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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y^{(4)} = p(t)\, y$

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The asymptotic behavior of solutions of the differential equation

$$(1) \qquad y'' + p(t)\, y = 0$$

under the hypothesis $p(t) \to \infty$ as $t \to \infty$ has been widely investigated. It is known, for instance, that if $p \in C'[a, \infty)$, $p'(t) \geqq 0$ and $\lim_{t\to\infty} p(t) = +\infty$, then (1) has at least one non-trivial solution which tends to zero as t tends to infinity. (See, for example, [2].) GALBRAITH, MCSHANE, and PARRISH [1] have recently shown that under the same hypotheses it need not be the case that all solutions tend to zero.

We shall call a non-trivial solution $y(t)$ of the differential equation

$$(\mathrm{L}) \qquad y^{(4)} = p(t)\, y$$

oscillatory if the set of zeros of $y(t)$ is not bounded above. The main purpose of this note is to show that the hypotheses $p \in C'[a, \infty]$, $p(t) > 0$, $p'(t) \geqq 0$ and $\lim_{t\to\infty} p(t) = +\infty$ imply that all oscillatory solutions of (L) tend to zero. We shall first show that the first three of the above conditions imply the existence of two independent oscillatory solutions.

Theorem 1. *Let $p \in C'[a, \infty)$, with $p(t) > 0$, $p'(t) \geqq 0$. Then there exist two independent oscillatory solutions of* (L) *which are bounded on $[a, \infty)$.*

Proof. We shall prove the theorem by using two lemmas.

Lemma 1.1. *Assuming the same hypotheses as in Theorem* 1, *if $y(t)$ is any solution of* (L) *with*

$$(2) \qquad y'(a) = y'(b) = 0\,, \quad b > a\,,$$

then

(3) $$\max_{t\in[a,b]} [y(t)]^2 \leqq [y(a)]^2 + \frac{[y''(a)]^2}{p(a)}.$$

Proof. Define

(4) $$H_y(t) = p(t)\,[y(t)]^2 - 2y'(t)\,y'''(t) + [y''(t)]^2 .$$

By differentiation,

(5) $$H_y(t) = H_y(a) + \int_a^t p'(s)\,[y(s)]^2\,\mathrm{d}s .$$

The assumption (2) implies that if

$$\max_{t\in[a,b]} [y(t)]^2 = [y(\bar{x})]^2 , \quad \bar{x} \in [a, b] ,$$

then

(6) $$y'(\bar{x}) = 0 .$$

If $\bar{x} = a$, (3) follows trivially; assume therefore that $\bar{x} > a$. By (2), (4), (5), and (6),

$$H_y(\bar{x}) = p(\bar{x})\,[y(\bar{x})]^2 + [y''(\bar{x})]^2 = H_y(a) + \int_a^{\bar{x}} p'(s)\,[y(s)]^2\,\mathrm{d}s \leqq$$

$$\leqq H_y(a) + \int_a^{\bar{x}} p'(s)\,[y(\bar{x})]^2\,\mathrm{d}s = H_y(a) + [p(\bar{x}) - p(a)]\,[y(\bar{x})]^2 .$$

Hence,

$$p(a)\,[y(\bar{x})]^2 + [y''(\bar{x})]^2 \leqq H_y(a)$$

and

$$[y(\bar{x})]^2 \leqq H_y(a)/p(a) = [y(a)]^2 + \frac{[y''(a)]^2}{p(a)} .$$

Lemma 1.2. *If* $p(t) > 0$, $p(t) \in C[a, \infty)$, *and* $y(t)$ *is any solution of* (L) *with*

(7) $$y(b) > 0, \quad y'(b) < 0 , \quad y''(b) > 0 , \quad y'''(b) < 0 , \quad b > a ,$$

then

(8) $$y(t) > 0 , \quad y'(t) < 0 , \quad y''(t) > 0 , \quad y'''(t) < 0$$

for all $t \in [a, b]$.

Proof. By continuity (8) holds on an interval (c, b). If (8) did not hold on $[a, b)$, there would exist $b_1 \in [a, b)$ such that (8) holds on (b_1, b) and if $w(t) \equiv y(t)\, y'(t)\,.$ $.\, y''(t)\, y'''(t)$, then $w(b_1) = 0$. But

$$w'(t) = [y'(t)]^2\, y''(t)\, y'''(t) + y(t)\,[y''(t)]^2\, y'''(t) + \\ + y(t)\, y'(t)\,[y'''(t)]^2 + p(t)\,[y(t)]^2\, y'(t)\, y''(t) < 0$$

for $t \in (b_1, b)$, and hence $w(b) = w(b_1) + \int_{b_1}^{b} w'(s)\, ds < 0$ which contradicts (7). This contradiction proves the lemma.

Proof of Theorem 1. Let z_0, z_1, z_2, z_3 be the solutions of (L) defined by the initial conditions

$$z_i^{(j)}(a) = \delta_{ij} = 0\,, \quad i \neq j \\ = 1\,, \quad i = j$$

$i, j = 0, 1, 2, 3$. For each integer $n > a$ let $b_{0n}, b_{3n}, c_{2n}, c_{3n}$ be numbers such that

$$b_{0n}^2 + b_{3n}^2 = c_{2n}^2 + c_{3n}^2 = 1 \tag{9}$$

and $b_{0n}\, z_0'(n) + b_{3n}\, z_3'(n) = c_{2n}\, z_2'(n) + c_{3n}\, z_3'(n) = 0$. Let u_n and v_n be the solutions of (L) defined by

$$u_n(t) = b_{0n}\, z_0(t) + b_{3n}\, z_3(t) \\ v_n(t) = c_{2n}\, z_2(t) + c_{3n}\, z_3(t)\,.$$

Since

$$u_n'(a) = u_n'(n) = v_n'(a) = v_n'(n) = 0\,,$$

it follows by Lemma 1.1 that for $t \in [a, n]$,

$$[u_n(t)]^2 \leqq [u_n(a)]^2 + \frac{[u_n''(a)]^2}{p(a)} = b_{0n}^2\,,$$

$$[v_n(t)]^2 \leqq [v_n(a)]^2 + \frac{[v_n''(a)]^2}{p(a)} = c_{2n}^2/p(a)\,.$$

Therefore, by (9), it follows that there exists a number A independent of n such that

$$[u_n(t)]^2 \leqq A\,, \quad [v_n(t)]^2 \leqq A \tag{10}$$

for $t \in [a, n]$.

By (9), there exists a sequence of integers $\{nj\}$ such that the sequences $\{b_{0nj}\}$, $\{b_{3nj}\}$, $\{c_{2nj}\}$ and $\{c_{3nj}\}$ converge respectively to numbers b_0, b_3, c_2, c_3 such that

$$b_0^2 + b_3^2 = c_2^2 + c_3^2 = 1\,. \tag{11}$$

Let u and v be the solutions of (L) defined by

$$u(t) = b_0\, z_0(t) + b_3\, z_3(t)\,, \quad v(t) = c_2\, z_2(t) + c_3\, z_3(t)\,. \tag{12}$$

By (11) u and v are not identically zero. Clearly the sequences $\{u_{n_j}(t)\}$ and $\{v_{n_j}(t)\}$ converge pointwise to $u(t)$ and $v(t)$, respectively, on $[a, \infty)$. From (10) it follows that $[u(t)]^2 \leqq A$ and $[v(t)]^2 \leqq A$ on $[a, \infty)$. If $u(t)$ and $v(t)$ were dependent, then from (12) it would follow that $u(t) = k\, z_3(t)$ for some $k \neq 0$. Since $z_3(a) = z_3'(a) = z_3''(a) = 0$, $z_3'''(a) = 1$, it follows from the assumptions of Theorem 1 that $\lim_{t\to\infty} |u(t)| = \infty$, which is a contradiction. This proves the independence of u and v.

Suppose that u is non-oscillatory. Without loss of generality we may assume that $u(t) > 0$ and hence $u^{(4)}(t) > 0$ for $t \geqq b \geqq a$. This implies that for large t none of the functions $u^{(4)}(t)$, $u'''(t)$, $u''(t)$, and $u'(t)$ change sign. If either $u^{(4)}(t)\, u'''(t) > 0$, $u'''(t)\, u''(t) > 0$, or $u''(t)\, u'(t) > 0$ from a certain point on, then $\lim_{t\to\infty} |u(t)| = \infty$, a contradiction. Hence, there exists a $c \geqq a$ such that for $t \geqq c$,

$$u'''(t) < 0\,, \quad u''(t) > 0\,, \quad u'(t) < 0\,, \quad u(t) > 0\,. \tag{13}$$

By Lemma 1.2 the inequalities (13) must hold on $[a, c]$, contrary to

$$u'(a) = b_0\, z_0'(a) + b_3\, z_3'(a) = 0\,.$$

This shows that u and, by the same token, v are oscillatory.

Remark 1. From the above proof it is clear that any non-trivial linear combination of u and v is oscillatory.

Remark 2. Using Lemma 1.2 and an argument similar to the one used in the proof of Theorem 1, one can establish the existence of a solution with property (8) on $[a, \infty)$. Using this and Theorem 1 it is easy to show that the hypotheses of Theorem 1 imply that the space of solutions of (L) which are bounded on $[a, \infty)$ has dimension three.

We now consider the behavior of oscillatory solutions if $p(t)$ tends monotonically to infinity.

Theorem 2. *If in addition to the hypotheses of Theorem 1 it is assumed that* $\lim_{t\to\infty} p(t) = +\infty$, *then for any oscillatory solution* $y(t)$ *of* (L), $\lim_{t\to\infty} y(t) = 0$.

Proof. We shall prove the theorem using two lemmas.

Lemma 2.1. *Suppose* $p(t) \in C'[a, \infty)$, $p(t) > 0$, $p'(t) \geqq 0$. *If* $y(t)$ *is any oscillatory solution of* (L), *then* $y'(t)$ *is bounded on* $[a, \infty)$.

Proof. Set

$$G_y(t) = \frac{[y'''(t)]^2}{p(t)} - 2y(t)\, y''(t) + [y'(t)]^2\,.$$

As may be verified by differentiation,

$$G_y(t) = G_y(a) - \int_a^t p'(s) \left[\frac{y'''(s)}{p(s)}\right]^2 ds \leqq G_y(a)\,. \tag{14}$$

Since $y(t)$ is oscillatory, $y'(t)$ has an unbounded set of zeros. Therefore it suffices to show that $y'(t)$ is bounded on the set of zeros of $y''(t)$, but if $y''(\bar{t}) = 0$, then $[y'(\bar{t})]^2 \leqq \leqq G_y(\bar{t}) \leqq G_y(a)$, completing the proof.

Now suppose that the oscillatory solution $y(t)$ does not tend to zero as t tends to infinity. We may assume without loss of generality that for some $\varepsilon > 0$ there is a sequence $\{t_n\}$ of relative maxima of $y(t)$ with $\lim_{n\to\infty} t_n = \infty$ and $y(t_n) \geqq \varepsilon$.

Here, we could invoke a comparison theorem of LEIGHTON and NEHARI [3, p. 340] to show that the distance between successive zeros of $y(t)$ tends to zero at infinity. However, the following lemma is sufficient for our purposes and has a simpler proof than the Leighton-Nehari result.

Lemma 2.2. *Under the above conditions, for each n let s_n be the last point before t_n at which $y(t) = \varepsilon/2$. Then $\lim_{n\to\infty} (t_n - s_n) = 0$.*

Proof. Since t_n is a relative maxima for $y(t)$, we have $y''(t_n) \leqq 0$. We shall show that

$$\frac{[y'''(t)]^2}{p(t)} \leqq k \equiv G_y(a) \quad \text{for} \quad s_n \leqq t \leqq t_n\,, \tag{15}$$

where $G_y(t)$ is defined as in Lemma 2.1. This follows from (14) at any point $t \in [s_n, t_n]$ at which $y''(t) \leqq 0$. Therefore, assume that $\sigma_n \geqq s_n$, where $\sigma_n = \sup\{t \mid t \leqq t_n, y''(t) \geqq 0\}$. We have to show that (15) holds on $[s_n, \sigma_n]$. Clearly $y'''(\sigma_n) \leqq 0$. Let r_n be the last zero of $y(t)$ before t_n. Since $y^{(4)}(t) > 0$ on (r_n, σ_n), we must have $y'''(t) \leqq 0$ on this interval. Hence, $[y'''(t)]^2$ is decreasing on this interval, and since $p'(t) \geqq 0$, $[y'''(t)]^2/p(t)$ has its maximum over $[r_n, \sigma_n]$ at r_n. From (14) and the definition of $G_y(t)$, $[y'''(r_n)]^2/p(r_n) \leqq k$ and this proves (15).

From (14) it also follows that on $[s_n, t_n]$, $y''(t) \geqq -k/\varepsilon$. Since $y^{(4)}(t) \geqq \frac{1}{2} p(s_n)\,\varepsilon$ on $[s_n, t_n]$, we have on this interval that

$$y'''(t) \geqq y'''(s_n) + (t - s_n)\, p(s_n) \frac{\varepsilon}{2} \geqq -\sqrt{(k\, p(s_n))} + (t - s_n)\, p(s_n) \frac{\varepsilon}{2},$$

and therefore

$$y''(t_n) \geqq y''(s_n) - \sqrt{(k p(s_n))}\,(t_n - s_n) + (t_n - s_n)^2\, p(s_n) \frac{\varepsilon}{4} \geqq$$

$$\geqq -\frac{k}{\varepsilon} - \sqrt{(k\, p(s_n))}\,(t_n - s_n) + (t_n - s_n)^2\, p(s_n) \frac{\varepsilon}{4}.$$

From the fact that $y''(t_n) \leqq 0$ and $\lim_{n\to\infty} p(s_n) = \infty$, we see at once that $\lim_{n\to\infty} (t_n - s_n) = 0$, proving Lemma 2.2.

To prove Theorem 2, we note that $y(t_n) - y(s_n) \geqq \varepsilon/2$, so by the mean value theorem, there is a point $h_n \in (t_n, s_n)$ such that $y'(h_n) \geqq \varepsilon/2(t_n - s_n)$. By Lemma 2.2, $\lim_{n\to\infty} y'(h_n) = +\infty$, but this contradicts Lemma 2.1, so that the conditions pricceding Lemma 2.2 cannot hold. Therefore, if $y(t)$ is oscillatory, $\lim_{t\to\infty} y(t) = 0$.

References

[1] *A. S. Galbraith, E. J. McShane,* and *Gene B. Parrish:* On the Solutions of Linear Second-order Differential Equations, Proc. Nat. Acad. Sci., Vol. *53* (1965) pp. 247—249.

[2] *A. C. Lazer:* On the Asymptotitc Behavior of Solutions of the Differential Equation $y'' + p(x)\, y = 0$, Proc. Amer. Math. Soc., Vol. *16* (1965) pp. 1295—1298.

[3] *Walter Leighton* and *Zeev Nehari:* On the Oscillations of Solutions of Self-adjoint Linear Differential Equations of the Fourth Order, Trans. Amer. Math. Soc., Vol. *89* (1958) pp. 325—377.

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