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MULTIPLE LAPLACE INTEGRAL

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In this paper we will build up a theory of multiple Laplace integrals analogous to the L_2-theory of Fourier integrals.

For brevity we use the following notation. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multiindex (α_k non-negative integer, $k = 1, 2, \ldots, n$), and $x \in R^n$, then we write $x^\alpha = \prod_{k=1}^{n} x_k^{\alpha_k}$, $|\alpha| = \sum_{k=1}^{n} \alpha_k$. By E we denote the multiindex $E = (1, 1, \ldots, 1)$. For $a, b \in R^n$, we write $a < b$, resp. $a \leqq b$, instead of $a_k < b_k$, resp. $a_k \leqq b_k$, $k = 1, 2, \ldots, n$. If $a, b \in R^n$, $a < b$, we write $\langle a, b\rangle^E = \prod_{k=1}^{n} \langle a_k, b_k\rangle$. If an integration of a function F on a set $\{u \in C^n : \operatorname{Re} u = \sigma\}$, where C is the set of all complex numbers and $\sigma \in R^n$, is to be performed then we use the notation $\int_{\sigma - i\infty}^{\sigma + i\infty} F(u) \, du$.

Let us mention some results of the Fourier transform theory which will be needed later. We will use the definition of Fourier transform proposed by Laurent Schwartz in [3]: Let $f \in L_2(R^n)$; then

$$\int_{\langle -R, R\rangle^E} f(x) \exp(-2\pi i \xi, x) \, dx \to F(\xi), \quad R \to \infty,$$

converges in the topology of $L_2(R^n)$ to an element $F \in L_2(R^n)$ which is called the Fourier image of f and denoted by $\mathscr{F} f = F$. Conversely, if $F = \mathscr{F} f$, $f \in L_2(R^n)$, then

$$\int_{\langle -R, R\rangle^E} F(\xi) \exp(2\pi i \xi, x) \, d\xi \to f(x), \quad R \to \infty,$$

in the topology of $L_2(R^n)$. We write $f = \mathscr{F}^{-1} F$.

Theorem A. *Fourier transform $\mathscr{F} : L_2(R^n) \to L_2(R^n)$ is a unitary mapping. (By a unitary mapping we understand a homeomorphism of a Hilbert space onto a Hilbert space which preserves the inner product).*

Theorem B. *Let $f \in L_2(R^n)$, $a, b \in R^n$, $a < b$; then*

$$\int_{\langle a,b\rangle^E} f(x)\,\mathrm{d}x = \int_{R^n} (\mathscr{F}f)(\xi) \prod_{k=1}^{n} \frac{\exp(2\pi i b_k \xi_k) - \exp(2\pi i a_k \xi_k)}{2\pi i \xi_k}\,\mathrm{d}\xi\,.$$

Definition. Let $f \in L_{loc}(\langle 0, \infty)^n)$, (locally Lebesgue integrable function on the interval $\langle 0, \infty)^n$). If, for some $u \in C^n$, the improper integral

$$(1) \qquad \int_{\langle 0,\infty)^n} f(x) \exp(-u, x)\,\mathrm{d}x$$

exists then we call it Laplace integral of f and denote it by $\mathscr{L}f$. The mapping $f \to \mathscr{L}f$ is called Laplace transform.

Now we recall some results of the Laplace transform theory for functions of several variables.

Definition. We shall say that the integral (1) boundedly converges at a point $u \in C^n$, if the function

$$\varphi(a) = \int_{\langle 0,a\rangle^E} f(x) \exp(-u, x)\,\mathrm{d}x\,, \quad a \in \langle 0, \infty)^n\,,$$

is bounded on $\langle 0, \infty)^n$ and $\lim_{a\to\infty} \varphi(a)$ exists.

The set of all points $u \in C^n$ at which (1) boundedly converges is called the domain of convergence and denoted by $\mathscr{K}_f$.

Theorem C. $u \in \mathscr{K}_f \Rightarrow \{v \in C^n : \operatorname{Re}(v - u) > 0\} \subset \mathscr{K}_f$.

Theorem D. *Let $\mathscr{K}_f \neq \emptyset$; then $\mathscr{L}f$ is holomorphic on* $\operatorname{int} \mathscr{K}_f$ *and for each multi-index α we have*

$$\left(\frac{\partial}{\partial u}\right)^{\alpha} (\mathscr{L}f)(u) = (-1)^{|\alpha|}\, \mathscr{L}(x^{\alpha} f(x))(u)\,, \quad u \in \operatorname{int} \mathscr{K}_f\,.$$

Theorem E. *Let $u \in \mathscr{K}_f$, and let an $a \in (0, \infty)^n$ exist such that, for all multi-indices α,*

$$(\mathscr{L}f)(u_1 + \alpha_1 a_1, u_2 + \alpha_2 a_2, \ldots, u_n + \alpha_n a_n) = 0\,.$$

Then $f(x) = 0$ a.e. on $\langle 0, \infty)^n$.

Theorem F. *Let $u \in C^n$ and the function*

$$\varphi(x) = \exp(-u, x) \int_{\langle 0,x\rangle^E} f(\xi)\,\mathrm{d}\xi\,, \quad x \in \langle 0, \infty)^n\,,$$

be bounded on $\langle 0, \infty)^n$. Then $\{v \in C^n : \operatorname{Re}(v - u) > 0\} \subset \mathscr{K}_f$.

Theorem G. *Let a function f be locally absolutely continuous in the variable x_1 on $(0,\infty)$ for almost all $(x_2, x_3, \ldots, x_n) \in (0, \infty)^{n-1}$. Let us write $g = \partial f/\partial x_1$ and let $u \in \mathscr{K}_g$, $\operatorname{Re} u > 0$. Then the integral $\mathscr{L}(f(x) - f(0, x_2, \ldots, x_n))(v)$ boundedly converges for all $v \in C^n$, $\operatorname{Re}(v - u) > 0$, and we have*

$$(\mathscr{L}_g)(v) = v_1 \, \mathscr{L}(f(x) - f(0, x_2, \ldots, x_n))(v) .$$

Lemma 1. *Let $\gamma \in R^n$ and let a function F of n complex variables be holomorphic for $\operatorname{Re} u > \gamma$ and bounded on the set $\{u \in C^n : \operatorname{Re} u > \vartheta\}$ for each $\vartheta > \gamma$. Let us denote*

$$(2) \qquad \Phi(x, \sigma) = \left(\frac{1}{2\pi i}\right)^n \int_{\sigma - i\infty}^{\sigma + i\infty} u^{-2E} F(u) \exp(u, x) \, du , \quad \sigma > \gamma , \quad \sigma > 0 , \quad x \in R^n .$$

Then: 1) *$\Phi(x, \sigma)$ does not depend on σ.*
2) *$\Phi(x, \sigma) = 0$ for $x \notin (0, \infty)^n$.*

Proof. 1) Let $\vartheta > \gamma$, $\vartheta > 0$. It is sufficient to show that $\Phi(x, \sigma) = \Phi(x, \vartheta_1, \sigma_2, \ldots, \sigma_n)$. Let, for instance, $\vartheta_1 > \sigma_1$. For every $R_1 > 0$ we define curves in complex plane by

$$\begin{aligned} \Gamma_1 &= \{u_1 : \operatorname{Re} u_1 \in \langle \sigma_1, \vartheta_1 \rangle, \ \operatorname{Im} u_1 = R_1\} , \\ \Gamma_2 &= \{u_1 : \operatorname{Re} u_1 = \vartheta_1, \ \operatorname{Im} u_1 \in \langle -R_1, R_1 \rangle\} , \\ \Gamma_3 &= \{u_1 : \operatorname{Re} u_1 \in \langle \sigma_1, \vartheta_1 \rangle, \ \operatorname{Im} u_1 = -R_1\} , \\ \Gamma_4 &= \{u_1 : \operatorname{Re} u_1 = \sigma_1, \ \operatorname{Im} u_1 \in \langle -R_1, R_1 \rangle\}. \end{aligned}$$

If we orientate the curves Γ_r, $r = 1, 2, 3, 4$, appropriately, we get

$$\int_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} u^{-2E} F(u) \exp(u, x) \, du_1 = 0 .$$

Let the constant $\varkappa$ majorize the function $|F|$ on the set $\{u \in C^n : \operatorname{Re} u \geqq \sigma\}$; then

$$|\Phi(x, \sigma) - \Phi(x, \vartheta_1, \sigma_2, \ldots, \sigma_n)| =$$

$$= \left| \lim_{R_1 \to \infty} \left(\frac{1}{2\pi i}\right)^n \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \cdots \int_{\sigma_n - i\infty}^{\sigma_n + i\infty} \left(\int_{\Gamma_1 \cup \Gamma_3} u^{-2E} F(u) \exp(u, x) \right) du \right| \leqq$$

$$\leqq \frac{2\varkappa}{(2\pi)^n} \left(\prod_{k=2}^n e^{\sigma_k x_k} \int_{-\infty}^{\infty} \frac{d\tau_k}{\sigma_k^2 + \tau_k^2} \right) \lim_{R_1 \to \infty} \int_{\sigma_1}^{\vartheta_1} \frac{d\lambda}{\lambda^2 + R_1^2} \leqq 2^{1-n} \varkappa \prod_{k=2}^n \frac{e^{\sigma_k x_k}}{|\sigma_k|} \lim_{R_1 \to \infty} \frac{1}{R_1} = 0 .$$

2) $\Phi(x, \sigma)$ is continuous in x on R^n. Hence, it is sufficient to prove that $\Phi(x, \sigma) = 0$ for $x \notin \langle 0, \infty)^n$. Let, for instance, $x_1 < 0$. Let us choose $\sigma_1 > \max(0, \gamma_1)$, $R_1 > 0$.

Then if we put $u_1 = R_1 \exp(i\varphi)$, $\operatorname{Re} u_1 \geqq \sigma_1$, we get

$$\left|\int_{\sigma_1 - iR_1}^{\sigma_1 + iR_1} u_1^{-2} F(u) \exp(u_1 x_1) \, du_1\right| =$$

$$= \left|\int_{-\Theta}^{\Theta} (R_1 \exp(i\varphi))^{-1} F(R_1 \exp(i\varphi), u_2, \ldots, u_n) \exp(x_1 R_1 \exp(i\varphi)) \, d\varphi\right| \leqq$$

$$\leqq \frac{\varkappa}{R_1} \int_{-\Theta}^{\Theta} \exp(x_1 R_1 \cos\varphi) \, d\varphi \leqq \frac{\varkappa}{R_1} \pi .$$

This implies that

$$\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} u_1^{-2} F(u) \exp(u_1 x_1) \, du_1 = 0 \quad \text{and} \quad \Phi(x, \sigma) = 0 .$$

Lemma 2. *Let the assumptions of Lemma* 1 *be satisfied. Let a function* $f \in L_{loc}(R^n)$ *exist such that*

$$\Phi(x) = \int_{\langle 0,x \rangle^E} \left(\int_{\langle 0,y \rangle^E} f(z) \, dz \right) dy , \quad x \in R^n , \tag{3}$$

(*here we write* $\Phi(x) = \Phi(x, \sigma)$ *due to Lemma* 1). *Moreover, assume that* $\vartheta \in \widetilde{\mathscr{K}}_f$, $\vartheta > \gamma$, $\vartheta > 0$.

Then $F(u) = (\mathscr{L}f)(u)$, $\operatorname{Re} u > \vartheta$.

Proof. According to Theorem C the assumption $\vartheta \in \widetilde{\mathscr{K}}_f$ implies that $\{u \in C^n : : \operatorname{Re} u > \vartheta\} \subset \mathscr{K}_f$; consequently, by Theorem G, $\{u \in C^n : \operatorname{Re} u > \vartheta\} \subset \mathscr{K}_\Phi$. Let us choose $\sigma > \vartheta$; then

$$\Phi(x) = \exp(\sigma, x) \int_{R^n} (\sigma + 2\pi i\tau)^{-2E} F(\sigma + 2\pi i\tau) \exp(2\pi i\tau, x) \, d\tau .$$

We have $(\sigma + 2\pi i\tau)^{-2E} F(\sigma + 2\pi i\tau) \in L_2(R^n)$ and, according to the mentioned, Fourier L_2-theory,

$$\int_{\langle -R,R \rangle^E} \exp(-\sigma, x) \, \Phi(x) \exp(-2\pi i\tau, x) \, dx \to \frac{F(\sigma + 2\pi i\tau)}{(\sigma + 2\pi i\tau)^{2E}}, \quad R \to \infty , \tag{4}$$

in the topology of $L_2(R^n)$. Since we already know that the integral on the left side of (4) converges, we can write

$$(\mathscr{L}f)(\sigma + 2\pi i\tau) = (\sigma + 2\pi i\tau)^{2E} (\mathscr{L}\Phi)(\sigma + 2\pi i\tau) =$$

$$= (\sigma + 2\pi i\tau)^{2E} \cdot \frac{F(\sigma + 2\pi i\tau)}{(\sigma + 2\pi i\tau)^{2E}} = F(\sigma + 2\pi i\tau) .$$

Lemma 3. *Let the assumptions of Lemma 1 be satisfied. Let* $\sigma > \gamma$, $\sigma > 0$, $r \in \in \langle 1, 2 \rangle$ *exist such that* $\int_{R^n} |F(\sigma + i\tau)|^r \, d\tau < +\infty$. *Then there exists a function* $f \in L_{loc}(R^n)$ *such that* $\sigma \in \mathscr{K}_f$ *and* $F(u) = (\mathscr{L}f)(u)$, $\operatorname{Re} u > \sigma$.

Proof. We verify the assumptions of Lemma 2. First, let us show that

$$(5) \qquad (\sigma + i\tau)^{-E} F(\sigma + i\tau) \in L_1(R^n) .$$

This is trivial for $r = 1$; thus, let $r > 1$. Then, according to Hölder's inequality,

$$\int_{R^n} \left| \frac{F(\sigma + i\tau)}{(\sigma + i\tau)^E} \right| d\tau \leqq \left(\int_{R^n} |F(\sigma + i\tau)|^r \, d\tau \right)^{1/r} \left(\int_{R^n} \prod_{k=1}^{n} |\sigma_k + i\tau_k|^{-q} \, d\tau \right)^{1/q} < +\infty .$$

This implies the existence of the derivative $\Psi(x) = (\partial/\partial x)^E \, \Phi(x)$ continuous on R^n, vanishing on $R^n - (0, \infty)^n$, for which

$$(6) \qquad \Psi(x) = \int_{R^n} (\sigma + 2\pi i\tau)^{-E} F(\sigma + 2\pi i\tau) \exp(\sigma + 2\pi i\tau, x) \, d\tau$$

holds. Further, we can write:

$$(7) \qquad \begin{aligned} \exp(-\sigma, x)\, \Psi(x) &= \exp(-\sigma, x) \left[\Psi(x) - \sum \Psi(0, x_2, \ldots, x_n) + \right. \\ &\quad \left. + \sum \Psi(0, 0, x_3, \ldots, x_n) - \ldots + (-1)^n \, \Psi(0) \right] = \\ &= \int_{R^n} (\sigma + 2\pi i\tau)^{-E} F(\sigma + 2\pi i\tau) \prod_{k=1}^{n} (\exp(2\pi i\tau_k x_k) - 1) \, d\tau = \\ &= \int_{R^n} F(\sigma + 2\pi i\tau) \prod_{k=1}^{n} \frac{2\pi i\tau_k}{\sigma_k + 2\pi i\tau_k} \prod_{k=1}^{n} \frac{\exp(2\pi i\tau_k x_k) - 1}{2\pi i\tau_k} \, d\tau . \end{aligned}$$

In the case $r = 1$ it is obvious that we can differentiate (7) with respect to x and the derivative $(\partial/\partial x)^E (\exp(-\sigma, x)\, \Psi(x))$ is continuous on R^n.

In the case $r > 1$ we have

$$\int_{R^n} |F(\sigma + 2\pi i\tau)|^2 \, d\tau =$$

$$= \int_{R^n} |F(\sigma + 2\pi i\tau)|^{2-r} \cdot |F(\sigma + 2\pi i\tau)|^r \, d\tau \leqq \varkappa^{2-r} \int_{R^n} |F(\sigma + 2\pi i\tau)|^r \, d\tau < +\infty ,$$

where $\varkappa$ majorizes $|F|$ on the set $\{u \in C^n : \operatorname{Re} u \geqq \sigma\}$. Hence,

$$G(\tau) = F(\sigma + 2\pi i\tau) \prod_{k=1}^{n} \frac{2\pi i\tau_k}{\sigma_k + 2\pi i\tau_k} \in L_2(R^n) ,$$

and, in accordance with Fourier L_2-theory, there is a function $g \in L_2(R^n)$ such that $G = \mathscr{F} g$.

According to theorem B,

$$\int_{\langle 0,x\rangle^E} g(x)\,\mathrm{d}x = \int_{R^n} G(\tau) \prod_{k=1}^{n} \frac{\exp(2\pi i\tau_k x_k) - 1}{2\pi i\tau_k}\,\mathrm{d}\tau = \exp(-\sigma, x)\,\Psi(x)\,.$$

Thus, we have proved the existence of a function $f \in L_{loc}(R^n)$ satisfying (3). It remains to verify that $\sigma \in \overline{\mathscr{K}_f}$. From (5), (6) it follows that

$$|\exp(-\sigma, x)\,\Psi(x)| \leqq \int_{R^n} |(\sigma + 2\pi i\tau)^{-E}\,F(\sigma + 2\pi i\tau)|\,\mathrm{d}\tau < +\infty\,, \quad x \in R^n\,,$$

and, according to theorem F, the assumption $\sigma \in \overline{\mathscr{K}_f}$ holds.

Lemma 4. Let $\gamma \in R^n$ and let a function F of n complex variables be holomorphic for $\operatorname{Re} u > \gamma$. Let an $r \in \langle 1, 2\rangle$ exist such that

$$\sup_{\sigma>\gamma} \int_{R^n} |F(\sigma + i\tau)|^r\,\mathrm{d}\tau < +\infty\,. \tag{8}$$

Then there exists $f \in L_{loc}(R^n)$ such that $\gamma \in \overline{\mathscr{K}_f}$ and $F(u) = (\mathscr{L}f)(u)$, $\operatorname{Re} u > \gamma$.

Proof. 1. Let $\gamma > 0$ then according to Lemma 3 and Theorem E it suffices to prove that, for every $\vartheta > \gamma$, the function F is bounded on $\{u \in C^n : \operatorname{Re} u \geqq \vartheta\}$. Let us choose u, $\operatorname{Re} u \geqq \vartheta$, $\varepsilon \in (0, \min_k (\vartheta_k - \gamma_k))$, $\varrho = (\varrho_1, \varrho_2, \ldots, \varrho_n)$, $\varrho_k \in (0, \varepsilon)$, $k = 1, 2, \ldots, n$. Let $\varkappa = \sup_{\sigma>\gamma} \int_{R^n} |F(\sigma + i\tau)|^r\,\mathrm{d}\tau$. Then

$$F(u) = \left(\frac{1}{2\pi i}\right)^n \int_{\substack{|v_k - u_k| = \varrho_k \\ k=1,2,\ldots,n}} (v - u)^{-E}\,F(v)\,\mathrm{d}v = \left(\frac{1}{2\pi}\right)^n \int_{\langle 0,2\pi\rangle^n} F(u + \varrho e^{i\varphi})\,\mathrm{d}\varphi\,,$$

$$(\tfrac{1}{2}\varepsilon^2)^n\,|F(u)| \leqq \left(\prod_{k=1}^{n} \int_0^{\varepsilon} \varrho_k\,\mathrm{d}\varrho_k\right) . |F(u)| \leqq \left(\frac{1}{2\pi}\right)^n \int_{\langle 0,2\pi\rangle^n \times \langle 0,\varepsilon\rangle^n} |F(u + \varrho e^{i\varphi})|\,.\,\varrho^E\,\mathrm{d}\varphi\,\mathrm{d}\varrho\,.$$

Further we distinguish two cases: i. $r = 1$. Then

$$(\tfrac{1}{2}\varepsilon^2)^n\,|F(u)| \leqq \left(\frac{1}{2\pi}\right)^n \int_{\substack{|(x_k + iy_k) - u_k| \leqq \varepsilon \\ k=1,2,\ldots,n}} |F(x + iy)|\,\mathrm{d}x\,\mathrm{d}y \leqq$$

$$\leqq \left(\frac{1}{2\pi}\right)^n \int_{\substack{|x_k - \sigma_k| \leqq \varepsilon \\ k=1,2,\ldots,n}} \left(\int_{R^n} |F(x + iy)|\,\mathrm{d}y\right) \mathrm{d}x \leqq \varkappa \left(\frac{\varepsilon}{\pi}\right)^n .$$

ii. $r > 1$. Then according to Hölder's inequality

$$(\tfrac{1}{2}\varepsilon^2)^n |F(u)| \leqq \left(\frac{1}{2\pi}\right)^n \int_{\langle 0,2\pi\rangle^n \times \langle 0,\varepsilon\rangle^n} |F(u + \varrho e^{i\varphi})| \cdot \varrho^{(1/r)E} \cdot \varrho^{(1-1/r)E} \, d\varphi \, d\varrho \leqq$$

$$\leqq \left(\frac{1}{2\pi}\right)^n \left(\int_{\langle 0,2\pi\rangle^n \times \langle 0,\varepsilon\rangle^n} |F(u + \varrho e^{i\varphi})|^r \varrho^E \, d\varphi \, d\varrho\right)^{1/r} \cdot \left(\int_{\langle 0,2\pi\rangle^n \times \langle 0,\varepsilon\rangle^n} \varrho^E \, d\varphi \, d\varrho\right)^{1-1/r} \leqq$$

$$\leqq \left(\frac{1}{2\pi}\right)^n (\varkappa(2\varepsilon)^n)^{1/r} \cdot (\pi\varepsilon^2)^{n(1-1/r)} = (\tfrac{1}{2}\varepsilon^2)^n \left(\varkappa \left(\frac{2}{\pi\varepsilon}\right)^n\right)^{1/r} .$$

2. Let us put $\gamma_+ = (\max(\gamma_1, 0), \ldots, \max(\gamma_n, 0))$, $\gamma_- = \gamma_+ - \gamma$, and for $\operatorname{Re} u > \gamma_+$ define a function $G(u) = F(u - \gamma_-)$, Then according to the first part of this proof there exists such $g \in L_{loc}(R^n)$ that $\gamma_+ \in \overline{\mathscr{K}_g}$ and $G(u) = (\mathscr{L}g)(u)$, $\operatorname{Re} u > \gamma_+$.

If we put $f(x) = g(x) \exp(-\gamma_-, x)$, $x \in R^n$, then $f \in L_{loc}(R^n)$, $\gamma \in \overline{\mathscr{K}_f}$ and for $\operatorname{Re} u > \gamma$ it holds $(\mathscr{L}f)(u) = (\mathscr{L}g)(u + \gamma_-) = G(u + \gamma_-) = F(u)$. The proof is complete.

Definition. Let $\gamma \in R^n$. Then we denote $L_{2,\gamma}$ the set of all complex functions f, measurable on $\langle 0, \infty)^n$, for which

$$\int_{\langle 0,\infty\rangle^n} |f(x)|^2 \exp(-2\gamma, x) \, dx < +\infty . \tag{9}$$

By $H_{2,\gamma}$ we denote the set of all functions F of n complex variables, holomorphic for $\operatorname{Re} u > \gamma$, for which

$$\sup_{\sigma > \gamma} \int_{R^n} |F(\sigma + i\tau)|^2 \, d\tau < +\infty . \tag{10}$$

Lemma 5. *Let $\gamma \in R^n$ be given. Then $\mathscr{L}(L_{2,\gamma}) = H_{2,\gamma}$.*

Proof. 1. Let $f \in L_{2,\gamma}$; then for $\sigma > \gamma$ we have

$$\int_{\langle 0,\infty\rangle^n} |f(x)| \exp(-\sigma, x) \, dx \leqq$$

$$\leqq \left(\int_{\langle 0,\infty)^n} |f(x)|^2 \exp(-2\gamma, x) \, dx\right)^{1/2} \cdot \left(\int_{\langle 0,\infty)^n} \exp(-2(\sigma - \gamma), x) \, dx\right)^{1/2} < +\infty .$$

Hence, $\gamma \in \overline{\mathscr{K}_f}$ and, according to Theorem D, the function $F = \mathscr{L}f$ is holomorphic on the set $\{u \in C^n : \operatorname{Re} u > \gamma\}$. According to Parseval's equality for Fourier transform,

$$\int_{\langle 0,\infty)^n} |f(x)|^2 \exp(-2\sigma, x) \, dx = \left(\frac{1}{2\pi}\right)^n \int_{R^n} |F(\sigma + i\tau)|^2 \, d\tau . \tag{11}$$

Then the inequality (10) is an immediate consequence of (11).

2. Let $F \in H_{2,\gamma}$; then, according to Lemma 4, there is a function $f \in L_{loc}(R^n)$ such that $\gamma \in \overline{\mathscr{K}_f}$ and $F(u) = (\mathscr{L}f)(u)$, $\operatorname{Re} u > \gamma$. Then the inequality (9) follows immediately from (11).

Lemma 6. *Let $\gamma \in R^n$ be given. Then,*

1. *for every function $F \in H_{2,\gamma}$ there exists a limit* $\lim\limits_{\substack{\sigma_k \to \gamma_k + \\ k=1,2,\ldots,n}} F(\sigma + i\tau)$ *in the topology of $L_2(R^n)$. Let us denote this limit by $F(\gamma + i\tau)$.*

2. *the function $\varphi(\sigma) = \int_{R^n} |F(\sigma + i\tau)|^2 \,\mathrm{d}\tau$ is continuous and nonincreasing in all its variables on the set $\{\sigma \in R^n : \sigma \geqq \gamma\}$. In particular,*

$$\sup_{\sigma > \gamma} \int_{R^n} |F(\sigma + i\tau)|^2 \,\mathrm{d}\tau = \int_{R^n} |F(\gamma + i\tau)|^2 \,\mathrm{d}\tau .$$

3. *$L_{2,\gamma}$, resp. $H_{2,\gamma}$, is a Hilbert space with an inner product*

$$(f, g)_L = \int_{\langle 0,\infty)^n} f(x)\, \overline{g(x)} \exp(-2\gamma, x) \,\mathrm{d}x\,, \quad f, g \in L_{2,\gamma}\,, \tag{12}$$

resp.

$$(F, G)_H = \left(\frac{1}{2\pi}\right)^n \int_{R^n} F(\gamma + i\tau)\, \overline{G(\gamma + i\tau)} \,\mathrm{d}\tau\,, \quad F, G \in H_{2,\gamma}\,. \tag{13}$$

Proof. 1. Let $F \in H_{2,\gamma}$; according to Lemma 5 there is $f \in L_{2,\gamma}$, $\mathscr{L}f \risingdotseq F$, and for $\sigma > \gamma$ the equality (11) holds. This implies that

$$\|F(\sigma_1 + i\tau) - F(\sigma_2 + i\tau)\|^2_{L_2(R^n)} =$$
$$= (2\pi)^n \int_{\langle 0,\infty)^n} |f(x)|^2 (\exp(-\sigma_1, x) - \exp(-\sigma_2, x))^2 \,\mathrm{d}x \to 0\,.$$

2. This is an immediate consequence of (11).

3. The statement concerning $L_{2,\gamma}$ is obvious. Let us show that $H_{2,\gamma}$ with the inner product (13) is complete. Let us have a sequence $F_p \in H_{2,\gamma}$, $p = 1, 2, \ldots$, $\|F_p - F_q\|_{H_{2,\gamma}} \to 0$, as $p, q \to \infty$. Let us take $f_p \in L_{2,\gamma}$ such that $\mathscr{L}f_p = F_p$, $p = 1, 2, \ldots$ Then

$$\|F_p - F_q\|^2_{H_{2,\gamma}} = \sup_{\sigma > \gamma} \left(\frac{1}{2\pi}\right)^n \int_{R^n} |F_p(\sigma + i\tau) - F_q(\sigma + i\tau)|^2 \,\mathrm{d}\tau \risingdotseq \tag{14}$$
$$= \sup_{\sigma > \gamma} \int_{\langle 0,\infty)^n} |f_p(x) - f_q(x)|^2 \exp(-2\sigma, x) \,\mathrm{d}x =$$
$$= \int_{\langle 0,\infty)^n} |f_p(x) - f_q(x)|^2 \exp(-2\gamma, x) \,\mathrm{d}x\,.$$

Hence, there is $f \in L_{2,\gamma}$ such that $f_p \to f$, $p \to \infty$, in the topology of $L_{2,\gamma}$. Then $\mathscr{L}f = F \in H_{2,\gamma}$ and from (14) ti follows that $\|F_p - F\|H_{2,\gamma} \to 0$, $p \to \infty$.

Theorem. *Let $\gamma \in R^n$ be given. Then Laplace transform $\mathscr{L} : L_{2,\gamma} \to H_{2,\gamma}$ is a unitary mapping.*

Proof. According to Lemmas 5 and 6 we have only to show that

$$f, g \in L_{2,\gamma} \Rightarrow (f, g)_{L_{2,\gamma}} = (\mathscr{L}f, \mathscr{L}g)_{H_{2,\gamma}} .$$

This, however, is exactly Parseval's equality for Fourier transform $f(x) \exp(-\gamma, x) \to \to (\mathscr{L}f)(\gamma + i\tau)$.

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