |x_m(t, \omega) - x_n(t, \omega)|_q \leqq \sum_{i=n}^{\infty} \varepsilon_i^{2q/(2q+1)} .$$

Obviously, the sequence $x_m(t, \omega)$ is fundamental and $\lim_{m \to \infty} x_m(t, \omega) = x(t, \omega)$ exists for $\omega \notin \Lambda$. Furthermore, we have the estimate

$$\text{(7.1)} \qquad \sup_t |x(t, \omega) - x_m(t, \omega)|_q \leqq \sum_{i=m}^{\infty} \varepsilon_i^{2q/(2q+1)} \quad \text{for} \quad m \geqq n_0(\omega) .$$

We shall show $P(\Lambda) = 0$. From the above inequalities it follows, that

$$\text{(7.2)} \qquad P(\Lambda) \leqq \lim_{k \to \infty} \sum_{m=k}^{\infty} P\{\omega : \sup_t |x_m(t, \omega) - x_{m+1}(t, \omega)|_q >$$

$$> |||x_m - x_{m+1}|||_q^{2q/(2q+1)}\} \leqq \lim_{k \to \infty} \sum_{m=k}^{\infty} |||x_m - x_{m+1}|||_q^{2q/(2q+1)} \leqq \lim_{k \to \infty} \sum_{m=k}^{\infty} \varepsilon_m^{2q/(2q+1)} = 0 .$$

Denote $A_n = \{\omega : n_0(\omega) \leqq n\}$. Since

$$\sqrt[2q]{\int_{A_m} \sup_t |x(t, \omega)|_q^{2q} \, \mathrm{d}P} \leqq \sqrt[2q]{\int_{A_m} \sup_t |x(t, \omega) - x_m(t, \omega)|_q^{2q} \, \mathrm{d}P} +$$

$$+ \sqrt[2q]{\int_{A_m} \sup_t |x_m(t, \omega)|_q^{2q} \, \mathrm{d}P} \leqq \sum_{k=m}^{\infty} \varepsilon_k^{2q/(2q+1)} + \sup_m |||x_m|||_q$$

and $\Omega = \Lambda + \bigcup_m A_m$, where $P(\Lambda) = 0$, we have $|||x(t, \omega)|||_q < \infty$. Because $x(t, \omega)$ is $\mathscr{L} \times \mathscr{F}$-measurable, $\mathscr{F}(t)$-measurable for every t and $|||x(t, \omega)||| < \infty$ it belongs to $\mathscr{M}$.

It remains to prove that x_m converge to x in the strong q-norm $|||\bullet|||_q$. Actually, we have (for brevity we denote $|||y|||_* = \sup_t |y(t, \omega)|_q$)

$$\sqrt[2q]{\int_{\Omega} |||x - x_m|||_*^{2q} \, \mathrm{d}P} \leqq \sqrt[2q]{\int_{A_m} |||x - x_m|||_*^{2q} \, \mathrm{d}P} +$$

$$+ \sqrt[2q]{\int_{\Omega - A_m} |||x|||_*^{2q} \, \mathrm{d}P} + \sqrt[2q]{\int_{\Omega - A_m} |||x_m|||_*^{2q} \, \mathrm{d}P} .$$

By (7.1) and by the definition of A_m we obtain

$$\sqrt[2q]{\int_{\Omega} |||x - x_m|||_*^{2q} \, \mathrm{d}P} \leqq \sum_{k=m}^{\infty} \varepsilon_k^{2q/(2q+1)} + \sqrt[2q]{\int_{\Omega - A_m} |||x|||_*^{2q} \, \mathrm{d}P} + \sqrt[2q]{\int_{\Omega - A_m} |||x_m|||_*^{2q} \, \mathrm{d}P} .$$

Since $|||x|||_q < \infty$, $\Omega - \bigcup_m A_m = \Lambda$, where $P(\Lambda) = 0$, the terms $\sqrt[2q]{\int_{\Omega - A_m} |||x|||_*^{2q} \, \mathrm{d}P}$ tend to zero; since $\Omega - A_1 \supset \Omega - A_2 \supset \ldots$, $P(\bigcap_m (\Omega - A_m)) = P(\Lambda) = 0$ and since

$|||x_m|||_q$ are bounded, the terms $\sqrt[2q]{\int_{\Omega - A_m} |||x_m|||_*^{2q}\, dP}$ tend to zero. This completes the proof of Lemma 7.

We return to the proof of Theorem 4. The successive approximations $x_m(t, \omega)$ have a limit $x(t, \omega)$ by Lemma 7. Obviously,

$$|||x(t, \omega) - x(0, \omega) - \int_0^t a(\tau, x(\tau))\, d\tau - \int_0^t B(\tau, x(\tau, \omega))\, dw(\tau)|||_q \leqq$$
$$\leqq |||x(t, \omega) - x_{m+1}(t, \omega)|||_q + K|||x(t, \omega) - x_m(t, \omega)|||_q +$$
$$+ \frac{2q}{2q - 1} H|||x(t, \omega) - x_m(t, \omega)|||_q \sqrt[2q]{[F(L) - F(0)]}$$

and the right-hand side tends to 0 as $m \to \infty$. Theorem 4 is proved.

Let $x^{(1)}(t)$, $x^{(2)}(t)$ be two solutions of (1). For their difference we have

$$\|x^{(1)}(t) - x^{(2)}(t)\|_q \leqq \|x^{(1)}(0) - x^{(2)}(0)\|_q + K \int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau)\|_q\, d\tau +$$
$$+ HK \sqrt[2q]{\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau)\|_q^{2q}\, dF(\tau)}$$

for $t \in \langle 0, L \rangle$, where $H = 2n \exp\{(1/q)\, 4^{q-1}(F(L) - F(0))\}$. This yields an estimate

$$\|x^{(1)}(t) - x^{(2)}(t)\|_q \leqq \tag{7,3}$$
$$\leqq 3\|x^{(1)}(0) - x^{(2)}(0)\|_q \exp\left\{\frac{1}{2q} 3^{2q-1} K^{2q}(t^{2q} + H^{2q}(F(t) - F(0))\right\}.$$

This estimate depends only on K, $F(t)$, q, $\|x^{(1)}(0) - x^{(2)}(0)\|_q$.

References

[1] *Vrkoč I.:* The exponential stability and periodic solutions of Ito stochastic equations with small stochastic terms. Czech. Mat. J. *18* (*93*) 1968, p. 301—314.

[2] *Kurzweil J.:* Exponentially stable integral manifolds, averaging principle and continuous dependence on a parameter Czech. Mat. J. *16* (*91*) 1966, p. 380—423.

[3] *Vrkoč I.:* Extension of the averaging method to stochastic equations Czech. Mat. J. *16* (*91*) 1966, p. 518—544.

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).