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# ON SINGULAR MATRICES

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**1.** PENROSE [7] discussed a generalized inverse for matrices, and he established the following theorem.

**Theorem A.** *For any matrix $A$, the four equations $AXA = A, XAX = X, (AX)^* = AX$, and $(XA)^* = XA$ have a unique solution $X$, where $A^*$ denotes the conjugate transpose of $A$.*

This unique solution $X$ is called the generalized inverse of $A$. If we remove the third and fourth equations in Theorem $A$ above, a solution $X$ (of the equations $XAX = X$ and $AXA = A$) is not in general unique.

Then the natural question is:

**Problem.** *What is the cardinal number of the set of all solutions $X$ of the equations $AXA = A$ and $XAX = X$ for a matrix $A$ in the set $M_n(F)$ of all $n$ by $n$ matrices over a field $F$?*

The purpose of this note is to prove (Theorem 1) that if $A \in M_n(F)$ then the cardinal number of the set of all solutions $X$ of the equations $AXA = A$ and $XAX = X$ is equal to $|F|^{2(\text{rank of } A)(n-(\text{rank of } A))}$.

This result leads to a new definition in the class of regular semigroups (see Definition 2) and gives new examples of regular semigroups with zero (see Theorem 2).

**2.** Let $F$ be a field. $M_n(F)$ denotes the set of all $n$ by $n$ matrices over the field $F$ with binary operation, the usual matrix multiplication. By Theorem A, $M_n(F)$ is a regular semigroup. We define $V(A) = \{X \in M_n(F) : AXA = A \text{ and } XAX = X\}$ which will be called an inverse set of $A$ in $M_n(F)$. $\varrho(A)$ denotes the rank of a matrix $A$ in $M_n(F)$, and $|T|$ denotes the cardinal number of a set $T$.

**Lemma 1.** *Let $A \in M_n(F)$ and let $X \in V(A)$. Then $\varrho(A) = \varrho(X)$.*

Proof. From $AXA = A$ and $XAX = X$, $\varrho(A) = \varrho(AXA) \leqq \varrho(X) = \varrho(XAX) \leqq \leqq \varrho(A)$ by Theorem 1.4 of [6, p. 83]; hence $\varrho(A) = \varrho(X)$.

**Lemma 2.** *The cardinal number of an inverse set $V(A)$ of a matrix $A$ in $M_n(F)$ is invariant under elementary row or column operation on $A$, that is, $|V(A)| = |V(EA)| = |V(AH)|$, where $E$ and $H$ are elementary matrices* (see Definition of elementary matrices on page 91 in [6]).

Proof. Let $A \in M_n(F)$ and let $E$ be an elementary matrix in $M_n(F)$. Let $X \in V(A)$ and let $E^{-1}$ be the inverse matrix of the non-singular matrix $E$. Then $EA = E(AXA) = EA(XE^{-1})\,EA$ and $XE^{-1} = (XAX)\,E^{-1} = XE^{-1}(EA)\,XE^{-1}$; hence $V(A)\,E^{-1} \subseteq V(EA)$ and $|V(A)| \leqq |V(EA)|$. Similarly, we obtain $V(EA)\,E \subset V(A)$ and $|V(EA)| \leqq |V(A)|$. Thus $|V(A)| = |V(EA)|$. Analogously, we have $|V(A)| = |V(AH)|$, where $H$ is an elementary matrix. This proves Lemma 2.

We need the following well known theorem.

**Theorem B.** *Every $m$ by $n$ matrix $A$ is equivalent to a matrix $C = (c_{ij})$ where $c_{ii} = 1$, $i = 1, 2, \ldots, \varrho(A)$, and $c_{ij} = 0$, otherwise. The matrix $C$ is called the canonical form of $A$* (see Theorem 3.4 on page 106 in [6]).

For $1 \leqq k \leqq n$ let $C_k = (d_{ij})$ where $d_{ii} = 1$ for $i = 1, 2, \ldots, k$ and $d_{ij} = 0$, otherwise.

According to Lemma 2 and Theorem B, to solve the problem we need only consider $C_k$, $k = 1, 2, \ldots, n$.

The main lemma follows.

**Lemma 3.** *Let $k$ and $n$ be positive integers with $k \leqq n$. Let $F_q$ be a Galois field with $q$ elements. If $C_k \in M_n(F_q)$, then $|V(C_k)| = q^{2k(n-k)} = q^{2(\varrho(C_k))(n-\varrho(C_k))}$.*

Proof. Let $k < n$. Let $X$ be an element of the inverse set $V(C_k)$. Then $C_k X C_k = C_k$ and $X C_k X = X$. By direct calculation, it is not hard to see that $X = (x_{ij})$ takes the form:

$$x_{ij} = \begin{cases} 1 & \text{if } \; i = j \; \text{ and } \; i = 1, 2, \ldots, k\,; \\ 0 & \text{if } \; i \neq j \; \text{ and } \; \{i, j\} \subset \{1, 2, \ldots, k\}\,; \\ x_{ij} & \text{if } \; i = 1, 2, \ldots, k \; \text{ and } \; j = k+1, k+2, \ldots, n\,; \\ x_{ij} & \text{if } \; i = k+1, k+2, \ldots, n \; \text{ and } \; j = 1, 2, \ldots, k\,; \\ \sum_{t=1}^{k} x_{it} x_{tj} & \text{if } \; \{i, j\} \subset \{k+1, k+2, \ldots, n\}\,, \end{cases}$$

where $x_{ij}$ above are arbitrary in $F_q$. Thus we are able to choose $2k(n-k)$ entries of $X$ arbitrary so that the cardinal number of the set $V(C_k)$ is equal to $q^{2k(n-k)}$. If $k = n$, then $V(C_n) = \{C_n\}$, and $|V(C_n)| = 1$. This proves Lemma 3.

**Theorem 1.** *If $A \in M_n(F)$, then the cardinal number of the inverse set $V(A)$ is equal to $|F|^{2\varrho(A)(n-\varrho(A))}$.*

Proof follows from Lemmas 2, 3 and Theorem B.

## 3. APPLICATIONS AND A QUESTION

**Definition 1.** A semigroup $S$ with 0 is said to be homogeneous $n$ regular if $|V(a)| = n$ for every $a \in S \setminus 0$ [4].

Let $n$ and $k$ be two positive integers with $k \leqq n$. We define $S_{n,k}(F) = \{X \in M_n(F) : : \varrho(X) \leqq k\}$, and let $S_{n,n-1}(F) = S_n(F)$.

We have corollaries and Theorem 2.

**Corollary 1.** *$S_2(F_q)$ is a homogeneous $q^2$ regular semigroup with 0, where $F_q$ is a finite field with q elements.*

*$S_3(F_q)$ is a homogeneous $q^4$ regular semigroup with 0.*
*$S_{n,1}(F_q)$ is a homogeneous $q^{2(n-1)}$ regular semigroup with 0.*

**Corollary 2.** *If F is a field of characteristic 0 then $S_n(F)$ is a homogeneous $\infty$ regular semigroup with 0.*

We have a new definition in the class of regular semigroups with 0.

**Definition 2.** Let $S$ be a regular semigroup with 0. $S$ is called a $[s, t]$ regular semigroup with 0 if $s \leqq |V(a)| \leqq t$ for every $a \in S \setminus 0$, where $s$ and $t$ are positive integers with $s < t$.

**Theorem 2.** *Let $F_q$ be a Galois field with q elements. Then $S_n(F_q)$ is a $[q^{2(n-1)}, q^{2[n/2](n-[n/2])}]$ regular semigroup with 0, where*

$$[n/2] = \begin{cases} n/2 & \text{if } n \text{ is even}, \\ (n-1)/2 & \text{if } n \text{ is odd}. \end{cases}$$

In $S_3(F_q)$, there are two non-zero idempotents $e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

with $ef = fe = e$ and $e \neq f$. Hence $f$ is not a primitive idempotent of the homogeneous $q^4$ regular semigroup $S_3(F_q)$. This example shows that the condition "every idempotent of $S$ is primitive" is not necessary for a regular semigroup $S$ with 0 to be homogeneous $n$ regular (see Theorems 1, 3, 7 and 8 in [4]).

Hence we raise the following question:

**Question.** *What are necessary and sufficient conditions for a regular semigroup S with 0 to be homogeneous n regular?*

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