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ORBITS OF TRANSFORMATION GROUPS ON CERTAIN GRASSMANN MANIFOLDS

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INTRODUCTION

Let a connected Lie group G act effectively on a manifold V^n of the class C^∞. Denote by $\mathfrak{g}$ the Lie algebra of G. Then G acts on $\mathfrak{g}$ as the adjoint group $\mathrm{Ad}(G)$. Let us denote by $Z^{[k]}$ the Grassmann manifold of all linear subspaces $\mathscr{P} \subseteq \mathfrak{g}$ of dimension k; then G acts canonically on $Z^{[k]}$. Consider a fixed Lie subgroup H of G and its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Put $h = \dim \mathfrak{h}$. For any $k \geqq h$, denote by $Z_k \subset Z^{[k]}$ the manifold of all subspaces $\mathscr{P} \subseteq \mathfrak{g}$ such that $\dim \mathscr{P} = k$ and $\mathscr{P} \supseteq \mathfrak{h}$. Then H acts on the manifold Z_k.

A. ŠVEC (see [1]) has occupied himself thoroughly with Cartan's method of specialization of frames in connection with the equivalence problem for surfaces. He has given a precise formulation of the equivalence problem and also an exact description of the gradual steps of the speciaization procedure; both in the terms of infinitesimal connections in fibre bundles. Roughly speaking, if we are given a surface M in a space V^n, then according to A. Švec, each step of the specialization procedure can be interpreted in the following way: we are given a manifold Z_k, a Lie group H acting on Z_k as above, and a system of orbits under H on Z_k. We have to construct a local submanifold of Z_k crossing each orbit of the system exactly at one point.

From this point of view it is a very important thing to obtain a survey of all orbits of Z_k under the group H. This problem has not been treated in the work [1]. In the present paper we propose a method, which leads, in some cases, to a complete classification of orbits of an arbitrary H-invariant manifold $Z \subseteq Z^{[k]}$. Our idea is the following one: we attach to any point $\mathscr{P}$ of the manifold Z a simple figure $R(\mathscr{P})$ of the original space V^n (a "representing frame") such that

a) the map $R : \mathscr{P} \to R(\mathscr{P})$ is one-to-one,

b) the group H acts in a similar way both on the manifold Z and on the set $R(Z)$ of figures of V^n.

Then the original problem is reduced to a new one: find all the orbits under H of the set $R(Z)$. The applicability of our method to problems of classical differential geometry is now guaranteed by the fact, that in those cases, V^n will be the n-dimensional affine (or projective) space and G, H will be some groups of affine (or projective) transformations. We can see that if the figures $R(\mathscr{P})$ are really very simple, for instance composed of linear subspaces, we obtain eventually a problem of the linear geometry.

In the first Part of this work our method will be fully discussed under the assumption that V^n is the n-dimensional affine space A^n and G is a subgroup of the group $GA(n)$. It seems that some results of this Part can be generalized still further. In particular it could be interesting to re-formulate them for V^n being a projective space and G a group of projective transformations.

As an example, we shall solve the following non-trivial problem: let G be the group $GA^+(2)$ = the component of unity of the whole affine group $GA(2)$, $\mathfrak{g}$ its Lie algebra, Γ_4 the Grassmann manifold of all 4-dimensional subspaces of $\mathfrak{g}$. *Find all orbits under G of the manifold* Γ_4. A complete classification of orbits of Γ_4 will be performed in the second Part of this paper. Although it is of no use for the classical differential geometry, it is nevertheless valuable for testing the efficiency of our method and for collecting a great deal of various material.

In the paper [6] our method has been applied to the equivalence problem of surfaces of the equiaffine space A_e^3. The representing frames constructed there are shown to be geometrical objects well-known from the affine differential geometry.

PART I. GENERALITIES

1. G-COVERING SETS

The words "differentiable" or "smooth" will be used to mean "differentiable of class C^∞". *A differentiable action φ of a Lie group G on a smooth manifold M* is a differentiable map $\varphi : G \times M \to M$ such that $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ and $\varphi(e, x) = x$ for each $g, h \in G$, $x \in M$, where e denotes the unit element in G. Each differentiable action φ induces a) a homomorphism $\Phi : G \to G(M)$ of G onto certain group of diffeomorphisms of the manifold M, and b) a homomorphism $\Phi_* : \mathfrak{g} \to \mathrm{X}(M)$ of the Lie algebra $\mathfrak{g}$ into the infinite Lie algebra of all smooth global vector fields on M. The vector fields of the form $\Phi_*(X)$, $X \in \mathfrak{g}$, are called *fundamental vector fields of M* with respect to φ, or else with respect to G, if φ is fixed. If Φ is an isomorphism, then Φ_* is an isomorphism, too. The Lie algebra $\Phi_*(\mathfrak{g}) \subset \mathrm{X}(M)$ will be called *the Lie algebra of the transformation group* $\Phi(G) = G(M)$. *The orbit under G of a point* $x \in M$ is the intransitivity class of x, i.e., the set $\varphi(G \times x) = \{\varphi(g, x) \mid g \in G\}$. If a fixed differentiable action $\varphi : G \times M \to M$ is preassigned, we say that G *acts* (differentiably) *on* M. We shall also write simply $g \,.\, x$ instead of $\varphi(g, x)$ in this case.

We shall also use analogous but more general formulations as "continuous action of a topological group G on a topological space X" or "a topological group G acts (continuously) on a topological space X".

Definition I. Let a topological group G act continuously on a topological space X. Let $U \subseteq Y \subseteq X$ be open subsets of X. The open set U will be called a *G-covering set* of Y if the following axiom holds: for any $x \in Y$, and any neighbourhood $V(e)$ of the unit element e of G, there is an element $g \in V(e)$ such that $g \,.\, x \in U$.

The following assertions result easily from the continuity of group actions and their proofs will be omitted.

Proposition I. *Let G act continuously on X and let $Y \subseteq X$ be an open set. Then*

a) If $U_1, U_2, \ldots, U_k$ are G-covering sets of Y, then so is their intersection $U_1 \cap \cap U_2 \cap \ldots \cap U_k$.

b) If U is a G-covering set of Y and $V \subseteq U$ is a G-covering set of U, then V is a G-covering set of Y.

Proposition II. *Let G act continuously on X and let U be a G-covering set of X. Then*

a) Each invariant subset of X, in particular each orbit under G, intersects U.

b) If $x_1, x_2, \ldots, x_k \in X$, and $W \subseteq G$ is open, then there is an element $g \in W$ such that $g \,.\, x_i \in U$ for $i = 1, 2, \ldots, k$.

c) For any $g \in G$, $g \,.\, U$ is a G-covering set of X.

d) If $x_1, \ldots, x_k \in X$ and $W \subseteq G$ is open, then there is a G-covering set $U_g = g \,.\, U$ of X such that $g \in W$ and $x_i \in U_g$ for $i = 1, 2, \ldots, k$.

Proposition III. *Let G act continuously on the spaces X, Y and consider its induced action on the product space $X \times Y$. If U (or V) is a G-covering set of X (or Y), then $U \times V$ is a G-covering set of $X \times Y$.*

Proposition IV. *Let X be a fibre bundle with a bundle projection $p : X \to B$. Let φ be a fibre preserving continuous action of G on X, and let $\tilde{\varphi}$ be the induced action of G on the basis B. Then $\tilde{U} \subseteq B$ is a G-covering set of B if and only if $p^{-1}(\tilde{U})$ is a G-covering set of X.*

In case of differentiable action, we have the following sufficient condition for a subset to be G-covering.

Proposition V. *Let a Lie group G act differentiably on a manifold M, let $N \subseteq M$ be some invariant subspace and $U \subseteq N$. Assume that for each point $p \in N$ there is*

1) a neighbourhood $O(p)$ with respect to M and differentiable functions $f_1(q)$, $f_2(q), \ldots, f_k(q)$ defined on $O(p)$ such that $q \in (N \dot{-} U) \cap O(p) \Leftrightarrow f_i(q) = 0$ for $i = = 1, 2, \ldots, k$,

2) *a fundamental vector field* $X \in \mathfrak{X}(M)$ *such that* $X_p(f_i) \neq 0$ *for some index* i. *Then* U *is a* G*-covering set of* N.

The proof can be performed by standard methods and it will be left to the reader. Let us remark only that $a)$ the property of U to be open and non-empty follows directly from conditions 1) and 2), $b)$ condition 2) may be weakened by the assumption that p runs through $N \dot{-} U$ instead of the whole N.

2. EQUIVARIANT OBJECTS AND REPRESENTING FRAMES

Definition II. Let a topological group G act on topological spaces X and Y. A map $O : Y \to \exp(X)$ will be called *an equivariant object on* Y *with values in* X if, for any $y \in Y$, $g \in G$, we have $O(g \,.\, y) = g \,.\, O(y)$. (Here $g \,.\, O(y)$ denotes the set $\{g \,.\, x \mid x \in O(y)\}$ and $\exp(X)$ the set of al subsets of X.)

Definition III. A finite set $\{O_1, O_2, \ldots, O_m\}$ of equivariant objects on the space Y with values in the space X is called a *representing frame on* Y *with values in* X, if the relations $y \in Y$, $y' \in Y$, $O_i(y') = O_i(y)$ for $i = 1, 2, \ldots, m$ always imply $y' = y$.

3. COMPLEX ELEMENTS

Let E be a vector space over a commutative field K; *we can associate to* E *exactly one affine space* A *over* K. E is then called *the space of free vectors of the affine space* A. (See Bourbaki [2]). Let A^n be the n-dimensional affine space over real numbers, E^n the corresponding vector space. Let us denote by CE^n the *complexification* $E^n + iE^n$ of the space E^n (See [3]). The affine space CA^n which is associated to CE^n will be called *the complexification of the real affine space* A^n. We have a canonical injection $A^n \to CA^n$. The elements of the set $CA^n \dot{-} A^n$ are called *imaginary points* of the space A^n. The one-dimensional complex affine subspaces of CA^n are said to be *complex lines* of the space A^n. The usual lines of A^n or the complex lines $CA^1 \subset CA^n$ such that $\dim_R (CA^1 \cap A^n) = 1$ will be referred to as *real lines of* A^n. The other complex lines will be called *imaginary* ones. As usual, we can introduce the notion of *complex conjugate lines*, and similar concepts for the subspaces of CA^n of higher dimensions.

Let H be a connected subgroup of the affine group $GA(n)$ acting on A^n. Then H acts on the space CA^n according to the rule $h(a + bi) = h \,.\, a + (h \,.\, b)\, i$ for $h \in H$, $a, b \in A^n$. Let $\mathfrak{h}$ be the Lie algebra of H, E the underlying vector space of $\mathfrak{h}$. By the *complexification* $C\mathfrak{h}$ *of the algebra* $\mathfrak{h}$ we mean the space CE together with the operation $[X + iY, X' + iY'] = [X, X'] - [Y, Y'] + i\{[X, Y'] + [Y, X']\}$. $C\mathfrak{h}$ is a complex Lie algebra and $\dim_C C\mathfrak{h} = \dim_R \mathfrak{h}$. The algebra $C\mathfrak{h}$ generates a connected complex Lie group CH of transformations of the space CA^n; the latter will be called the *complexification of the group* H. Every complex one-dimensional subspace ξ^C of $C\mathfrak{h}$ determines a one-dimensional complex Lie subgroup in CH.

Let A^n be a real affine space of dimension n. Any coordinate system in A^n can be described as a diffeomorphism $\mathfrak{R}^\alpha : A^n \to \mathbf{R}^n$ onto a fixed coordinate space $\mathbf{R}^n$. Any coordinate system $\mathfrak{R}^\alpha$ in A^n induces canonically a coordinate system $C\mathfrak{R}^\alpha : CA^n \to \to \mathbf{C}^n$. Let $g \in GA(n)$ be some affine transformation of A^n; then the coordinate system $\mathfrak{R}^\beta : x \to \mathfrak{R}^\alpha(g \,.\, x)$ will be denoted by $\mathfrak{R}^\alpha \circ g$. Let $G \subseteq GA(n)$ be some subgroup. Two coordinate systems $\mathfrak{R}^\alpha$, $\mathfrak{R}^\beta$ in A^n will be referred to as *belonging to the same coordinate G-type* if $\mathfrak{R}^\beta = \mathfrak{R}^\alpha \circ g$ for some $g \in G$. If a fixed subgroup G and a fixed coordinate G-type $\mathfrak{R}(G)$ is chosen, then all coordinate systems $\mathfrak{R}^\alpha \in \mathfrak{R}(G)$ will be called *admissible*. Let $\mathfrak{g}$ be the Lie algebra of the group G, $\dim \mathfrak{g} = r$. Let us choose once for all $n \times r$ functions $\eta_i^j(\xi_1, \ldots, \xi_n)$ on the coordinate space $\mathbf{R}^n$ such that the vector fields

$$\sum_{j=1}^{n} \eta_i^j(x_1^\alpha, \ldots, x_n^\alpha) \frac{\partial}{\partial x_j^\alpha},$$

$i = 1, 2, \ldots, r$, form a basis of $\mathfrak{g}$ with respect to any admissible coordinate system $\mathfrak{R}^\alpha(x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha)$. For instance, if $G = GA(n)$, we can put

$$\text{(I)} \qquad (\eta_i^j) = \begin{pmatrix} 1, \ldots, 0, \xi_1, 0, \ldots, 0, \ \ldots, \xi_n, \ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0, \ldots, 1, \ 0, \ 0, \ldots, \xi_1, \ldots, 0, \ \ldots, \xi_n \end{pmatrix}.$$

Let $\mathfrak{R}^\alpha$ be an admissible coordinate system in A^n, $\mathfrak{R}^\alpha x = (x_1^\alpha, \ldots, x_n^\alpha)$ for $x \in A^n$. Put $\mathfrak{S}_\alpha X = (a_\alpha^1, \ldots, a_\alpha^r)$ for any $X \in \mathfrak{g}$, where the numbers $a_\alpha^1, \ldots, a_\alpha^r$ are determined by the relation

$$X = \sum_{i,j} a_\alpha^i \eta_i^j(x_1^\alpha, \ldots, x_n^\alpha) \frac{\partial}{\partial x_j^\alpha}$$

with respect to the coordinate system $\mathfrak{R}^\alpha$. The map $\mathfrak{S}_\alpha : \mathfrak{g} \to \mathbf{R}^r$ just introduced will be called *an admissible coordinate system in* $\mathfrak{g}$ *induced by* $\mathfrak{R}^\alpha$. We can see easily that whenever $\mathfrak{R}^\beta = \mathfrak{R}^\alpha \circ g$, then $\mathfrak{S}_\beta = \mathfrak{S}_\alpha \circ \operatorname{Ad}(g)$. The numbers $a_\alpha^1, \ldots, a_\alpha^r$ will be sometimes called briefly *the coordinates of vector $X \in \mathfrak{g}$ with respect to the coordinate system $\mathfrak{R}^\alpha$.*

Let $Z^{[k]}$ be the Grassmann manifold of all subspaces of dimension k of the algebra $\mathfrak{g}$; then for $k \leqq r$ $\dim Z^{[k]} = k(r - k)$. For any admissible $\mathfrak{R}^\alpha$ let us denote by $E_{i_1,\ldots,i_{r-k}}^\alpha$ the subspace determined in $\mathfrak{g}$ by the vectors

$$\sum_{j=1}^{n} \eta_{i_1}^j(x_1^\alpha, \ldots, x_r^\alpha) \frac{\partial}{\partial x_j^\alpha}, \ldots, \sum_{j=1}^{n} \eta_{i_{r-k}}^j(x_1^\alpha, \ldots, x_n^\alpha) \frac{\partial}{\partial x_j^\alpha}.$$

Further let $U_{i_1,\ldots,i_{r-k}}^\alpha$ be the set of all subspaces $\mathscr{P} \in Z^{[k]}$ such that $\mathscr{P} \cap E_{i_1,\ldots,i_{r-k}}^\alpha = = 0$. Let $\mathscr{P} \in Z^{[k]}$ be given by the system of equations

$$\text{(II)} \qquad \sum_{i=1}^{r} m_i^l a_\alpha^i = 0\,, \quad l = 1, 2, \ldots, r - k\,,$$

with respect to $\mathfrak{R}^\alpha$. Then $\mathscr{P} \in U^\alpha_{i_1,\dots,i_{r-k}}$ if and only if the determinant $[m^l_{i_1}, m^l_{i_2}, \dots, m^l_{i_{r-k}}] \neq 0$. Thus the open sets $U^\alpha_{i_1,\dots,i_{r-k}}$, where $\{i_1, \dots, i_{r-k}\} \subset \{1, 2, \dots, r\}$, form a finite covering of $Z^{[k]}$. In each $U^\alpha_{i_1,\dots,i_{r-k}}$ a local coordinate system

$$\mathfrak{S}^\alpha_{i_1,\dots,i_{r-k}} : U^\alpha_{i_1,\dots,i_{r-k}} \to \mathbf{R}^{k(r-k)}$$

is given as follows: for each $\mathscr{P} \in U^\alpha_{i_1,\dots,i_{r-k}}$ we have

$$\mathfrak{S}^\alpha_{i_1,\dots,i_{r-k}}(\mathscr{P}) = \begin{pmatrix} u^\alpha_{1,1}, & \dots, u^\alpha_{1,k} \\ \dots & \dots \\ u^\alpha_{r-k,1}, & \dots, u^\alpha_{r-k,k} \end{pmatrix}$$

if and only if the system of equations (II) being solved with respect to $a^{i_1}_\alpha, \dots, a^{i_{r-k}}_\alpha$ assumes the form

$$\left.\begin{array}{l} a^1_\alpha = u^\alpha_{1,1} a^{j_1}_\alpha + \dots + u^\alpha_{1,k} a^{j_k}_\alpha \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a^{i_{r-k}}_\alpha = u^\alpha_{r-k,1} a^{j_1}_\alpha + \dots + u^\alpha_{r-k,k} a^{j_k}_\alpha \end{array}\right\} \quad \begin{array}{l} j_1 < j_2 < \dots < j_k\,, \\ \{i_1, \dots, i_{r-k}, j_1, \dots, j_k\} = \{1, 2, \dots, r\}\,. \end{array}$$

All possible local coordinate systems $\mathfrak{S}^\alpha_{i_1,\dots,i_{r-k}}$ will be called *admissible*; only the coordinate systems of this form will be used in the following. If, in a particular problem, only $\mathfrak{S}^\alpha_B$ with a fixed multi-index $B = (i_1, \dots, i_{r-k})$ are used, then the coordinates $u^\alpha_{i,j}$ with respect to $\mathfrak{S}^\alpha_B$ will be called briefly *coordinates with respect to* $\mathfrak{R}^\alpha$. If $\mathfrak{R}^\beta = \mathfrak{R}^\alpha \circ g$, we shall also write $\mathfrak{S}^\beta_B = \mathfrak{S}^\alpha_B \circ g$. We can see easily that $\mathfrak{S}^\beta_B = \mathfrak{S}^\alpha_B \circ g$ implies $U^\beta_B = g^{-1} \,.\, U^\alpha_B$. The subspaces $\mathscr{P} \subset \mathfrak{g}$ belonging to a manifold $Z^{[k]}$ in question will be called *k-blocks* or simply *blocks in* $Z^{[k]}$.

5. G-COVERING SETS AND EQUIVARIANT OBJECTS IN $Z^{[k]}$

Proposition VI. Assumption A. *Let $H \subset G$ be two subgroups of the whole affine group $GA(n)$, $\mathfrak{g}$ the Lie algebra of G, $\dim \mathfrak{g} = r$, and let $Z^{[k]}$ be the Grassmann manifold $\{\mathscr{P} \subset \mathfrak{g} \mid \dim \mathscr{P} = k\}$. Let $\mathfrak{M} \subset Z^{[k]}$ be a subspace which is invariant under H, or more precisely, under* Ad (H). *Let $\mathfrak{R}(H)$ be a coordinate H-typ in A^n and let us suppose there is a coordinate system $\mathfrak{R}^0 \in \mathfrak{R}(H)$ and a multi-index $B = (i_1, \dots, i_{r-k})$ such that $\mathfrak{M} \cap U^0_B$ is an H-covering set of $\mathfrak{M}$.*

Assumption B. *Let* $O : \mathfrak{M} \to \exp(A^n)$ $(\mathfrak{M} \to \exp(CA^n))$ *be a map. Let us suppose that*:

1) *To each $u \in \mathfrak{M}$ there is given a non-empty open set $U(H, u) \subset \mathfrak{R}(H)$, $\mathfrak{R}(H)$ being provided with the topology of the group H.*

2) *There is an open set $\bar{U} \subset \mathbf{R}^{k(r-k)}$ and s functions $F_i(\xi_1, \dots, \xi_r, u_{1,1}, \dots, u_{r-k,r})$, $(i = 1, 2, \dots, s)$ given on $\mathbf{R}^n \times \bar{U}$ (or $\mathbf{C}^n \times \bar{U}$) such that*

a) for $u \in \mathfrak{M}$, $\mathfrak{R}^\alpha \in \mathfrak{R}(H)$, $u \in U_B^\alpha$ *we have* $\mathfrak{R}^\alpha \in U(H, u)$ *if and only if* $\mathfrak{S}_B^\alpha(u) \in \bar{U}$,

b) whenever $u \in \mathfrak{M}$, $\mathfrak{R}^\alpha \in U(H, u)$, $u \in U_B^\alpha$ *and* $x \in A^n(x \in CA^n)$, *then* $x \in O(u)$ *if and only if the the relations*

$$\text{(III)} \qquad \begin{aligned} &F_i(\xi_1^\alpha, \ldots, \xi_n^\alpha, u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha) = 0\,, && i = 1, 2, \ldots, t \\ &F_j(\xi_1^\alpha, \ldots, \xi_n^\alpha, u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha) \geqq 0\,, && j = t + 1, \ldots, v \\ &F_l(\xi_1^\alpha, \ldots, \xi_n^\alpha, u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha) \neq 0\,, && l = v + 1, \ldots, s \end{aligned}$$

hold, where $(\xi_1^\alpha, \ldots, \xi_n^\alpha) = \mathfrak{R}^\alpha(x)$ (or $C\mathfrak{R}^\alpha(x)$), $(u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha) = \mathfrak{S}_B^\alpha(u)$. *Here the functions* F_i, F_l *assume values from* $\mathbf{R}$ (or $\mathbf{C}$) *and the functions* F_j *merely from* $\mathbf{R}$.

Under these assumptions, the map O *is an equivariant object on* $\mathfrak{M}$ *with values in* A^n (or in CA^n) *with respect to the actions of* H *on* $\mathfrak{M}$ *and on* A^n (on CA^n).

Proof. Let be given $u \in \mathfrak{M}$, $h \in H$. Put $W = \{g \in H \mid \mathfrak{R}^0 \circ g^{-1} \in U(H, hu)\}$, then $W \subset H$ is an open set. Because $\mathfrak{M} \cap U_B^0$ is supposed to be H-covering set of $\mathfrak{M}$, there is an element $h' \in W$ such that $h \,.\, u \in h'(\mathfrak{M} \cap U_B^0) = \mathfrak{M} \cap h'(U_B^0)$ (Proposition II, d)). Thus there is a coordinate system $\mathfrak{R}^\beta \in U(H, hu)$ such that $h \,.\, u \in (\mathfrak{M} \cap U_B^0)$. We can put $\mathfrak{R}^\beta = \mathfrak{R}^0 \circ h'^{-1}$. Let us choose $x \in O(u)$ and denote $\mathfrak{R}^\alpha = \mathfrak{R}^\beta \circ h$. Then $u \in \mathfrak{M} \cap h^{-1}(U_B^\beta)$; hence $u \in \mathfrak{M} \cap U_B^\alpha$, $\mathfrak{S}_B^\alpha(u) = \mathfrak{S}_B^\beta(h \,.\, u) \in \bar{U}$ and consequently $\mathfrak{R}^\alpha \in U(H, u)$. If we put $\mathfrak{R}^\alpha(x) = (\xi_1^\alpha, \ldots, \xi_n^\alpha)$, $\mathfrak{S}_B^\alpha(u) = (u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha)$, the relations (III) hold. Now $\mathfrak{R}^\beta(h \,.\, x) = \mathfrak{R}^\alpha(x)$. According to the former relation $\mathfrak{S}_B^\beta(h \,.\, u) = \mathfrak{S}_B^\alpha(u) \in \bar{U}$ we can see that the system (III) is satisfied by the β-coordinates of the points $h \,.\, x \in A^n$, $h \,.\, u \in \mathfrak{M}$. Because of $h \,.\, u \in U_B^\beta$, $\mathfrak{R}^\beta \in U(H, h \,.\, u)$ we have $h \,.\, x \in O(h \,.\, u)$. Thus the inclusion $h \,.\, O(u) \subset O(h \,.\, u)$ is proved. If we convert the parts of the points $u, h \,.\, u \in \mathfrak{M}$, we obtain $h^{-1}O(h \,.\, u) \subset O(u)$ and this completes our proof.

The complex case can be discussed in the same way.

Note. We shall often use a particular case of the Proposition, where $U(H, u) = \mathfrak{R}(H)$ for any $u \in \mathfrak{M}$ and $\bar{U} = \mathbf{R}^{k(r-k)}$.

Proposition VII. *Let us suppose that the conditions* **A** *of Proposition VI are fulfilled. Let* $O_1, \ldots, O_m$ *be equivariant objects on* $\mathfrak{M}$ *with values in a space* X. *Suppose that there exists an integer* $s \geqq 0$ *with the following property: to each* $u_0 \in \mathfrak{M}$ *we can assign a non-empty open subset* $U(H, u_0) \in \mathfrak{R}(H)$ *such that for any* $\mathfrak{R}^\alpha \in U(H, u_0)$ *we have* $u_0 \in U_B^\alpha$ *and there are exactly* s *points* $u_j \neq u_0$ *in* $\mathfrak{M} \cap U_B^\alpha$ *such that* $O_i(u_j) = O_i(u_0)$ *for* $i = 1, 2, \ldots, m$, $j = 1, \ldots, s$. *Then for any* $u_0 \in \mathfrak{M}$ *there are exactly* s *points* $u_j \neq u_0$ *in* $\mathfrak{M}$ *such that* $O_i(u_j) = O_i(u_0)$ *for* $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, s$. *Particularly, if* $s = 0$, *then the objects* $O_1, O_2, \ldots, O_n$ *form a representing frame on* $\mathfrak{M}$ *with respect to the group* H.

Proof. Let us suppose there are $s + 1$ points $u_1, \ldots, u_s, u_{s+1} \in \mathfrak{M}$ such that $u_j \neq u_0$, $O_i(u_j) = O_i(u_0)$ *for* $j = 1, 2, \ldots, s + 1$, $i = 1, 2, \ldots, m$. Let us denote by $W \subset H$ the open set determined by the relation $U(H, u_0) = \mathfrak{R}^0 \circ W$. According to Proposition II, d), there is an element $h \in W$ such that $u_0, u_1, \ldots, u_{s+1} \in h^{-1}(U_B^0)$. If we put $\mathfrak{R}^\alpha = \mathfrak{R}^0 \circ h$ then $\mathfrak{R}^\alpha \in U(H, u_0)$ and $u_0, u_1, \ldots, u_{s+1} \in \mathfrak{M} \cap U_B^\alpha$. But this is a contradiction.

6. IMPROPER ELEMENTS

Let A^n be a real affine space, E^n the associated vector space of free vectors. The one-dimensional linear subspaces $\xi^\infty \subset E^n$ will be referred to as *improper points of A^n*. The set of all improper points will be denoted by A_∞^{n-1}.

Suppose $\mathfrak{R}^\alpha$ is an admissible coordinate system in A^n, ξ^∞ an improper point. The directional parameters $v_1^\alpha, v_2^\alpha, \ldots, v_n^\alpha$ of a line which is parallel to ξ^∞ are called *homogeneous coordinates* of ξ^∞ with respect to $\mathfrak{R}^\alpha$.

If $G \subset GA(n)$, then the action of G in A^n determines an action in E^n, and in A_∞^{n-1} as well. For any $g \in G$, $\xi^\infty \in A_\infty^{n-1}$, the homogeneous coordinates of the improper point $g \,.\, \xi^\infty$ with respect to $\mathfrak{R}^\alpha$ are the same as the homogeneous coordinates of ξ^∞ with respect to $\mathfrak{R}^\beta = \mathfrak{R}^\alpha \circ g$.

If CA^n is the complexification of A^n, we can introduce the set CA_∞^{n-1} of improper points of CA^n and the homogeneous complex coordinates in CA_∞^{n-1}.

The following modification of Proposition VI can be easily proved:

Proposition VIII. *Let us suppose that the conditions* **A** *of Proposition VI are fulfilled.* Let $O : \mathfrak{M} \to \exp(A_\infty^{n-1})$ (or $O : \mathfrak{M} \to \exp(CA_\infty^{n-1})$) *be a map. Further suppose that*

1) *To each $u \in \mathfrak{M}$ there is given a non-empty open set $U(H, u) \subset \mathfrak{R}(H)$, $\mathfrak{R}(H)$ being provided with the topology of the group H.*

2) *There is an open set $\bar{U} \subset \mathbf{R}^{k(r-k)}$ and s functions $F_i(\xi_1, \ldots, \xi_n, u_{1,1}, \ldots, u_{r-k,r})$, $i = 1, 2, \ldots, s$ given on $\mathbf{R}^n \times \bar{U}$ (or on $\mathbf{C}^n \times \bar{U}$) with values in $\mathbf{R}$ (or in $\mathbf{C}$) such that*

a) $F_i(\lambda\xi_1, \ldots, \lambda\xi_n, u_{1,1}, \ldots, u_{r-k,k}) = \lambda^{k_i} F_i(\xi_1, \ldots, \xi_n, u_{1,1}, \ldots, u_{r-k,k})$ *identically on* $\mathbf{R} \times \mathbf{R}^n \times \bar{U}$ *(or on* $\mathbf{C} \times \mathbf{C}^n \times \bar{U}$*),*

b) for $u \in \mathfrak{M}$, $\mathfrak{R}^\alpha \in \mathfrak{R}(H)$, $u \in U_B^\alpha$ we have $\mathfrak{R}^\alpha \in U(H, u)$ if and only if $\mathfrak{S}_B^\alpha(u) \in \bar{U}$,

c) whenever $u \in \mathfrak{M}$, $\mathfrak{R}^\alpha \in U(H, u)$, $u \in U_B^\alpha$ and $\xi^\infty \in A_\infty^{n-1}$ (or ${}^c\xi^\infty \in CA_\infty^{n-1}$), then $\xi^\infty \in O(u)$ if and only if the relations

$$\text{(III}'\text{)} \qquad \begin{aligned} &F_i(v_1^\alpha, \ldots, v_n^\alpha, u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha) = 0\,, \quad i = 1, 2, \ldots, t \\ &F_j(v_1^\alpha, \ldots, v_n^\alpha, u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha) \neq 0\,, \quad j = t + 1, \ldots, s \end{aligned}$$

hold, where $(u_{1,1}^\alpha, \ldots, u_{r-k,k}^\alpha) = \mathfrak{S}_B^\alpha(u)$ and $v_1^\alpha, \ldots, v_n^\alpha$ are homogeneous coordinates of the improper point ξ^∞ with respect to $\mathfrak{R}^\alpha$ (or homogeneous coordinates of ${}^c\xi^\infty$ with respect to $C\mathfrak{R}^\alpha$). Under these assumptions the map O is an equivariant object on $\mathfrak{M}$ with values in A_∞^{n-1} (or in CA_∞^{n-1}).

If we have to classify the orbits of a manifold $\mathfrak{M} \subseteq Z^{[k]}$ under a group $H \subset G \subset \subset GA(n)$, our main task is to accomplish a suitable invariant decomposition $\mathfrak{M} = = \bigcup_i \mathfrak{M}_i$ and to construct a representing frame under H on each $\mathfrak{M}_i$, with values in $CA^n \cup CA^{n-1}_\infty$. As a rule, we start with some open subspace $\mathfrak{N}$ of $\mathfrak{M}$ consisting of "general" points. Then each further step of the mentioned decomposition will arise from the construction of representing frames at the preceding stage. The elements $\mathfrak{M}_i$ of the decomposition will be, as a rule, some manifolds. On each manifold $\mathfrak{M}_i$ we first construct a sufficient number of equivariant objects. In order to obtain an equivariant object we shall often proceed as follows: in the first place we find a map $\xi : \mathfrak{M}_i \to \bar{\mathfrak{g}}$ (or $\xi : \mathfrak{M}_i \to C\bar{\mathfrak{g}}$) into the manifold of all 1-dimensional subspaces („directional elements") of the algebra $\mathfrak{g}$ (or its complexification $C\mathfrak{g}$). This map must not depend on the coordinates. The image $\xi(\mathscr{P})$ of $\mathscr{P} \in \mathfrak{M}_i$ will be usually somehow related with the subspace $\mathscr{P}$ in the algebra $\mathfrak{g}$ or $C\mathfrak{g}$. For instance, $\xi(\mathscr{P})$ will belong to $\mathscr{P}$ or $C\mathscr{P}$, or to a prolongation of $\mathscr{P}$ or $C\mathscr{P}$ by the bracket operation. We shall also use some generalized maps such that the image $\xi(\mathscr{P})$ of $\mathscr{P} \in \mathfrak{M}_i$ will be a finite or infinite system of directional elements. Having obtained a directional element $\xi(\mathscr{P})$ (or a system of directional elements) we construct a subset $O(\mathscr{P}) \subset CA^n \cup CA^{n-1}$. As a rule, $O(\mathscr{P})$ will be the set of all singularities of $\xi(\mathscr{P})$ or some invariant set under the 1-dimensional transformation group determined by $\xi(\mathscr{P})$. Then we choose a suitable multi-index $B = (i_1, \dots, i_{r-k})$, and thus a kind of local coordinates $\mathfrak{S}^\alpha_B$ in $Z^{[k]}$, such that $U_B \cap \mathfrak{M}_i$ is an H-covering set of $\mathfrak{M}_i$. Now we must express the object $O(\mathscr{P})$ analytically using coordinate systems $\mathfrak{R}^\alpha$, $\mathfrak{S}^\alpha_B$, and show that this analytic expression has an invariant form. (See Propositions VI and VIII). So $O(\mathscr{P})$ is proved to be an equivariant object.

Provided we have already found all required equivariant objects $O_1, \dots, O_m$, it remains to show the one-to-one-property of the map $\mathscr{P} \to R(\mathscr{P}) = \{O_1(\mathscr{P}), \dots \dots, O_m(\mathscr{P})\}$ on $\mathfrak{M}_i$. We use the equations (or more generally, the analytic expressions) of the objects $O_1, \dots, O_m$ and Proposition VII for this purpose. At the same time we find the domain of values of the map $\mathscr{P} \to R(\mathscr{P})$. It remains to perform the classification of orbits of the domain $R(\mathfrak{M}_i)$ of values. Let us remark that the evaluation of $R(\mathfrak{M}_i)$ and also the proof of the one-to-one property will become easier if the employed coordinate systems $\mathfrak{S}^\alpha_B$ are not only H-covering but even H_1-covering, where $H_1 \subset H$ is a subgroup. Then we can confine ourselves to a coordinate H_1-type and the coordinates may be specialized in such a way that our analytic expressions take a simpler form. In the optimal case $H_1 = e$, $\mathfrak{S}^\alpha_B$ are global coordinate systems on $\mathfrak{M}_i$. Provided the equivariance property has been already proved, we can work from now on with a unique "canonical" coordinate system.

In some special cases we shall not be able to find a representing frame on $\mathfrak{M}_i$; here we use the method of "reducing the dimension". Let us suppose we have constructed equivariant objects $O_1, \dots, O_m$ on $\mathfrak{M}_i$ such that the domain of values of the map

$\mathscr{P} \to \{O_1(\mathscr{P}), \ldots, O_m(\mathscr{P})\}$ is a manifold of dimension $p < \dim \mathfrak{M}_i$. Then in order to obtain the complete classification of orbits on $\mathfrak{M}_i$ it suffices to classify orbits of a manifold of dimension $(\dim \mathfrak{M}_i - p)$. If this number is sufficiently small the latter problem can be solved directly.

As for the example discussed in the second Part, we shall always be able to obtain a representing frame or, in unfavourable cases, to reduce the dimension to $\dim \mathfrak{M}_i - p = 1$.

PART II. APPLICATION

In this Part we shall occupy ourselves with the following problem: *Let $G = GA^+(2)$ be the group of all positive affine transformations of the affine plane A^2 and Γ_4 the Grassmann manifold of all 4-dimensional subspaces of the Lie algebra $\mathfrak{g} = \mathfrak{ga}(2)$ of the group G. Find all orbits of Γ_4 under G.*

Let us introduce the following *notation*:

G	the group of all positive affine transformations of A^2
$\mathfrak{g}$	the Lie algebra of G
T	the group of all translations of A^2
$\mathfrak{t}$	the Lie algebra of T
G_e	the subgroup of all positive equiaffine transformations of A^2
$\mathfrak{g}_e$	the Lie algebra of G_e
$G_c(p)$	the isotropy group of a point $p \in A^2$ with respect to G
$\mathfrak{g}_c(p)$	the Lie algebra of $G_c(p)$
$G_{ec}(p)$	the isotropy group of a point $p \in A^2$ with respect to G_e
$\mathfrak{g}_{ec}(p)$	the Lie algebra of $G_{ec}(p)$
$\mathfrak{g}^0$	the set of all vectors $X \in \mathfrak{g}$, $X = u(\partial/\partial x) + v(\partial/\partial y) + ax(\partial/\partial x) + bx(\partial/\partial y) + cy(\partial/\partial x) + dy(\partial/\partial y)$ such that the invariant relation $ad - bc = 0$ is satisfied.

Let us introduce, in the first place, the admissible coordinate systems in $\mathfrak{g}$. For any admissible $\mathfrak{R}^\alpha(x^\alpha, y^\alpha)$ in A^2 we consider the ordered basis of the Lie algebra $\mathfrak{g}$ consisting of the vector fields $\partial/\partial x^\alpha$, $\partial/\partial y^\alpha$, $x^\alpha(\partial/\partial x^\alpha)$, $x^\alpha(\partial/\partial y^\alpha)$, $y^\alpha(\partial/\partial x^\alpha)$, $y^\alpha(\partial/\partial y^\alpha)$. For $X \in \mathfrak{g}$ put $\mathfrak{S}_\alpha(X) = (u^\alpha, v^\alpha, a^\alpha, b^\alpha, c^\alpha, d^\alpha)$ if and only if

$$(1) \qquad X = u^\alpha \frac{\partial}{\partial x^\alpha} + v^\alpha \frac{\partial}{\partial y^\alpha} + a^\alpha x^\alpha \frac{\partial}{\partial x^\alpha} + b^\alpha x^\alpha \frac{\partial}{\partial y^\alpha} + c^\alpha y^\alpha \frac{\partial}{\partial x^\alpha} + d^\alpha y^\alpha \frac{\partial}{\partial y^\alpha}.$$

So we have obtained an admissible coordinate system $\mathfrak{S}_\alpha : \mathfrak{g} \to \mathbf{R}^6$ on $\mathfrak{g}$ induced by $\mathfrak{R}^\alpha$.

In the following we shall often delete the index α in such formulae as (1). This note concerns also the local coordinates on Γ_4 with respect to the coordinate systems $\mathfrak{S}^\alpha_B$ (see below).

The canonical homomorphism $G \to \mathrm{Ad}\,(G)$ is an isomorphism and it induces a Lie algebra isomorphism $\mathfrak{g} \to \mathrm{ad}\,(\mathfrak{g})$. The latter is given, in arbitrary coordinate systems $\mathfrak{R}^\alpha$, $\mathfrak{S}_\alpha$, by the correspondence

$$(2)\qquad \frac{\partial}{\partial x} \to -a\frac{\partial}{\partial u} - b\frac{\partial}{\partial v}, \quad \frac{\partial}{\partial y} \to -c\frac{\partial}{\partial u} - d\frac{\partial}{\partial v},$$

$$x\frac{\partial}{\partial x} \to u\frac{\partial}{\partial u} - b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c},$$

$$x\frac{\partial}{\partial y} \to u\frac{\partial}{\partial v} + (a-d)\frac{\partial}{\partial b} + c\left(\frac{\partial}{\partial d} - \frac{\partial}{\partial a}\right),$$

$$y\frac{\partial}{\partial x} \to v\frac{\partial}{\partial u} + b\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial d}\right) + (d-a)\frac{\partial}{\partial c},$$

$$y\frac{\partial}{\partial y} \to v\frac{\partial}{\partial v} + b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c}.$$

In the following notes we want to introduce some conventions and to remind the most simple properties of the algebra $\mathfrak{g}$.

Note 1. The one-dimensional subspaces of the algebra $\mathfrak{g}$ will be referred to as *directional elements* or simply *d-elements* of $\mathfrak{g}$. For any vector $X \in \mathfrak{g}$, the d-element ξ determined by X will be denoted by $\xi = (X)$. The 1-dimensional subgroup of the group G determined by a vector X or by the corresponding d-element (X) will be denoted by $G[X]$ or $G[(X)]$. We shall use the analogous notation in the complex case (see Section 3 of Part I). Each invariant subset under $G[X]$ of the space $CA^2 \cup \cup CA^1_\infty$ will be called a *singular set of the vector* $X \in \mathfrak{g}$ or else of the d-element $\xi = (X)$. The singular points of a vector X will be briefly called *singularities of that vector,* or else of the d-element (X). (Cf. [6]).

Note 2. Let $A_1, \ldots, A_k$ be vectors or subspaces of a vector space V; then $(A_1, \ldots \ldots, A_k)$ will denote its linear closure in V. If $A, B \subset \mathfrak{g}$ are subspace, then by their *bracket* $[A, B]$ is meant the linear subspace spanned by all brackets $[X, Y]$, $X \in A$, $Y \in B$. We can see that if $\xi, \eta \subset \mathfrak{g}$ are d-elements, then either $[\xi, \eta] = 0$ or $[\xi, \eta]$ is a d-element.

Note 3. If $X \in \mathfrak{g}$, then its ordinary coordinates u, v, a, b, c, d with respect to an admissible coordinate system $\mathfrak{S}_\alpha$ will be referred to as *homogeneous coordinates of the d-element* (X) with respect to $\mathfrak{S}_\alpha$.

Note 4. Let $X \in \mathfrak{g}$, $X = (u, v, a, b, c, d)$. It is known (see [4]) that the characteristic roots of the matrix $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$ are invariants under the transformation group $\mathrm{Ad}\,(G)$. Thus the functions $ad - bc$, $a + d$ are invariants. Each admissible $\mathfrak{S}_\alpha$ in $\mathfrak{g}$ determines a coordinate system $C\mathfrak{S}_\alpha$ in $C\mathfrak{g}$, which is called *admissible* again. The complex coordinates u, v, a, b, c, d of a vector $X^c \in C\mathfrak{g}$ are given by means of the formula (1). The expressions $ad - bc$, $a + d$ are invariants with respect to the induced action of G on the complexification $C\mathfrak{g}$.

Note 5. Let given $X \in \mathfrak{g}$. We have $X \in \mathfrak{t}$ if and only if $a = b = c = d = 0$ with respect to any $\mathfrak{S}_\alpha$. We have $X \in \mathfrak{g}_e$ if and only if $a + d = 0$. Finally the relation $ad - bc = 0$ implies $X \in \mathfrak{g}^0$ (see the list of notation).

The most simple results concerning singular subsets under 1-dimensional subgroups of G will be summarized in the following Theorem.

Theorem 1. *Let $X \in \mathfrak{g}$, $X = (u, v, a, b, c, d)$ with respect to an arbitrary admissible coordinate system $\mathfrak{S}_\alpha$. Then the following assertions hold:*

a) $G[X] \subset T$ *if and only if* $a = b = c = d = 0$.

b) $G[X]$ *is a group of dilatations from a point if and only if* $a = d \neq 0$, $b = c = 0$.

c) *If* $ad - bc < 0$, $a + d = 0$, *then* $G[X]$ *is a group of hyperbolic rotations.*

d) *If* $ad - bc > 0$, $a + d = 0$, *then* $G[X]$ *is a group of elliptic rotations.*

e) *If* $ad - bc = 0$, $a + d = 0$ *and the vector* X *admits a singularity in* A^2, *then* $G[X]$ *is a group of shear transformations.*

f) *If* $ad - bc = 0$, $a + d \neq 0$ *and the vector* X *admits a singularity in* A^2, *then* $G[X]$ *is a group of dilatations from a line.*

g) *In the case* **a)** X *has a singularity in* A^1_∞; *in the case* **b)** X *has a singularity in* A^2.

h) *In the case* **c)**, X *possesses exactly one singularity* $[x_0, y_0]$ *in* A^2 *and two mutually different real singular lines passing through the point* $[x_0, y_0]$. *(Common asymptotes of hyperbolic trajectories.) In the case* **d)**, X *possesses exactly one singularity* $[x_0, y_0]$ *in* A^2 *and two imaginary conjugate singular lines passing through the point* $[x_0, y_0]$. *(Common asymptotes of elliptic trajectories.) In the case* **e)**, X *possesses a real line of singularities, ("a pointwise singular line"), namely the directional line of the shear transformations.*

i) *In both cases* **c)** *and* **d)** *the corresponding couple of singular lines is given by the equation*

$$(3) \qquad b(x - x_0)^2 - 2a(x - x_0)(y - y_0) + c(y - y_0)^2 = 0\,.$$

j) *In the case* **e)** *the equation* (3) *expresses the double directional line of shear transformations; the point* $[x_0, y_0]$ *can be an arbitrary point of that line.*

k) *In the case* **f)** *the vector* X *admits:*

1) *a pointwise singular line (the basic line of the dilatations),*

2) *an improper singularity in* A^1_∞, *which is different from the improper point of the former line.*

For any point $[x_0, y_0] \in A^2$ we have exactly one singular line joining that point with the improper singularity.

l) *In the cases* **e)** *and* **f)**, *the pointwise singular line is given by each of the equations*

$$(4) \qquad a(x - x_0) + c(y - y_0) = 0\,, \quad \text{or} \quad b(x - x_0) + d(y - y_0) = 0\,,$$

here $[x_0, y_0]$ denotes an arbitrary point of the line. (At least one of the equations (4) does not vanish identically). In the case **f)**, *another singular line passing through the point $[x_0, y_0]$ is given by each of the equations*

$$(5) \qquad a(y - y_0) - b(x - x_0) = 0 \quad \text{or} \quad c(y - y_0) - d(x - x_0) = 0\,.$$

m) *The paragraph* **f)** *and the formulae (4), (5) are valid in the complex case, too.*

Proof. All assertions of the Theorem can be verified with the help of "canonical coordinates". For the corresponding procedure see [4].

Let us remark that the formulae (3) and (4) describe some equivariant objects; each of them is defined on a submanifold of $\mathfrak{g}$ and assumes values in CA^2.

Let us consider an invariant decomposition of the manifold Γ_4

$$(6) \qquad \Gamma_4 = \Gamma_4^0 \cup \Gamma_4^1 \cup \Gamma_4^2\,,$$

where $\Gamma_4^i = \{\mathscr{P} \in \Gamma_4, \dim(\mathscr{P} \cap \mathfrak{t}) = i\}\,, \quad i = 0, 1, 2\,.$

Each of these components will be discussed separately.

1. THE MANIFOLD Γ_4^0

Γ_4^0 consists of all blocks $\mathscr{P} \in \Gamma_4$ such that $\mathscr{P} \cap \mathfrak{t} = 0$ in $\mathfrak{g}$. It is an open submanifold of Γ_4, $\dim \Gamma_4^0 = \dim \Gamma_4 = 8$. Let $\mathfrak{S}_\alpha$ be an admissible coordinate system in $\mathfrak{g}$, $\mathfrak{S}_\alpha(X) = (u, v, a, b, c, d)$. In our notation (see Section 4 of Part I) we have $\mathfrak{t} = E_{12}^\alpha$ and thus $\Gamma_4^0 = U_{12}^\alpha$ for arbitrary α. Any block $\mathscr{P} \in \Gamma_4^0$ is given by equations of the form

$$(7) \qquad u = u_1 a + u_2 b + u_3 c + u_4 d\,, \quad v = v_1 a + v_2 b + v_3 c + v_4 d$$

where we delete the index α. The coordinate systems $\mathfrak{S}_{12}^\alpha : \Gamma_4^0 \to \mathbf{R}^8$ make Γ_4^0 a linear space; we have $\mathfrak{S}_{12}^\alpha(\mathscr{P}) = (u_1, \ldots, u_4, v_1, \ldots, v_4)$. The fact that $\mathfrak{S}_{12}^\alpha$ are global coordinate systems will be fully exploited in the following computations (uniqueness problems). For the present, let us suppose $\mathfrak{R}^\alpha$ to be arbitrary.

Denote by $N \subset G$ the subgroup consisting of all translations and of all positive dilatations from a point in A^2; let $\mathfrak{n} \subset \mathfrak{g}$ be the corresponding Lie algebra. Then we have $\dim \mathfrak{n} = 3$, $\mathfrak{n} \supset \mathfrak{t}$. If $\mathscr{P} \in \Gamma_4^0$, then $\dim(\mathscr{P} \cap \mathfrak{n}) = 1$ and $\xi(\mathscr{P}) = \mathscr{P} \cap \mathfrak{n}$ is a real d-element with homogeneous coordinates $a = d = 1$, $b = c = 0$, (see the point **b)** of Theorem 1). From the system (7) we obtain $u = u_1 + u_4$, $v = v_1 + v_4$.

The d-element $\xi(\mathscr{P})$ possesses a single singularity (the centre of dilatations) given by

$$(8) \qquad H(\mathscr{P}) : x = -(u_1 + u_4)\,, \quad y = -(v_1 + v_4)\,.$$

The map $\mathscr{P} \to H(\mathscr{P})$ is obviously an equivariant object on Γ_4^0 with values in A^2.

We shall construct another point object from Γ_4^0 into A^2. Let be given $\mathscr{P} \in \Gamma_4^0$ and put $\mathscr{P}_e = \mathscr{P} \cap \mathfrak{g}_e$; then $\dim \mathscr{P}_e = 3$. (See Note 5.) We can see from (7) that the subspace $\mathscr{P}_e$ has a basis of the form

$$(9) \qquad \begin{aligned} X^\alpha &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (u_1 - u_4)\frac{\partial}{\partial x} + (v_1 - v_4)\frac{\partial}{\partial y}\,, \\ Y^\alpha &= x\frac{\partial}{\partial y} + u_2\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\,, \\ Z^\alpha &= y\frac{\partial}{\partial x} + u_3\frac{\partial}{\partial x} + v_3\frac{\partial}{\partial y}\,. \end{aligned}$$

We can see that there are exactly two linearly independent vectors in the set $\mathscr{P}_e \cap \mathfrak{g}^0$, for example the vectors Y^α, Z^α.

Proposition 1. *Let U_1, U_2 be two linearly independent vectors in $\mathscr{P}_e \cap \mathfrak{g}^0$. Then there is exactly one vector $W \in \mathscr{P}_e$ such that $W \equiv \lambda[U_1, U_2]$, $[U_1, W] \equiv 2U_1$, $[U_2, W] \equiv 2U_2$ (mod t). The vectors $U_1 + [U_1, W]$, $U_2 + [U_2, W]$ have a single common singularity $Q(\mathscr{P}) \in A^2$. The point $Q(\mathscr{P})$ is independent of the choice of linearly independent vectors U_1, U_2 and it is given by the equations*

$$(10) \qquad Q(\mathscr{P}) : x = \tfrac{1}{3}(u_4 - u_1 - 2v_2)\,, \quad y = \tfrac{1}{3}(v_1 - v_4 - 2u_3)\,.$$

Proof. Let U_1, U_2 meet the demands of the Proposition. Then there are real numbers $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $D = \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, $U_i = \alpha_i\beta_i X^\alpha + \beta_i^2 Y^\alpha - \alpha_i^2 Z^\alpha$, $i = 1, 2$. Now we obtain

$$[U_1, U_2] = D\{\beta_1\beta_2[X^\alpha, Y^\alpha] + \alpha_1\alpha_2[X^\alpha, Z^\alpha] + (\alpha_1\beta_2 + \alpha_2\beta_1)[Y^\alpha, Z^\alpha]\}\,.$$

From (9) follow the congruences

$$(11) \qquad [X^\alpha, Y^\alpha] \equiv 2Y^\alpha\,, \quad [X^\alpha, Z^\alpha] \equiv -2Z^\alpha\,, \quad [Y^\alpha, Z^\alpha] \equiv X^\alpha \pmod{t}\,.$$

Thus the wanted vector W has to satisfy the relation

$$W = \mu\{2\beta_1\beta_2 Y^\alpha - 2\alpha_1\alpha_2 Z^\alpha + (\alpha_1\beta_2 + \alpha_2\beta_1) X^\alpha\}\,, \quad \mu \neq 0\,.$$

Further we obtain

$$[U_i, W] = \mu D\{\beta_i^2[X^\alpha, Y^\alpha] + \alpha_i^2[X^\alpha, Z^\alpha] + 2\alpha_i\beta_i[Y^\alpha, Z^\alpha]\}$$

and with regard to (11) the condition $[U_i, W] \equiv 2U_i \,(\mathrm{mod}\, \mathfrak{t})$ yields $\mu = 1/D$. Hence $U_i + [U_i, W] = [\alpha_i\beta_i(3x - A) - \alpha_i^2(3y - B)] (\partial/\partial x) + [\beta_i^2(3x - A) - \alpha_i\beta_i(3y - B)] (\partial/\partial y)$, where $A = u_4 - u_1 - 2v_2$, $B = v_1 - v_4 - 2u_3$. For a common singularity of the vectors $U_1 + [U_1, W]$, $U_2 + [U_2, W]$ we have the following conditions:

$$\beta_1(3x - A) - \alpha_1(3y - B) = 0\,, \quad \beta_2(3x - A) - \alpha_2(3y - B) = 0\,.$$

Regarding $D \neq 0$ we obtain (10), q.e.d.

The object $Q(\mathscr{P})$ admits the following simple interpretation: Let ξ_1^α be the d-element given $a = d = 0$, $b + c = 0$, ξ_2^α the d-element given by $a + d = 0$, $b = c = 0$ and ξ_3^α the d-element given by $a = d = 0$, $b - c = 0$. Then $G[\xi_1^\alpha]$ is a group of elliptic rotations about the point $A^\alpha : x = v_3 - v_2$, $y = u_2 - u_3$, $G[\xi_2^\alpha]$ is a group of hyperbolic rotations about the point $B^\alpha : x = u_4 - u_1$, $y = v_1 - v_4$ and $G[\xi_3^\alpha]$ is also a group of hyperbolic rotations about the point $C^\alpha : x = -v_2 - v_3$, $y = -u_2 - u_3$.

The points A^α, B^α, C^α depend on the coordinate system and they are no equivariant objects. But the centre of gravity of the triangle $\Delta A^\alpha B^\alpha C^\alpha$ does not depend on $\mathfrak{R}^\alpha$; it is exactly the point $Q(\mathscr{P})$. It would be interesting to find a visualisable invariant construction of the object (10).

Let $X \in \mathfrak{g}$; according to (1) we have $X = (u + ax + cy)(\partial/\partial x) + (v + bx + dy)(\partial/\partial y)$ and all singularities of X are given by the relations

$$u + ax + cy = 0\,, \quad v + bx + dy = 0\,. \tag{12}$$

If $X \in \mathscr{P}$, $\mathscr{P} \in \Gamma_4^0$, then the coordinates u, v satisfy the relations (7). If moreover $X \in \mathscr{P}_e$, i.e., $a + d = 0$, then the equations (12) assume the form

$$\begin{aligned} a(x + u_1 - u_4) + bu_2 + c(y + u_3) &= 0\,,\\ a(v_1 - v_4 - y) + b(x + v_2) + cv_3 &= 0\,. \end{aligned} \tag{13}$$

Let $\mathbf{x}(x, y) \in A^2$ be an arbitrary point; let us put

$$\begin{aligned} &L(\mathbf{x}) = x + u_1 - u_4\,, \quad M(\mathbf{x}) = u_2\,, \quad N(\mathbf{x}) = y + u_3\,,\\ &R(\mathbf{x}) = v_1 - v_4 - y\,, \quad S(\mathbf{x}) = x + v_2\,, \quad T(\mathbf{x}) = v_3\\ &D_1(\mathbf{x}) = \begin{vmatrix} M(\mathbf{x}), & N(\mathbf{x}) \\ S(\mathbf{x}), & T(\mathbf{x}) \end{vmatrix}, \quad D_2(\mathbf{x}) = -\begin{vmatrix} L(\mathbf{x}), & N(\mathbf{x}) \\ R(\mathbf{x}), & T(\mathbf{x}) \end{vmatrix},\\ &D_3(\mathbf{x}) = \begin{vmatrix} L(\mathbf{x}), & M(\mathbf{x}) \\ R(\mathbf{x}), & S(\mathbf{x}) \end{vmatrix}, \quad D(\mathbf{x}) = D_1^2(\mathbf{x}) + D_2(\mathbf{x})\, D_3(\mathbf{x})\,. \end{aligned} \tag{14}$$

Here we must keep in view that the functions (14) depend on the choice of a subspace $\mathscr{P}$, and in addition, on the choice of the coordinate system $\mathfrak{R}^\alpha$. These facts will be not marked explicitly. If a point $\mathbf{x} \in A^2$ is given the system (13) has a geometrical signification: it determines all d-elements $\xi \subset \mathscr{P}_e = \mathscr{P} \cap \mathfrak{g}_e$ having a singularity at $\boldsymbol{x}$. For any block $\mathscr{P} \in \Gamma_4^0$ we obtain an invariant decomposition of the plane A^2

$$A^2 = \mathscr{P}^\square \cup \mathscr{P}^* , \tag{15}$$

where we put $\boldsymbol{x} \in \mathscr{P}^\square$, if the rank of the system (13) is 2, and $\boldsymbol{x} \in \mathscr{P}^*$ otherwise. We obviously have

$$\boldsymbol{x} \in \mathscr{P}^* \Leftrightarrow D_1(\boldsymbol{x}) = D_2(\boldsymbol{x}) = D_3(\boldsymbol{x}) = 0 . \tag{16}$$

Proposition 2. *Let be given $\mathscr{P} \in \Gamma_4^0$, $\boldsymbol{x}_0 \in \mathscr{P}^\square$. Then there is exactly one d-element $\xi \subset \mathscr{P}_e$ having a singularity at $\boldsymbol{x}_0$. Moreover:*

a) For $D(\boldsymbol{x}_0) > 0$, $G[\xi]$ is a group of hyperbolic rotations about $\boldsymbol{x}_0$.

b) For $D(\boldsymbol{x}_0) < 0$, $G[\xi]$ is a group of elliptic rotations about $\boldsymbol{x}_0$.

c) For $D(\boldsymbol{x}_0) = 0$, $G[\xi]$ is a group of shear transformations with directional line passing through the point $\boldsymbol{x}_0$.

Proof. With regard to the assumption $\boldsymbol{x}_0 \in \mathscr{P}^\square$ and to the notation (14) the wanted d-element $\xi \subset \mathscr{P}_e$ has homogeneous coordinates $a = \lambda D_1(\boldsymbol{x}_0)$, $b = \lambda D_2(\boldsymbol{x}_0)$, $c = \lambda D_3(\boldsymbol{x}_0)$, $d = -a$; $\lambda \neq 0$. The values of u, v are given by (7). It is obvious that $ad - bc = -\lambda^2 D(\boldsymbol{x}_0)$. Now our assertion follows from the points **c**), **d**), **e**) of Theorem 1. As we have seen, the sign of $D(\boldsymbol{x}_0)$ has a geometrical signification, and hence we have an invariant decomposition

$$\mathscr{P}^\square = \mathscr{P}^\square(1) \cup \mathscr{P}^\square(-1) \cup \mathscr{P}^\square(0) \tag{17}$$

in accordance with sgn $D(\boldsymbol{x}_0) = 1, -1, 0$.

Proposition 3. *Let us denote by $\varkappa(\mathscr{P}) \subset CA^2$ the set determined by the equation $D(\boldsymbol{x}) = 0$. Then*

a) the set $\varkappa(\mathscr{P})$ does not depend on the coordinate systems $\mathfrak{R}^\alpha$,

b) $\varkappa(\mathscr{P})$ is either the whole complex plane CA^2 or a cubic which decomposes into three lines in CA^2,

c) the map $\mathscr{P} \to \varkappa(\mathscr{P})$ is an equivariant object on Γ_4^0 with values in CA^2.

Proof. First of all we must provide an invariant definition of the set $\varkappa(\mathscr{P})$. We have a complex analogue of the decomposition (15): $CA^2 = \mathscr{P}^{c\square} \cup \mathscr{P}^{c*}$, where $\mathscr{P}^{c*} = \{\boldsymbol{x} \in CA^2 \mid D_i(\boldsymbol{x}) = 0,\ i = 1, 2, 3\}$. Let us denote by $C\mathfrak{g}^0$ the set of all complex vectors ${}^cX \in C\mathfrak{g}$, ${}^cX = (u, v, a, b, c, d)$, such that $ad - bc = 0$. The set $C\mathfrak{g}^0$ is independent of the coordinates (see Note 4). Now, to each $\boldsymbol{x} \in \mathscr{P}^{c\square}$ there is exactly one complex d-element $\xi^c \subset C\mathscr{P}_e$ having a singularity at $\boldsymbol{x}$. Put $\mathscr{P}^{c\square}(0) = \{\boldsymbol{x} \in \mathscr{P}^{c\square} \mid {}^c\xi \subset C\mathfrak{g}^0\}$. We have an obvious relation $\varkappa(\mathscr{P}) = \mathscr{P}^{c\square}(0) \cup \mathscr{P}^{c*}$ and hence follows the assertion a). By a detailed analysis of the system $D_1(\boldsymbol{x}) = D_2(\boldsymbol{x}) = D_3(\boldsymbol{x}) = 0$ we obtain that *the set $\mathscr{P}^{c*}$ is always finite.*

If $\boldsymbol{x} \in \mathscr{P}^{c\square}$, we can see as in the real case that $\boldsymbol{x} \in \mathscr{P}^{c\square}(0)$ if and only if the corresponding d-element ${}^c\xi$ admits a pointwise singular line (real or imaginary). By a direct computation we find that either the equation $D(\boldsymbol{x}) = 0$ vanishes identically or it is of 3-rd degree. Thus $\varkappa(\mathscr{P})$ is either the whole plane or a cubic.

Let $\varkappa(\mathscr{P})$ be a cubic and choose a point $\boldsymbol{x}_0 \in \varkappa(\mathscr{P})$. We can achieve that $\boldsymbol{x}_0 \in \mathscr{P}^{c\square}$ because the set $\mathscr{P}^{c*}$ is finite. The corresponding d-element ${}^c\xi$ admits a pointwise singular line $p(\boldsymbol{x}_0)$. If $\boldsymbol{x} \in p(\boldsymbol{x}_0) \cap \mathscr{P}^{c\square}$, then the d-element ${}^c\xi$ corresponds to $\boldsymbol{x}$ and hence $\boldsymbol{x} \in \mathscr{P}^{c\square}(0) \subset \varkappa(\mathscr{P})$. Because of $\mathscr{P}^{c*} \subset \varkappa(\mathscr{P})$ we obtain $p(\boldsymbol{x}_0) \subset \varkappa(\mathscr{P})$. The cubic $\varkappa(\mathscr{P})$ decomposes into a line and a conic. If $\varkappa(\mathscr{P}) \dot{-} p(\boldsymbol{x}_0) \neq \emptyset$, we can choose a point $\boldsymbol{x}_1 \in (\varkappa(\mathscr{P}) \dot{-} p(\boldsymbol{x}_0)) \cap \mathscr{P}^{c\square}(0)$, and so forth. Hence we obtain the assertion b).

Finally the statement c) follows immediatelly from Proposition VI, because the form of the equation $D(\boldsymbol{x}) = 0$ does not depend on the coordinates, q.e.d.

Proposition 4. *The mappings $\mathscr{P} \to \mathscr{P}^{\square}(1)$, $\mathscr{P} \to \mathscr{P}^{\square}(-1)$, $\mathscr{P} \to \mathscr{P}^{\square}(0)$, $\mathscr{P} \to \mathscr{P}^*$ are equivariant objects on Γ_4^0 with values in A^2.*

Proof. Each of the named objects is described by a system of relations (equations or inequalities) of invariant form (see Theorem VI).

Let $\mathscr{P} \in \Gamma_4^0$, $\boldsymbol{x}_0 \in \mathscr{P}^{\square}$ and let ξ be the d-element corresponding to $\boldsymbol{x}_0$ on the basis of Proposition 2. Then ξ admits two singular lines in CA^2, which are real and different or imaginary conjugate or real and coincident, according to the sign of $D(\boldsymbol{x}_0)$. In any case, the equation of that couple of lines can be written in the form

$$D_2(\boldsymbol{x}_0)(x - x_0)^2 - 2D_1(\boldsymbol{x}_0)(x - x_0)(y - y_0) - D_3(\boldsymbol{x}_0)(y - y_0)^2 = 0 . \tag{18}$$

(Cf. the equation (3) and Proposition 2.)

Let us introduce the following conventions and denotations: *If $\boldsymbol{x} = H(\mathscr{P})$* (see (8)), *then the corresponding values of the terms* (14) *will be marked by omitting the argument. The corresponding values at the point $\boldsymbol{x} = Q(\mathscr{P})$ will be marked by omitting the argument and by a twiddle over the capital. Further let us denote*

$$\Delta_1 = \begin{vmatrix} D_2 & D_3 \\ \tilde{D}_2 & \tilde{D}_3 \end{vmatrix}, \quad \Delta_2 = \dot{-} \begin{vmatrix} D_1 & D_3 \\ \tilde{D}_1 & \tilde{D}_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} D_1 & D_2 \\ \tilde{D}_1 & \tilde{D}_2 \end{vmatrix} . \tag{19}$$

$$\Delta = \Delta_1^2 + 4\Delta_2\Delta_3 . \tag{20}$$

The introduced symbols depend on $\mathscr{P} \in \Gamma_4^0$, and on the choice of a coordinate system $\mathfrak{R}^\alpha$.

Now let us put

$$\alpha = x_Q - x_H = \tfrac{2}{3}(2u_4 + u_1 - v_2) , \quad \beta = y_Q - y_H = \tfrac{2}{3}(2v_1 + v_4 - u_3) \tag{21}$$

(Cfr. (8) and (10)).

From (14) we obtain immediately (taking into account (8) and (10))

$$(22)\qquad \begin{aligned} L &= -2u_4\,, & R &= 2v_1\,,\\ M &= u_2\,, & S &= v_2 - u_1 - u_4\,,\\ N &= u_3 - v_1 - v_4\,, & T &= v_3\,. \end{aligned}$$

$$(23)\qquad \begin{aligned} \tilde{L} &= L + \alpha\,, & \tilde{R} &= R - \beta\,,\\ \tilde{M} &= M\,, & \tilde{S} &= S + \alpha\,,\\ \tilde{N} &= N + \beta\,, & \tilde{T} &= T\,. \end{aligned}$$

$$(24)\qquad \begin{aligned} \tilde{D}_1 &= D_1 - \alpha N - \beta S - \alpha\beta\,,\\ \tilde{D}_2 &= D_2 + \beta(R - N) - \alpha T - \beta^2\,,\\ \tilde{D}_3 &= D_3 + \beta M + \alpha(L + S) + \alpha^2\,. \end{aligned}$$

$$(25)\qquad \alpha = -\tfrac{1}{3}(2S + L)\,, \quad \beta = \tfrac{1}{3}(R - 2N)\,.$$

$$(26)\qquad LD_1 + MD_2 + ND_3 = 0\,, \quad RD_1 + SD_2 + TD_3 = 0\,.$$

$$(27)\qquad \tilde{L}\tilde{D}_1 + \tilde{M}\tilde{D}_2 + \tilde{N}\tilde{D}_3 = 0\,, \quad \tilde{R}\tilde{D}_1 + \tilde{S}\tilde{D}_2 + \tilde{T}\tilde{D}_3 = 0\,.$$

From (23) and (27) follows

$$(28)\qquad \begin{aligned} L\tilde{D}_1 + M\tilde{D}_2 + N\tilde{D}_3 &= -\alpha\tilde{D}_1 - \beta\tilde{D}_3\,,\\ R\tilde{D}_1 + S\tilde{D}_2 + T\tilde{D}_3 &= \beta\tilde{D}_1 - \alpha\tilde{D}_2\,. \end{aligned}$$

We can write (25) in the form

$$(25')\qquad L = -2S - 3\alpha\,, \quad R = 2N + 3\beta\,,$$

and from (26) we obtain

$$(29)\qquad \begin{aligned} -2SD_1 + MD_2 + ND_3 &= 3\alpha D_1\,,\\ 2ND_1 + SD_2 + TD_3 &= -3\beta D_1\,. \end{aligned}$$

Similarly we obtain from (28) the relations

$$(30)\qquad \begin{aligned} -2S\tilde{D}_1 + M\tilde{D}_2 + N\tilde{D}_3 &= 2\alpha\tilde{D}_1 - \beta\tilde{D}_3\,,\\ 2N\tilde{D}_1 + S\tilde{D}_2 + T\tilde{D}_3 &= -2\beta\tilde{D}_1 - \alpha\tilde{D}_2\,. \end{aligned}$$

Finally, let us introduce the following notation: Let be given a matrix of the form $\begin{pmatrix} d_1, d_2, d_3 \\ \tilde{d}_1, \tilde{d}_2, \tilde{d}_3 \end{pmatrix}$. Then we put

$$(31) \qquad \Delta_1(d) = \begin{vmatrix} d_2, d_3 \\ \tilde{d}_2, \tilde{d}_3 \end{vmatrix}, \quad \Delta_2(d) = - \begin{vmatrix} d_1, d_3 \\ \tilde{d}_1, \tilde{d}_3 \end{vmatrix}, \quad \Delta_3(d) = \begin{vmatrix} d_1, d_2 \\ \tilde{d}_1, \tilde{d}_2 \end{vmatrix}$$

$$\Delta(d) = \Delta_1^2(d) + 4\,\Delta_2(d)\,\Delta_3(d)\,.$$

In case that $d_i = D_i$, $\tilde{d}_i = \tilde{D}_i$ for any $\mathscr{P} \in \Gamma_4^0$ we have obviously $\Delta_i(d) = \Delta_i$, $\Delta(d) = \Delta$.

Proposition 5. *Let k_1, k_2 be two couples of lines in CA^2 given by the equations*

$$(32) \qquad \begin{aligned} k_1 &\equiv d_2(x - x_1)^2 - 2d_1(x - x_1)(y - y_1) - d_3(y - y_1)^2 = 0\,, \\ k_2 &\equiv \tilde{d}_2(x - x_2)^2 - 2\tilde{d}_1(x - x_2)(y - y_2) - \tilde{d}_3(y - y_2)^2 = 0\,. \end{aligned}$$

Then the following two assertions are equivalent to each other:

a) $\Delta(d) = 0$.

b) There is a line of the couple k_1 which is parallel to a line of the couple k_2.

Proof. We can see easily that $\Delta(d)$ is the resultant of the quadratic forms $d_2x^2 - 2d_1xy - d_3y^2$, $\tilde{d}_2x^2 - 2\tilde{d}_1xy - \tilde{d}_3y^2$ (See [5]). Thus we have $\Delta(d) = 0$ if and only if these forms have a common linear factor. Hence follows our assertion.

Now, let us denote

$$(33) \qquad \begin{aligned} \mathfrak{M}_H^\square &= \{\mathscr{P} \in \Gamma_4^0 \mid H(\mathscr{P}) \in \mathscr{P}^\square\}\,, \\ \mathfrak{M}_Q^\square &= \{\mathscr{P} \in \Gamma_4^0 \mid Q(\mathscr{P}) \in \mathscr{P}^\square\}\,, \\ \mathfrak{M}^\square &= \mathfrak{M}_H^\square \cap \mathfrak{M}_Q^\square\,. \end{aligned}$$

Taking into account that $H(\mathscr{P})$, $Q(\mathscr{P})$, $\mathscr{P}^\square$ are equivariant objects, we can see that the sets (33) are invariant under G. In case that $\mathscr{P} \in \mathfrak{M}_H^\square$ denote by $k_H(\mathscr{P})$ the couple of lines (18), where $\mathbf{x}_0 = H(\mathscr{P})$. In case that $\mathscr{P} \in \mathfrak{M}_Q^\square$ denote by $k_Q(\mathscr{P})$ the couple of lines (18), where $\mathbf{x}_0 = Q(\mathscr{P})$. We have obtained two *equivariant objects* of the form $k_H : \mathfrak{M}_H^\square \to CA^2$, $k_Q : \mathfrak{M}_Q^\square \to CA^2$ given by

$$(34) \qquad k_H(\mathscr{P}) \equiv D_2(x - x_H)^2 - 2D_1(x - x_H)(y - y_H) - D_3(y - y_H)^2 = 0$$

$$(35) \qquad k_Q(\mathscr{P}) \equiv D_2(x - x_Q)^2 - 2D_1(x - x_Q)(y - y_Q) - D_3(y - y_Q)^2 = 0\,.$$

$k_H(\mathscr{P})$ is a couple of lines, which are real and different, or imaginary conjugate, or real and coincident if $H(\mathscr{P}) \in \mathscr{P}^\square(1)$ or $H(\mathscr{P}) \in \mathscr{P}^\square(-1)$ or $H(\mathscr{P}) \in \mathscr{P}^\square(0)$, respectively. Similarly for $k_Q(\mathscr{P})$.

*

Let us consider an invariant decomposition

$$\Gamma_4^0 = \Gamma_4^0(1) \cup \Gamma_4^0(0) \tag{36}$$

where $\Gamma_4^0(1) \equiv \{\mathscr{P} \in \Gamma_4^0 \mid H(\mathscr{P}) \not\equiv Q(\mathscr{P})\}$.

Let us start with the manifold $\Gamma_4^0(1)$. *Put*

$$\mathfrak{M} = \mathfrak{M}^\square \cap \Gamma_4^0(1), \quad \mathfrak{M} = \mathfrak{M}^* \cup \mathfrak{M}^\circ, \quad \mathfrak{M}^* = \{\mathscr{P} \in \mathfrak{M} \mid \Delta \neq 0\}. \tag{37}$$

Considering Proposition 5 we see that the relation $\mathscr{P} \in \mathfrak{M}^*$ has a geometrical signification: it means that *neither line of the couple* $k_H(\mathscr{P})$ *is parallel with a line of the couple* $k_Q(\mathscr{P})$. $\mathfrak{M}^*$ and $\mathfrak{M}^\circ$ are G-invariant submanifolds of Γ_4^0 as well.

Proposition 6. *Let* $H_0 \not\equiv Q_0$ *be two points of* A^2 *and* k_1, k_2 *two line couples in* CA^2 *with the following properties:* H_0 *is a double point of the couple* k_1, Q_0 *is a double point of the couple* k_2; *neither line of the couple* k_1 *is parallel with a line of the couple* k_2. *Then there is at most one block* $\mathscr{P} \in \mathfrak{M}^*$ *such that* $H(\mathscr{P}) \equiv H_0$, $Q(\mathscr{P}) \equiv Q_0$, $k_H(\mathscr{P}) \equiv k_1$, $k_Q(\mathscr{P}) \equiv k_2$.

Proof. Let $\mathfrak{R}^\alpha$ be an arbitrary admissible coordinate system and let the couples k_1, k_2 be given by the equations (32) with respect to $\mathfrak{R}^\alpha$, where $H_0 = [x_1, y_1]$, $Q_0 = [x_2, y_2]$. Put $\alpha_0 = x_2 - x_1$, $\beta_0 = y_2 - y_1$. Then the conditions $H(\mathscr{P}) \equiv H_0$, $Q(\mathscr{P}) \equiv Q_0$ can be re-written in the form $H(\mathscr{P}) \equiv H_0$, $\alpha = \alpha_0$, $\beta = \beta_0$ (see (21)), and with respect to (8), (22) and (25) we obtain

$$u_1 = \frac{L}{2} - x_1, \quad v_4 = -\frac{R}{2} - y_1, \tag{38}$$

$$L = -2S - 3\alpha_0, \quad R = 2N + 3\beta_0. \tag{39}$$

From (22) we find easily that the coordinates $u_1, \ldots, u_4, v_1, \ldots, v_4$ are uniquely determined by the values $L, M, N, R, S, T, u_1, v_4$. With regard to (38), (39) it remains to show that the functions M, N, S, T are uniquely determined by the conditions $k_H(\mathscr{P}) \equiv k_1$, $k_Q(\mathscr{P}) \equiv k_2$. If we compare (34) and (35) with (32), we see that the triplets (D_1, D_2, D_3), (d_1, d_2, d_3) and also the triplets $(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3)$, $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)$ shall be proportional. From (29) and (30) we obtain

$$\begin{aligned} d_2M + d_3N - 2d_1S \quad &= 3\alpha_0 d_1, \\ 2d_1N + d_2S + d_3T &= -3\beta_0 d_1, \\ \tilde{d}_2M + \tilde{d}_3N - 2\tilde{d}_1S \quad &= 2\alpha_0\tilde{d}_1 - \beta_0\tilde{d}_3, \\ 2\tilde{d}_1N + \tilde{d}_2S + \tilde{d}_3T &= -2\beta_0\tilde{d}_1 - \alpha_0\tilde{d}_2. \end{aligned} \tag{40}$$

We find easily that the determinant of the system is $\Delta(d)$ and thus non-zero according to Proposition 5. This completes the proof.

From Proposition VII we obtain

Theorem 2. *The equivariant objects $H(\mathscr{P})$, $Q(\mathscr{P})$, $k_H(\mathscr{P})$, $k_Q(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{M}^*$ with values in CA^2.*

We have not yet determined the domain of values of that representing frame. Let us consider an invariant decomposition

$$\mathfrak{M}^* = \bigcup_{i,j} \mathfrak{M}^*(i, j), \quad i, j = 1, -1, 0, \tag{41}$$
$$\mathfrak{M}^*(i, j) = \{\mathscr{P} \in \mathfrak{M}^* \mid H(\mathscr{P}) \in \mathscr{P}^{\square}(i),\ Q(\mathscr{P}) \in \mathscr{P}^{\square}(j)\}.$$

We shall bring out all necessary calculations for the manifold $\mathfrak{M}^*(1, 1)$; the other cases will be discussed quite concisely. Let us denote by $\mathfrak{A}(1, 1)$ the domain of values of the representing frame $\{H(\mathscr{P}), Q(\mathscr{P}), k_H(\mathscr{P}), k_Q(\mathscr{P})\}$ on the manifold $\mathfrak{M}^*(1, 1)$. Let $\mathfrak{B}(1, 1)$ be the manifold of all configurations (H_0, Q_0, k_1, k_2) like those from Proposition 6 and such that both k_1 and k_2 consist of real mutually different real lines. Then $\mathfrak{B}(1, 1)$ is a differentiable manifold of dimension 8. We wish to show that $\mathfrak{A}(1. 1)$ is an open submanifold of $\mathfrak{B}(1, 1)$.

Let be given a configuration $\{H_0, Q_0, k_1, k_2\} \in \mathfrak{B}(1, 1)$ and let us choose the lines of the couple k_1 as coordinate axes for $\mathfrak{R}^{\alpha}$. Then we can write

$$k_1 \equiv (x - x_1)(y - y_1) = 0,$$
$$k_2 \equiv \tilde{d}_2(x - x_2)^2 - 2\tilde{d}_1(x - x_2)(y - y_2) - \tilde{d}_3(y - y_2)^2 = 0.$$

The system (40) takes a simpler form because of $d_2 = d_3 = 0$ and after calculating M, N, T, S from it we obtain

$$D_1 = MT - NS = \frac{(\beta_0\tilde{d}_3 - 2\alpha_0\tilde{d}_1)(\alpha_0\tilde{d}_2 + 2\beta_0\tilde{d}_1) - 9\alpha_0\beta_0\tilde{d}_2\tilde{d}_3}{4\tilde{d}_2\tilde{d}_3}.$$

It is obvious that the coordinate system $\mathfrak{R}^{\alpha}$ can be specialized in the unique way such that $d_2 = 1$, $d_3 = \pm 1$, $\alpha_0 = 1$. Then β_0 and d_1 are continuous local functions on $\mathfrak{B}(1, 1)$. Denote by $V(D_1) \subset \mathfrak{B}(1, 1)$ the subset of all configurations $\mathscr{K} \in \mathfrak{B}(1, 1)$ such that the determinant D_1 computed from (40) is non-zero with respect to the canonical coordinate system for $\mathscr{K}$. Then $V(D_1)$ is a non-empty open subset of $\mathfrak{B}(1, 1)$. In the similar way we obtain open subsets $V(D_2)$, $V(D_3)$, ..., $V(\tilde{D}_3)$ of $\mathfrak{B}(1, 1)$; put $\mathfrak{A}^*(1, 1) = [V(D_1) \cup V(D_2) \cup V(D_3)] \cap [V(\tilde{D}_1) \cup V(\tilde{D}_2) \cup V(\tilde{D}_3)]$, $\mathfrak{A}^*(1, 1) \subset \mathfrak{B}(1. 1)$ is an open subset and we have obviously $\mathfrak{A}(1, 1) \subset \mathfrak{A}^*(1, 1)$. If $\mathscr{K} \in \mathfrak{A}^*(1, 1)$, then the corresponding block $\mathscr{P} \in \Gamma_4^0$ belongs to $\mathfrak{M}^{\square}$; thus the objects $H(\mathscr{P})$, $Q(\mathscr{P})$, $k_H(\mathscr{P})$, $k_Q(\mathscr{P})$ are defined. Because the coordinates $M, N, \ldots, \tilde{T}$ of the block $\mathscr{P}$ satisfy (39) und (40), they must also satisfy the relations

$$d_1L + d_2M + d_3N = 0, \quad d_1R + d_2S + d_3T = 0,$$
$$\tilde{d}_1\tilde{L} + \tilde{d}_2\tilde{M} + \tilde{d}_3\tilde{N} = 0, \quad \tilde{d}_1\tilde{R} + \tilde{d}_2\tilde{S} + \tilde{d}_3\tilde{T} = 0.$$

Thus the non-zero triplets (D_1, D_2, D_3), $(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3)$ are proportional to the triplets (d_1, d_2, d_3), $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3)$. (Cf. (26), (27)). Hence it follows easily, that the block $\mathscr{P}$ satisfies the requirements of Proposition 6 not only formally but also geometrically. Thus $\mathfrak{A}(1, 1) = \mathfrak{A}^*(1, 1)$ is an open subset which was to be proved.

Now it is obvious that $\mathfrak{A}(1, 1)$ consists merely of orbits of dimension 6. We shall show that the corresponding orbit space $\overline{\mathfrak{A}}(1, 1)$ is diffeomorphic to an open subset of R^2. In this case we say that $\mathfrak{A}(1, 1)$ *consists of* ∞^2 *orbits of dimension* 6. Formulations like that will be often used in the following.

Let us denote by $\overline{\mathfrak{B}}(1, 1)$ the orbit space of $\mathfrak{B}(1, 1)$. Let $\mathscr{K} \in \mathfrak{B}(1, 1)$ be a configuration. (Let us remind that the plane A^2 is oriented.) We denote the lines of the couple k_1 by h_1 and h_2 such that the following condition is fulfilled: if the line H_0Q_0 rotates about H_0 in the positive sense, then it coincides successively with h_1 and h_2. Then we denumerate the lines q_1, q_2 of k_2 in a similar way. Let us denote by A, B, C, D the intersection points $h_1 \times q_1$, $h_1 \times q_2$, $h_2 \times q_1$, $h_2 \times q_2$ in this order. Then the ratia $\lambda = \overrightarrow{HA} : \overrightarrow{AB}$, $\mu = \overrightarrow{HC} : \overrightarrow{CD}$ communicate a diffeomorphism $\overline{\mathfrak{B}}(1, 1) \to R^2$. $\overline{\mathfrak{A}}(1, 1)$ is an open subset of $\overline{\mathfrak{B}}(1, 1)$, pan consequently, its image is an open subset of R^2, q.e.d.

We can discuss all submanifolds $\mathfrak{M}^*(\mathrm{i}, \mathrm{j})$ except $\mathfrak{M}^*(0, 0)$ in the same manner. We can see without difficulties that each of the manifolds $\mathfrak{M}^*(1, -1)$, $\mathfrak{M}^*(-1, 1)$, $\mathfrak{M}^*(-1, -1)$ consists of ∞^2 orbits of dimension 6 and each of the manifolds $\mathfrak{M}^*(0, 1)$, $\mathfrak{M}^*(0, -1)$, $\mathfrak{M}^*(1, 0)$, $\mathfrak{M}^*(-1, 0)$ consists of ∞^1 orbits of dimension 6.

As for $\mathfrak{M}^*(0, 0)$, let us prove the following proposition:

Proposition 7. *If* $\mathscr{P} \in \mathfrak{M}^*(0, 0)$, *then neither of the double lines* $k_H(\mathscr{P})$, $k_Q(\mathscr{P})$ *coincides with the line* $H(\mathscr{P})\, Q(\mathscr{P})$.

Proof. If the double line (35) coincides with $H(\mathscr{P})\, Q(\mathscr{P}) \equiv \beta(x - x_Q) - \alpha(y - y_Q) = 0$ (see (21)), we obtain $\alpha\tilde{D}_1 + \beta\tilde{D}_3 = 0$, $\beta\tilde{D}_1 - \alpha\tilde{D}_2 = 0$, and from (26), (28) follows $D_1 : D_2 : D_3 = \tilde{D}_1 : \tilde{D}_2 : \tilde{D}_3$ and $\Delta = 0$, which is a contradiction with the assumption $\mathscr{P} \in \mathfrak{M}^*$. We show in a similar way that $k_H(\mathscr{P}) \not\equiv H(\mathscr{P})\, Q(\mathscr{P})$. Now, let us denote by $\mathfrak{B}(0, 0)$ the set of all configurations $\{H_0, Q_0, k_1, k_2\} \subset A^2$, where $k_1 \ni H_0$, $k_2 \ni Q_0$, $H_0 \not\equiv Q_0$ k_1, k_2 are real and non-parallel lines such that $k_1 \not\equiv H_0Q_0 \not\equiv k_2$. Then $\mathfrak{B}(0, 0)$ is the domain of values of the representing frame $\{H, Q, k_H, k_Q\}$ on the manifold $\mathfrak{M}^*(0, 0)$. Indeed, let us choose a coordinate system $\mathfrak{R}^\alpha$ such that the coordinate axes $\vec{x}$, $\vec{y}$ are parallel to k_1 and k_2. Then we have $k_1 \equiv (x - x_1)^2 = 0$, $k_2 \equiv (y - y_2)^2 = 0$. We can put $d_1 = d_3 = 0$, $\tilde{d}_1 = \tilde{d}_2 = 0$ in the system (40) and hence $M = S = T = 0$, $N = -\beta_0$, $R = \beta_0$, $L = -3\alpha_0$, $D_2 = RN - LT = -\beta_0^2 \neq 0$, $\tilde{D}_3 = \tilde{L}\tilde{S} - \tilde{M}\tilde{R} = -2\alpha_0^2 \neq 0$. Thus $\mathscr{P} \in \mathfrak{M}^\square$ and we obtain easily that $\mathscr{P} \in \mathfrak{M}^*(0, 0)$. Since G preserves orientation of A^2 we can see that $\mathfrak{B}(0, 0)$ consists of two orbits of dimension 6 under G.

Theorem 3. *The manifold* $\mathfrak{M}^* = \bigcup \mathfrak{M}^*(i, j)$ *is composed of the following types of orbits*: $\mathfrak{M}^*(1, 1)$, $\mathfrak{M}^*(-1, 1)$, $\mathfrak{M}^*(1, -1)$, $\mathfrak{M}^*(-1, -1)$ *consist of* ∞^2 *orbits of dimension* 6 *each*; $\mathfrak{M}^*(0, 1)$, $\mathfrak{M}^*(0, -1)$, $\mathfrak{M}^*(1, 0)$, $\mathfrak{M}^*(-1, 0)$ *consist of* ∞^1 *orbits of dimension* 6 *each*; $\mathfrak{M}^*(0, 0)$ *consists of two orbits of dimension* 6.

*

Let us go on and consider the submanifold $\mathfrak{M}^0$ determined in $\mathfrak{M}$ by the invariant relaiton $\Delta = 0$ (see (37)). We have an invariant decomposition

$$\mathfrak{M}^0 = \bigcup_{i,j} \mathfrak{M}^0(i, j), \quad i, j = 1, -1, 0, \tag{42}$$

which is analogous to the decomposition (41).

Proposition 8. *Suppose* $\mathscr{P} \in (\mathfrak{M}^0 \dot{-} \mathfrak{M}^0(0, 0))$. *Then*

a) there is a line of the couple $k_H(\mathscr{P})$ *which is parallel to a line of the couple* $k_Q(\mathscr{P})$,

b) there is a line, either in $k_H(\mathscr{P})$ *or in* $k_Q(\mathscr{P})$, *which is not parallel to any line of the other couple.*

Proof. The first assertion follows from the geometrical signification of the relation $\Delta = 0$. It remains to prove the second one. It is obviously valid for each submanifold $\mathfrak{M}^0(i, j)$ such that either $i = 0$ or $j = 0$. Let us suppose that $ij \neq 0$ and b) is false. Then there is a translation of the complex plane CA^2 which transfers $k_H(\mathscr{P})$ into $k_Q(\mathscr{P})$. From (34), (35) follows $D_1 : D_2 : D_3 = \tilde{D}_1 : \tilde{D}_2 : \tilde{D}_3$. Taking into account (26) and (28) we obtain $\alpha\tilde{D}_1 + \beta\tilde{D}_3 = 0$, $\beta\tilde{D}_1 - \alpha\tilde{D}_2 = 0$. The determinant of the last system is $\tilde{D} \neq 0$ and hence $\alpha = \beta = 0$ and $H(\mathscr{P}) \equiv Q(\mathscr{P})$ – a contradiction, q.e.d.

Let us consider the manifold $\mathfrak{M}^0(1, 1)$. As a consequence of Proposition 8, we can distinguish four equivariant objects $p_1(\mathscr{P})$, $p_2(\mathscr{P})$, $p_3(\mathscr{P})$, $p_4(\mathscr{P})$ on $\mathfrak{M}^0(1, 1)$. These objects are determined by the conditions $\{p_1(\mathscr{P}), p_2(\mathscr{P})\} \equiv k_H(\mathscr{P})$, $\{p_3(\mathscr{P}), p_4(\mathscr{P})\} \equiv$ $\equiv k_Q(\mathscr{P})$, $p_1(\mathscr{P}) \parallel p_3(\mathscr{P})$, and consequently $p_2(\mathscr{P}) \nparallel p_4(\mathscr{P})$. A coordinate system $\mathfrak{R}^\alpha$ will be called *canonical for* $\mathscr{P} \in \mathfrak{M}^0(1, 1)$ if $\vec{x} \equiv p_1(\mathscr{P})$, $\vec{y} \equiv p_2(\mathscr{P})$. In canonical coordinates we have $k_H(\mathscr{P}) \equiv xy = 0$, $k_Q(\mathscr{P}) \equiv -2D_1(x - x_Q)(y - y_Q) - D_3(y -$ $- y_Q)^2 = 0$; hence $D_2 = D_3 = 0$, $\tilde{D}_2 = 0$, $D_1 \neq 0$, $\tilde{D}_1\tilde{D}_3 \neq 0$.

Let us choose a fixed $\mathscr{P} \in \mathfrak{M}^0(1, 1)$ and a fixed canonical coordinate system $\mathfrak{R}^\alpha$ for $\mathscr{P}$. Let us denote by the letters d_i, $\tilde{d}_i$, α_0, β_0 the corresponding values of the functions D_i, $\tilde{D}_i$, α, β. Then the system (40) has the form

$$-2d_1S = 3\alpha_0 d_1, \quad 2d_1N = -3\beta_0 d_1, \tag{43}$$

$$\tilde{d}_3N - 2\tilde{d}_1S = 2\alpha_0\tilde{d}_1 - \beta_0\tilde{d}_3, \quad 2\tilde{d}_1N + \tilde{d}_3T = -2\beta_0\tilde{d}_1.$$

From the first two equations we obtain $S = -\frac{3}{2}\alpha_0$, $N = -\frac{3}{2}\beta_0$ and the third equation implies

$$(44) \qquad 2\alpha_0 \tilde{d}_1 - \beta_0 \tilde{d}_3 = 0\,.$$

Finally, the last relation (43) yields $T = \beta_0 \tilde{d}_1 / \tilde{d}_3 = (\beta_0)^2 / 2\alpha_0$ and $L = R = 0$; hence $u_1 = u_4 = v_1 = v_4 = 0$ and

$$(45) \qquad D_1 = \frac{(\beta_0)^2}{2\alpha_0} M - \tfrac{9}{4}\alpha_0\beta_0\,, \quad \tilde{D}_1 = \frac{(\beta_0)^2}{2\alpha_0} M - \tfrac{1}{4}\alpha_0\beta_0\,.$$

We establish easily a geometrical signification of the condition (44):

$$(44') \qquad R(p_2(\mathscr{P}), p_3(\mathscr{P}), p_4(\mathscr{P}), HQ) = -1\,,$$

where $R(p_1, p_2, p_3, p_4)$ will always denote *the cross ratio of the four improper points of lines* p_1, p_2, p_3, p_4. (Also imaginary elements are admitted.)

Thus the lines $p_1(\mathscr{P})$, $p_3(\mathscr{P})$ depend on the other objects $p_2(\mathscr{P})$, $p_4(\mathscr{P})$, $H(\mathscr{P})$, $Q(\mathscr{P})$. The system (43) lets the function M indeterminate; thus the constructed objects are not sufficient for a representation of the manifold $\mathfrak{M}^0(1, 1)$. We have to construct an additional object independent of the previous ones. Let $\mathscr{P} \in \mathfrak{M}^0(1, 1)$. There is exactly one d-element $\tau(\mathscr{P}) \subset \mathrm{t}$ having as its singularity the improper point of the line $H(\mathscr{P})\, Q(\mathscr{P})$. We have $\tau(\mathscr{P}) = (\alpha(\partial/\partial x) + \beta(\partial/\partial y))$ with respect to any $\mathfrak{R}^\alpha$. We wish to find all complex d-elements ${}^c\xi \subset C\mathscr{P}$ such that

a) $[\tau(\mathscr{P}), {}^c\xi] = 0$,

b) ${}^c\xi$ admits a singularity in CA^2.

The requirement *a)* leads to the relations $\alpha a + \beta c = 0$, $\alpha b + \beta d = 0$ for homogeneous coordinates a, b, c, d of ${}^c\xi$, and we can put $a = \mu\beta$, $c = -\mu\alpha$, $b = \lambda\beta$, $d = -\lambda\alpha$, where λ, μ are suitable complex numbers. The conditions (12) then have the form

$$(12') \qquad u + \mu(\beta x - \alpha y) = 0\,, \quad v + \lambda(\beta x - \alpha y) = 0$$

and the condition *b)* implies $\lambda u - \mu v = 0$. According to (7) we obtain

$$(46) \qquad \lambda^2[u_2\beta - u_4\alpha] - \lambda\mu[u_3\alpha - v_4\alpha + v_2\beta - u_1\beta] + \mu^2[v_3\alpha - v_1\beta] = 0\,.$$

We can see that there are two d-elements ${}^c\xi_1, {}^c\xi_2 \subset C\mathscr{P}$ corresponding to the solutions $\lambda_1 : \mu_1$, $\lambda_2 : \mu_2$ of (46), which meet our demands. Because of $ad - bc = 0$ each of those d-elements admits a pointwise singular line. From (12') we obtain a common equation of both singular lines in canonical coordinates:

$$(47) \qquad \gamma(\mathscr{P}) \equiv (\alpha y - \beta x)^2 + \alpha\beta D_1 = 0\,.$$

The equivariant object $\gamma(\mathscr{P})$ we have just constructed consists of two parallels (real or imaginary conjugate), which correspond to each other in a reflection of CA^2 in the line $H(\mathscr{P})\ Q(\mathscr{P})$. Let us remark that after (44') the lines $p_1(\mathscr{P})$, $p_2(\mathscr{P})$ are not parallel to HQ and, in canonical coordinates, we have $\alpha\beta \neq 0$. Because $D_1 \neq 0$ the lines (47) are always mutually different. We see from (45) that $\tilde{D}_1 = D_1 + 2\alpha\beta$, and the condition $\tilde{D}_1 \neq 0$ has the following geometrical signification: provided that the couple $\gamma(\mathscr{P})$ is real, let us denote by C, C' the points of intersection of $\gamma(\mathscr{P})$ with $p_2(\mathscr{P})$, and by B_3, B_4 the points of intersection $p_3(\mathscr{P}) \times p_2(\mathscr{P})$, $p_4(\mathscr{P}) \times p_2(\mathscr{P})$. Then the condition $\tilde{D}_1 \neq 0$ is equivalent to the geometrical condition

$$(48) \qquad \overline{HC}^2 \neq \overline{HC'}^2 + \overline{HB}_3 \,.\, \overline{HB}_4\,, \quad or \quad \overline{HC}^2 \neq \overline{HC'}^2 + \tfrac{1}{2}\overline{HB}_4^2\,.$$

Proposition 9. *Let two real points $H_0 \neq Q_0$ and two real non-parallel lines $p_2 \ni H_0$, $p_4 \ni Q_0$ be given such that $p_2 \neq H_0Q_0 \neq p_4$. Let γ be a couple of parallels, either real or imaginary conjugate, which correspond to each other in a reflection of CA_2 in the line H_0Q_0. In the real case let us suppose that $\gamma \neq H_0Q_0$ and that the analogue of the second relation* (48) *holds. Then there is exactly one block $\mathscr{P} \in \mathfrak{M}^0(1, 1)$ such that $H(\mathscr{P}) \equiv H_0$, $Q(\mathscr{P}) \equiv Q_0$, $p_2(\mathscr{P}) \equiv p_2$, $p_4(\mathscr{P}) \equiv p_4$, $\gamma(\mathscr{P}) \equiv \gamma$.*

Proof. Let us construct a real line $p_3 \ni Q_0$ such that $R(p_2, p_3, p_4, H_0Q_0) = -1$. Let us draw a parallel p_1 to p_3 through the point H_0; put $k_1 \equiv \{p_1, p_2\}$, $k_2 \equiv \{p_3, p_4\}$. Then the conditions $H(\mathscr{P}) \equiv H_0$, $Q(\mathscr{P}) \equiv Q_0$, $p_2(\mathscr{P}) \equiv p_2$, $p_4(\mathscr{P}) \equiv p_4$ can be replaced by the conditions $k_H(\mathscr{P}) \equiv k_1$, $k_Q(\mathscr{P}) \equiv k_2$. Let us choose a coordinate system $\mathfrak{R}^\alpha$ such that $\boldsymbol{x} \equiv p_1$, $\boldsymbol{y} \equiv p_2$; then $k_1 \equiv xy = 0$, $k_2 \equiv -2d_1(x - \alpha_0)(y - \beta_0) - d_3(y - \beta_0)^2 = 0$, where α_0, β_0 are the coordinates of the point Q_0. The conditions of coincidence $k_H(\mathscr{P}) \equiv k_1$, $k_Q(\mathscr{P}) \equiv k_2$ lead now to the system of equations formed by (38), (39), (43). Our construction obviously quarantees that (44) holds and the system (43) possesses a single solution S, T, N as above. With respect to (38), (39) it remains to determine M. The line couple γ can be described in the form $(\alpha_0 y - \beta_0 x)^2 + c\alpha_0\beta_0 = 0$, where $c \neq 0$. A comparison with (47) shows that $D_1 = c$ and from the first equation (45) we obtain a unique value of M. According to our construction we have $D_1 \neq 0$. Owing to the formula (44) and to the analogue of the second relation (48) we see that $c + 2\alpha_0\beta_0 \neq 0$. Hence and from the second relation (45) we obtain $\tilde{D}_1 \neq 0$. Thus the block $\mathscr{P}$ determined above belongs to $\mathfrak{M}^\square$. One can verify easily that $\mathscr{P} \in \mathfrak{M}^0(1, 1)$ and that the primary geometrical requirements of the Proposition are satisfied.

Theorem 4. *The equivariant objects $H(\mathscr{P})$, $Q(\mathscr{P})$, $p_2(\mathscr{P})$, $p_4(\mathscr{P})$, $\gamma(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{M}^0(1, 1)$ with values in CA^2. The manifold $\mathfrak{M}^0(1, 1)$ naturally decomposes into four systems of ∞^1 orbits of dimension* 6.

Proof. In Proposition 9 we have proved the one-to-one property and also found the domain of values of the frame. The mentioned systems of orbits will arise if we combine the real or imaginary case of the couple $\gamma(\mathscr{P})$ with the case of positive or negative orientation of the triangle $H(\mathscr{P})\ Q(\mathscr{P})\ B(\mathscr{P})$, where $B(\mathscr{P})$ denotes the intersection point $p_2(\mathscr{P}) \times p_4(\mathscr{P})$. In each orbit system the orbits are described by a suitable division ratio of three points, where the line $p_2(\mathscr{P})$ intersectes the parallels of $\gamma(\mathscr{P})$ and the line $p_4(\mathscr{P})$.

Theorem 5.

a) $\mathfrak{M}^0(1, -1) = \mathfrak{M}^0(-1, 1) = \mathfrak{M}^0(0, -1) = \mathfrak{M}^0(-1, 0) = \emptyset$
b) $\mathfrak{M}^0(-1, -1) = \emptyset$
c) $\mathfrak{M}^0(0, 1) = \mathfrak{M}^0(1, 0) = \emptyset$

Proof. The assertion *a)* follows from the part *a)* of Proposition 7 and the assertion *b)* will be obtained by combining both parts of that Proposition. Assume that $\mathscr{P} \in \mathfrak{M}^0(1, 0)$. We put the coordinate axes $\vec{x}, \vec{y}$ into the lines of the couple $k_H(\mathscr{P})$ in such a way that, in addition, the axis $\vec{x}$ is parallel to the double line $k_Q(\mathscr{P})$. Then we have $k_H(\mathscr{P}) \equiv xy = 0$, $k_Q(\mathscr{P}) \equiv (y - y_Q)^2 = 0$.

By comparing that with (34) and (35) we obtain $D_2 = D_3 = \tilde{D}_1 = \tilde{D}_2 = 0$, $D_1\tilde{D}_3 \neq 0$. From (26), (27) follows $L = R = 0$, $\tilde{N} = \tilde{T} = 0$; from (23) we have $N + \beta = 0$ and from the second equation (25′) we obtain $\beta = 0$; hence $N = 0$. Now we have $D_1 = MT - SN = M\tilde{T} - SN = 0$ – a contradiction. Consequently $\mathfrak{M}^0(1, 0) = \emptyset$. The second relation *c)* can be proved in a similar way, q.e.d.

*

It remains to discuss the submanifold $\mathfrak{M}^0(0, 0)$. It will be advantageous to investigate this manifold together with a new one. Consider an invariant decomposition

$$(49) \qquad \Gamma_4^0(1) \dot{-} \mathfrak{M} = \mathfrak{L}_{12} \cup \mathfrak{L}_{21} \cup \mathfrak{L}_{11} \cup \mathfrak{L}_{02} \cup \mathfrak{L}_{20} \cup \mathfrak{L}_{10} \cup \mathfrak{L}_{01} \cup \mathfrak{L}_{00}\,,$$

where $\mathfrak{L}_{ij}$ denotes the submanifold of all blocks $\mathscr{P} \in \Gamma_4^0(1)$ such that the rank of the system (13) is i at the point $H(\mathscr{P})$ and j at the point $Q(\mathscr{P})$.

For $\max\{i, j\} = 2$ consider another invariant decomposition

$$(50) \qquad \mathfrak{L}_{ij} = \mathfrak{L}_{ij}(1) \cup \mathfrak{L}_{ij}(-1) \cup \mathfrak{L}_{ij}(0)$$

corresponding to the decomposition (17). (Let us remind that, in this case, exactly one of the points $H(\mathscr{P})$, $Q(\mathscr{P})$ belongs to $\mathscr{P}^\square$.)

Proposition 10. *Let* $\mathscr{P} \in \Gamma_4^0(1)$, $H(\mathscr{P}) \in \varkappa(\mathscr{P})$, $Q(\mathscr{P}) \in \mathscr{P}^\square(0)$, *and assume that the double line* $k_Q(\mathscr{P})$ *coincides with* $H(\mathscr{P})\,Q(\mathscr{P})$. *Then the cubic* $\varkappa(\mathscr{P})$ (see Proposition 3) *decomposes into the double line* $H(\mathscr{P})\,Q(\mathscr{P})$ *and into another line*

$$(51) \quad l(\mathscr{P}) \equiv T(x - x_H) + \left(\frac{2\alpha}{\beta}T - 3N - 3\beta\right)(y - y_H) - \tfrac{3}{2}(D_2 + T\alpha - \beta N) = 0\,.$$

Proof. As $k_Q(\mathscr{P})$ is a double line and $k_Q(\mathscr{P}) \equiv H(\mathscr{P})\,Q(\mathscr{P})$, we have obvious relations

$$(52) \qquad \alpha\tilde{D}_1 + \beta\tilde{D}_3 = 0\,, \quad \alpha\tilde{D}_2 - \beta\tilde{D}_1 = 0\,.$$

The cubic $\varkappa(\mathscr{P})$ is given by the equation

$$(53)\qquad \begin{aligned} D(\boldsymbol{x}) = T\bar{x}^3 - 3(N+\beta)\,\bar{x}^2\bar{y} - 3(S+\alpha)\,\bar{x}\bar{y}^2 + M\bar{y}^3 - \\ - \tfrac{3}{2}(D_2 + T\alpha - \beta N)\,\bar{x}^2 + 3(D_1 + \alpha N + \beta S + 3\alpha\beta)\,\bar{x}\bar{y} + \\ + \tfrac{3}{2}(D_3 + \alpha S - \beta M)\,\bar{y}^2 = 0\,, \end{aligned}$$

which follows from (52), (26), (27) and from the assumption $H(\mathscr{P}) \in \varkappa(\mathscr{P})$, i.e., $D = 0$. For the double line $H(\mathscr{P})\,Q(\mathscr{P})$ we have $(\beta\bar{x} - \alpha\bar{y})^2 = 0$. (We put for brevity $\bar{x} = x - x_H$, $\bar{y} = y - y_H$.) Our statement must be now verified by a direct calculation. It remains to show that (51) cannot vanish identically. But in case that (51) vanishes identically so does (53) and, in particular, $T = N + \beta = S + \alpha = M = 0$. The last relations can be written in the form $\tilde{T} = \tilde{N} = \tilde{S} = \tilde{M} = 0$, and from (25) we obtain $\tilde{L} = \tilde{R} = 0$. Thus the system (13) is of rank 0 at the point $Q(\mathscr{P})$ – a contradiction.

Proposition 11. *The map $\mathscr{P} \to l(\mathscr{P})$ is an equivariant object on the manifold $\mathfrak{M}^0(0, 0) \cup \mathfrak{L}_{12}(0)$.*

Proof. It is obvious that $H(\mathscr{P}) \in \varkappa(\mathscr{P})$, $Q(\mathscr{P}) \in \mathscr{P}^{\square}(0)$ holds on both components of the manifold. It remains to show that $k_Q(\mathscr{P}) \equiv H(\mathscr{P})\,Q(\mathscr{P})$.

a) Let $\mathscr{P} \in \mathfrak{M}^0(0, 0)$, then the double lines $k_H(\mathscr{P})$, $k_Q(\mathscr{P})$ are parallels and thus the non-zero triplets (D_1, D_2, D_3), $(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3)$ are proportional. From (26), (28) we have (52) and consequently $k_Q(\mathscr{P}) \equiv k_H(\mathscr{P}) \equiv H(\mathscr{P})\,Q(\mathscr{P})$.

b) Let $\mathscr{P} \in \mathfrak{L}_{12}(0)$ and let us suppose $k_Q(\mathscr{P}) \not\equiv H(\mathscr{P})\,Q(\mathscr{P})$. Choose a "canonical" coordinate system $\mathfrak{R}^\alpha$ such that $\vec{x} \equiv k_Q(\mathscr{P})$, $\vec{y} \equiv H(\mathscr{P})\,Q(\mathscr{P})$. In canonical coordinates we obtain $\tilde{D}_1 = \tilde{D}_2 = 0$, $\alpha = 0$, $\tilde{D}_3 \neq 0$, $\beta \neq 0$. From (27) follows $\tilde{N} = \tilde{T} = 0$, and from (25′) we obtain $\tilde{R} = 2\tilde{N} = 0$. If we substitute these values into the second equation (24), then with respect to $\tilde{D}_2 = 0$, $D_2 = 0$, we obtain $\beta = 0$ – a contradiction.

Proposition 12. *For each $\mathscr{P} \in \mathfrak{M}^0(0, 0) \cup \mathfrak{L}_{12}(0)$ we have*

a) $l(\mathscr{P}) \nparallel H(\mathscr{P})\,Q(\mathscr{P})$;

b) $Q(\mathscr{P}) \notin l(\mathscr{P})$.

Proof. ad *a*). Let us suppose $l(\mathscr{P}) \parallel H(\mathscr{P})\,Q(\mathscr{P})$. Choose a coordinate system $\mathfrak{R}^\alpha$ such that $\vec{y} \parallel l(\mathscr{P})$. Then $\alpha = 0$, $\beta \neq 0$, $\tilde{D}_1 = \tilde{D}_3 = 0$, $\tilde{D}_2 \neq 0$ follows from the relation $k_Q(\mathscr{P}) \parallel H(\mathscr{P})\,Q(\mathscr{P}) \parallel \vec{y}$. From (27) we obtain $\tilde{M} = \tilde{S} = 0$, and from (25′) $\tilde{L} = 0$. After (51) we have $N + \beta = 0$, and (25′) yields $R = \beta$, i.e., $\tilde{R} = 0$. Thus $\tilde{D}_2 = \tilde{R}\tilde{N} - \tilde{L}\tilde{T} = 0$ – a contradiction.

ad *b*). Let us choose a coordinate system $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$ such that $\vec{x} \parallel l(\mathscr{P})$, $\vec{y} \parallel H(\mathscr{P})\,Q(\mathscr{P}) \parallel k_Q(\mathscr{P})$. Then as above, we obtain successively $\alpha = 0$, $\beta \neq 0$, $\tilde{D}_1 = \tilde{D}_3 = 0$, $\tilde{D}_2 \neq 0$, $\tilde{M} = 0$, $\tilde{S} = \tilde{L} = 0$; hence $M = S = L = 0$ and $D_2 = RN$. The equation (51) transforms into $-3\tilde{N}y - \frac{3}{2}(RN - \beta N) = 0$. We see from (25′) that $R - \beta = 2\tilde{N}$, and finally, we obtain

$$(51') \qquad l(\mathscr{P}) \equiv y + N = 0\,, \quad \tilde{N} \neq 0\,.$$

Since $Q(\mathscr{P}) = [0, \beta]$, $\tilde{N} \neq 0$ we have $Q(\mathscr{P}) \notin l(\mathscr{P})$, q.e.d.

Proposition 13. *Let $H_0 \not\equiv Q_0$ be two points of A^2 and $l \subset A^2$ a real line such that $l \not\parallel H_0Q_0$, $l \not\ni Q_0$. Then there is exactly one block $\mathscr{P} \in \mathfrak{M}^0(0, 0) \cup \mathfrak{L}_{12}(0)$ such that $H(\mathscr{P}) \equiv H_0$, $Q(\mathscr{P}) \equiv Q_0$, $l(\mathscr{P}) \equiv l$. Moreover, $\mathscr{P} \in \mathfrak{L}_{12}(0)$ if and only if l intersects H_0Q_0 at the point H_0, or at the point Z_0 given by $\overrightarrow{H_0Z_0} : \overrightarrow{H_0Q_0} = 3 : 2$.*

Proof. Let us choose an $\mathfrak{R}^\alpha$ with the origin H_0 and such that $\vec{y} \equiv H_0Q_0$, $\vec{x} \parallel l$. Now we can specialize $\mathfrak{R}^\alpha$ so that $Q_0 = [0, 1]$. Firstly we obtain four conditions (38) and (39), where we put $x_1 = y_1 = 0$, $\alpha_0 = 0$, $\beta_0 = 1$. It remains to determine M, N, S, T. The condition $\mathscr{P} \in \mathfrak{M}^0(0, 0) \cup \mathfrak{L}_{12}(0)$ implies $k_Q(\mathscr{P}) \equiv H(\mathscr{P})\, Q(\mathscr{P})$. Thus $k_Q(\mathscr{P}) \parallel \vec{y}$ and we obtain, as usual, $\tilde{M} = \tilde{S} = 0$. Besides that we have $\alpha = 0$, $\beta = 1$ and thus $M = S = 0$. Further we can write $l \equiv y - y_0 = 0$, and the condition $l(\mathscr{P}) \equiv l$ implies, taking into account (51) and (51′), $T = 0$, $N = -\beta_0$. The block $\mathscr{P}$ is uniquely determined. We can see easily that $L = 0$, $R = 3 - 2y_0$, $\tilde{D}_1 = \tilde{D}_3 = D_1 = D_3 = 0$ and $D_2 = (2y_0 - 3)\, y_0$, $\tilde{D}_2 = 2(1 - y_0)^2$.

As $Q_0 \notin l$ we have $y_0 - 1 \neq 0$ and $\tilde{D}_2 \neq 0$; thus $Q(\mathscr{P}) \in \mathscr{P}^\square(0)$. Further the condition $\mathscr{P} \in \mathfrak{L}_{12}(0)$ is equivalent to $D_2 = 0$; hence either $y_0 = 0$ or $y_0 = \frac{3}{2}$, which our Proposition asserts. It is obvious that all geometrical conditions are really satisfied.

Theorem 6. *The equivariant objects $H(\mathscr{P})$, $Q(\mathscr{P})$, $l(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{M}^0(0, 0) \cup \mathfrak{L}_{12}(0)$. This manifold decomposes into ∞^1 orbits of dimension* 5; *each orbit is given by an invariant*

$$\lambda = \frac{\overrightarrow{H(\mathscr{P})\, Z(\mathscr{P})}}{\overrightarrow{H(\mathscr{P})\, Q(\mathscr{P})}} = \frac{D_2 + T\alpha - \beta N}{2(\alpha T - \beta N - \beta^2)}\,, \quad \lambda \neq 1\,,$$

where $Z(\mathscr{P})$ denotes the intersection point of lines $l(\mathscr{P})$, $H(\mathscr{P})\, Q(\mathscr{P})$. An orbit belongs to $\mathfrak{L}_{12}(0)$ if and only if, either $\lambda = 0$ or $\lambda = \frac{3}{2}$.

Proof. Our assertion follows from the preceding Propositions; the formula can be deduced from (51).

Let us remark that Theorem 6 completes our investigation of the manifold $\mathfrak{M}$ (see (37)).

Theorem 7. *Each of the manifolds $\mathfrak{L}_{12}(1)$, $\mathfrak{L}_{21}(1)$, $\mathfrak{L}_{12}(-1)$, $\mathfrak{L}_{21}(-1)$ is an orbit of dimension* 6.

Proof. We start with $\mathscr{P} \in \mathfrak{L}_{21}(1)$. Let us put the coordinate axes of a canonical coordinate system $\mathfrak{R}^\alpha$ into the lines of the real couple $k_H(\mathscr{P})$. From (34) we get

$D_1 \neq 0$, $D_2 = D_3 = 0$ in canonical coordinates. From (26) follows $L = R = 0$ and (25') implies $N = -3\beta/2$, $S = -3\alpha/2$. If we substitute these values and also the value $\tilde{D}_1 = 0$ into the first equation (24) we obtain $D_1 = 2\alpha\beta$. Since $D_1 \neq 0$ we have $\alpha\beta \neq 0$, and thus *neither of the lines of* $k_H(\mathscr{P})$ *coincides with* $H(\mathscr{P})\, Q(\mathscr{P})$.

Let now two points $H_0 \neq Q_0$ and a couple k of real non-parallel lines crossing at H_0 be given, $Q_0 \notin k$. Choose a coordinate system having the lines of the couple k as coordinate axes. Then the conditions $H(\mathscr{P}) \equiv H_0$, $Q(\mathscr{P}) \equiv Q_0$, $k_H(\mathscr{P}) \equiv k$ will be satisfied in a unique way by the values $L = R = 0$, $N = -3\beta_0/2$, $S = -3\alpha_0/2$ and $T = (\beta_0)^2/2\alpha_0$, $M = (\alpha_0)^2/2\beta_0$. (The last relations follow from the second and third equation (24) with respect to $D_2 = \tilde{D}_2 = D_3 = \tilde{D}_3 = 0$. We also make use of the relation (38)). We can see easily that $\mathscr{P} \in \mathfrak{L}_{21}(1)$, and the geometrical conditions of coincidence are actually satisfied. Now it is obvious that $\{H(\mathscr{P}), Q(\mathscr{P}), k_H(\mathscr{P})\}$ is a representing frame on the manifold $\mathfrak{L}_{21}(1)$ and the domain of values of the frame is an orbit of dimension 6.

The manifold $\mathfrak{L}_{12}(1)$ can be treated analogously.

Let us suppose $\mathscr{P} \in \mathfrak{L}_{21}(-1)$; then the couple $k_H(\mathscr{P})$ is imaginary conjugate and $\tilde{D}_1 = \tilde{D}_2 = \tilde{D}_3 = 0$. Let us examine the uniqueness. Choose two points $H_0 \neq Q_0$ and a couple of non-parallel imaginary conjugate lines with the common point H_0. We can see easily that the coordinate axes can be chosen so that $\vec{y} \equiv H_0Q_0$ and the couple k is given by $k \equiv dx^2 + y^2 = 0$, $d > 0$. The conditions $H(\mathscr{P}) \equiv H_0$, $\mathscr{P} \in \mathfrak{L}_{21}(-1)$, $Q(\mathscr{P}) \equiv Q_0$, $k_H(\mathscr{P}) \equiv k$ lead to the relations $D_1 = 0$, $\alpha = 0$. From the first equation (24) we obtain $S = 0$, from (25') $L = 0$, and from (26) $T = 0$. Hence $D_2 = RN$. Making necessary substitutions in the second equation (24) we obtain $(R - \beta)(N + \beta) = 0$. Here $R - \beta = 2(N + \beta)$ according to (25'), and finally $R - \beta = N + \beta = 0$. If we put $Q_0 = [0, \beta_0]$ we obtain $N = -\beta_0$, $R = \beta_0$. The condition of coincidence $k_H(\mathscr{P}) \equiv k$ yields, with regard to (34), $D_2 + dD_3 = 0$. Taking into account the values we have computed so far, we obtain $D_2 = -\beta_0^2$, $D_3 = -M\beta_0$ and thus $M = \beta_0/d$. We make sure easily that the block $\mathscr{P}$ just determined meets our demands. Thus the objects $H(\mathscr{P})$, $Q(\mathscr{P})$, $k_H(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{L}_{21}(-1)$ and the considered configurations form an orbit of dimension 6. The proof is quite similar for the manifold $\mathfrak{L}_{12}(-1)$.

Theorem 8. *The manifold* $\mathfrak{L}_{21}(0)$ *consists of two orbits of dimension* 5. *The manifold* $\mathfrak{L}_{20}(0)$ *is an orbit of dimension* 4.

Proof. Let $\mathscr{P} \in \mathfrak{L}_{21}(0) \cup \mathfrak{L}_{20}(0)$. An argument like that which we have used in the proof of Proposition 11 shows that $k_H(\mathscr{P}) \equiv H(\mathscr{P})\, Q(\mathscr{P})$. Unfortunately, the cubic $\varkappa(\mathscr{P})$ is of no use in this case because it becomes the 3-fold line HQ. Therefore, some direct computations are necessary. Let us choose two points $H_0 \neq Q_0$. A coordinate system $\mathfrak{R}^\alpha$ will be called *canonical* if $H_0 = [0, 0]$, $Q_0 = [0, 1]$ with respect to $\mathfrak{R}^\alpha$. From the requirements $H(\mathscr{P}) \equiv H_0$, $Q(\mathscr{P}) \equiv Q_0$ and $\mathscr{P} \in \mathfrak{L}_{21}(0) \cup \mathfrak{L}_{20}(0)$ follows $k_H(\mathscr{P}) \equiv H_0Q_0$, and hence $\alpha = 0$, $\beta = 1$, $D_1 = D_3 = 0$. The equations (26) imply $M = S = 0$ and from (25′) and the second equation (24) we obtain $(N + 1)(R - 1) = 0$; thus $N = -1$, $R = 1$. Hence the blocks $\mathscr{P} \in \mathfrak{L}_{21}(0) \cup \cup \mathfrak{L}_{20}(0)$ with a prescribed position of the basic points $H(\mathscr{P})$, $Q(\mathscr{P})$ are, with respect to arbitrary canonical coordinates, uniquely determined by the values of the parameter T. Denote by $\mathfrak{M}(H_0, Q_0)$ the set of all blocks satisfying the preceding requirements. Obviously each transformation from $G = GA^+(2)$ leaving the points H_0, Q_0 invariant belongs to a connected subgroup H, $\dim H = 2$. The corresponding Lie algebra $\mathfrak{h}$ is given by $\mathfrak{h} = (x(\partial/\partial x), x(\partial/\partial y))$ in arbitrary canonical coordinates. If we denote by φ the representation of the algebra $\mathfrak{h}$ in the space $\mathrm{X}(\mathfrak{L}_{21}(0) \cup \mathfrak{L}_{20}(0))$ we obtain, in canonical coordinates

$$\varphi\left(x \frac{\partial}{\partial x}\right) T = -T, \quad \varphi\left(x \frac{\partial}{\partial y}\right) T = R + N = 0 \quad \text{for each} \quad \mathscr{P} \in \mathfrak{M}(H_0, Q_0).$$

By the integration we obtain that, for any transformation $h \in H$, the parameters T, T' of two blocks $\mathscr{P}$, $h \,.\, \mathscr{P} \in \mathfrak{M}(H_0, Q_0)$ differ only by a positive factor. Having fixed a canonical $\mathfrak{R}^\alpha$, we can see that $\mathfrak{M}(H_0, Q_0)$ decomposes into three orbits given by the relations $T > 0$, $T = 0$, $T < 0$. As the group H is connected, these relations hold out any variation of canonical coordinates.

Obviously we have $T = 0$ if and only if $\mathscr{P} \in \mathfrak{L}_{20}(0)$. Because $GA^+(2)$ acts transitively on the couples $\{H_0, Q_0\}$, we obtain both assertions of the Theorem without difficulties.

Note. The following criterion can be deduced very easily: Let $\mathscr{P}, \mathscr{P}' \in \mathfrak{L}_{21}(0) \cup \cup \mathfrak{L}_{20}(0)$ be given. Let $\mathfrak{R}^\alpha$ be a canonical coordinate system with respect to the basic points $H(\mathscr{P})$, $Q(\mathscr{P})$ and let $\mathfrak{R}^\beta$ be a canonical coordinate system with respect to the basic points $H(\mathscr{P}')$, $Q(\mathscr{P}')$. Then $\mathscr{P}$ and $\mathscr{P}'$ belong to the same orbit if and only if $\operatorname{sgn} T = \operatorname{sgn} T'$ in the corresponding canonical coordinate systems.

Theorem 9.
$$\mathfrak{L}_{00} = \mathfrak{L}_{01} = \mathfrak{L}_{02} = \emptyset,$$
$$\mathfrak{L}_{10} = \mathfrak{L}_{11} = \mathfrak{L}_{20}(1) = \mathfrak{L}_{20}(-1) = \emptyset.$$

Consequently, the manifold $\Gamma_0^4(1)$ *does not contain any orbits except those we have found before.*

Proof. If the system (13) is of rank 0 at the point $H(\mathscr{P})$ we obtain $M = N = L = R = S = T = 0$, hence $\alpha = \beta = 0$ and $\mathscr{P} \in \Gamma_4^0(0)$ (see (36)). Thus $\mathfrak{L}_{00} = \mathfrak{L}_{01} = \mathfrak{L}_{02} = \emptyset$. If the system (13) is of rank 0 at the point $Q(\mathscr{P})$, then from (23) $L = -\alpha$, $M = 0$, $N = -\beta$, $R = \beta$, $S = -\alpha$, $T = 0$, $D_1 = \alpha\beta$, $D_2 = -\beta^2$, $D_3 = \alpha^2$. As far as $\mathscr{P} \in \Gamma_4^0(1)$, at least one of the determinants D_i is non-zero. Because $D = 0$, we obtain $\mathscr{P} \in \mathfrak{L}_{20}(0)$, a case treated in the preceding Theorem. Thus $\mathfrak{L}_{10} = \mathfrak{L}_{20}(1) = \mathfrak{L}_{20}(-1) = \emptyset$. Let us suppose finally that $\mathscr{P} \in \mathfrak{L}_{11}$, i.e., $D_i = \tilde{D}_i = 0$. Choose a coordinate system $\mathfrak{R}^\alpha$ such that $\vec{y} \equiv H(\mathscr{P})\, Q(\mathscr{P})$, then $\alpha = 0$. From the equations (24) and (25′) we obtain $S = M = L = 0$, $N = -2\beta$, $R = -\beta$, $D_2 = RN - LT = 2\beta^2 = 0$, and hence $\beta = 0$ — a contradiction.

*

Let us pass over to the manifold $\Gamma_4^0(0)$ given by the condition $H(\mathscr{P}) \equiv Q(\mathscr{P})$. Let us consider an invariant decomposition

$$(54) \qquad \Gamma_4^0(0) = \mathfrak{N}_2 \cup \mathfrak{N}_1 \cup \mathfrak{N}_0$$

built up after the rank of the system (13) at the point $H(\mathscr{P})$, and another decomposition

$$(55) \qquad \mathfrak{N}_2 = \mathfrak{N}_2(1) \cup \mathfrak{N}_2(-1) \cup \mathfrak{N}_2(0)$$

where $\mathfrak{N}_2(i) = \{\mathscr{P} \in \mathfrak{N}_2 \mid H(\mathscr{P}) \in \mathscr{P}^{\square}(i)\}$, $i = 1, -1, 0$.

Because $\alpha = \beta = 0$ on the manifold $\Gamma_4^0(0)$, the equation of the cubic $\varkappa(\mathscr{P})$ assumes a simple form

$$(56) \qquad \varkappa(\mathscr{P}) \equiv T\bar{x}^3 - 3N\bar{x}^2\bar{y} - 3S\bar{x}\bar{y}^2 + M\bar{y}^3 - \tfrac{3}{2}D_2\bar{x}^2 + \\ + 3D_1\bar{x}\bar{y} + \tfrac{3}{2}D_3\bar{y}^2 - D = 0\,, \quad \bar{x} = x - x_H\,, \quad \bar{y} = y - y_H\,.$$

Let $\mathscr{P} \in \mathfrak{N}_2(1)$. If we put the coordinate axes into the lines of the couple $k_H(\mathscr{P})$, we obtain $k_H(\mathscr{P}) \equiv xy = 0$. In our canonical coordinates $D_2 = D_3 = 0$, $D_1 \neq 0$, and from the equations (26), (25′) follows $L = R = N = S = 0$. We obtain

$$(56a) \qquad \varkappa(\mathscr{P}) \equiv Tx^3 + My^3 + 3D_1xy - D_1^2 = 0\,.$$

Here $T \neq 0$, $M \neq 0$ since $D_1 = TM \neq 0$. $k_H(\mathscr{P})$ and $\varkappa(\mathscr{P})$ possess exactly two real common points, namely

$$(57) \quad A_1(\mathscr{P}) = [0, \sqrt[3]{(D_1^2/M)}],\ A_2(\mathscr{P}) = [\sqrt[3]{(D_1^2/T)}, 0]\,, \quad A_1(\mathscr{P}) \not\equiv H(\mathscr{P}) \not\equiv A_2(\mathscr{P})\,.$$

(The numeration depends on the coordinate system employed.)

Theorem 10. *The point $H(\mathscr{P})$ and the (non-ordered) couple $\{A_1(\mathscr{P}), A_2(\mathscr{P})\}$ form together a representing frame on the manifold $\mathfrak{N}_2(1)$. That manifold is an orbit of dimension 6.*

Proof. Let us choose three mutually different points H_0, A_{10}, A_{20} in A^2. We alter the numeration of the points A_{10}, A_{20} if necessary so that for an *admissible* coordinate system $\mathfrak{R}^\alpha$ we have $H_0 = [0, 0]$, $A_{10} = [0, 1]$, $A_{20} = [1, 0]$. Such a coordinate system is uniquely determined. The conditions of coincidence $H(\mathscr{P}) \equiv H_0$, $\{A_1(\mathscr{P}), A_2(\mathscr{P})\} \equiv \{A_{10}, A_{20}\}$ can be written in the form $H(\mathscr{P}) \equiv H_0$, $A_1(\mathscr{P}) \equiv A_{10}$, $A_2(\mathscr{P}) \equiv A_{20}$. In the coordinate system $\mathfrak{R}^\alpha$ we obtain $u_1 = v_4 = 0$, $L = R = N = S = 0$ and $D_1^2/M = D_1^2/T = 1$ (Cf. (57)). Hence $M = T = 1$, $D_1 = TM = 1 \neq 0$. Thus the block $\mathscr{P}$ is uniquely determined and it belongs to $\mathfrak{N}_2(1)$. Consequently our conditions are geometrically satisfied. The domain of values of our representing frame consists of configurations $\{H_0, A_{10}, A_{20}\}$, where $H_0A_{10}A_{20}$ is a positively oriented triangle; thus we obtain a single orbit of dimension 6.

Let now $\mathscr{P} \in \mathfrak{N}_2(-1)$. Choose a coordinate system $\mathfrak{R}^\alpha$ such that the imaginary line couple $k_H(\mathscr{P})$ is given by the equation $x^2 + y^2 = 0$. Then $D_2 = -D_3$, $D_1 = 0$; and from (29) follows $M = N$, $S = T$. The equation (56) assumes the form

$$\text{(56b)} \qquad Tx^3 - 3Mx^2y - 3Txy^2 + My^3 - 3(M^2 + T^2)(x^2 + y^2) + \\ + 4(T^2 + M^2)^2 = 0 .$$

According to Proposition 3, b), the cubic $\varkappa(\mathscr{P})$ decomposes into three lines in CA^2 because (56b) can not vanish identically.

a) The lines of $\varkappa(\mathscr{P})$ are mutually different.

In fact, we find easily that the intersection points of $k_H(\mathscr{P})$ and $\varkappa(\mathscr{P})$ are, in canonical coordinates defined above, given by the relations

$$x^3 = (T^2 + M^2)(T + Mi), \quad y = \pm ix .$$

We have 6 different intersection points, which proves our assertion.

b) The lines of $\varkappa(\mathscr{P})$ are all real.

It is obvious that among the canonical coordinate systems there are coordinate systems with any prescribed direction of the axis $\vec{x}$. Particularly, the axis $\vec{x}$ can be chosen so as to intersect $\varkappa(\mathscr{P})$ at three different points of CA^2. Then if we put $y = 0$ in (56b), the corresponding cubic equation has the discriminant $D = 532M^2(T^2 + M^2)^4T^{-4}$. According to our choice we have $D \neq 0$, thus $D > 0$, and the equation has three real roots x_1, x_2, x_3. Hence the *real* line $\vec{x}$ intersects $\varkappa(\mathscr{P})$ at three *real* points, which proves the assertion b).

c) The three lines of $\varkappa(\mathscr{P})$ have no common point in A^2. If such a point existed, we could choose a (non-canonical) coordinate system with the origin at that point. In our coordinates we should obtain, by comparison with (56), $D_1 = D_2 = D_3 = 0$ — a contradiction.

d) The three lines of $\varkappa(\mathscr{P})$ have no common improper point. Let us choose a canonical $\mathfrak{R}^\alpha$ for which the axis $\vec{x}$ is parallel to a line of $\varkappa(\mathscr{P})$. If the other lines of $\varkappa(\mathscr{P})$ were also parallel with $\vec{x}$, then among the leading terms of (56b) only the term My^3 would be non-zero — a contradiction.

e) All lines of the cubic $\varkappa(\mathscr{P})$ are mutually non-parallel. The last assertion will be proved as a part of the following Proposition:

Proposition 14. *Let $\varkappa_0$ be a triplet of real lines having no common, proper or improper, point. Then*

a) If all lines of $\varkappa_0$ are mutually non-parallel, then there is exactly one block $\mathscr{P} \in \mathfrak{N}_2(-1)$ such that $\varkappa(\mathscr{P}) \equiv \varkappa_0$.

b) If two of the lines of $\varkappa_0$ are parallels, then there is not any block $\mathscr{P} \in \mathfrak{N}_2(-1)$ with $\varkappa(\mathscr{P}) \equiv \varkappa_0$.

Proof. Let us put the coordinate axes $\vec{x}, \vec{y}$ into two non-parallel lines of $\varkappa_0$. Here we can assume that the axis $\vec{x}$ is not parallel to the third line of the triplet. Then the separate lines of $\varkappa_0$ are given by equations $x = 0$, $y = 0$, $x + ky + q = 0$, $q \neq 0$, and for the whole $\varkappa_0$ we obtain

$$x^2y + kxy^2 + qxy = 0 . \tag{58}$$

In our (non-canonical) coordinates we obtain (by comparing (56) with (58)) $T = 0$, $M = 0, N = -\lambda$, $S = -k\lambda$, where $\lambda \neq 0$. According to (25′) we have $L = 2k\lambda$, $R = -2\lambda$ and hence $D_1 = -k\lambda^2$, $D_2 = 2\lambda^2$, $D_3 = -2k^2\lambda^2$. If two of the lines of $\varkappa_0$ are parallels, then $k = 0$ and $D_1 = D_3 = 0$. Hence $D = 0$, which is contrary to the condition $\mathscr{P} \in \mathfrak{N}_2(-1)$. In the opposite case the equation (56) takes the form

$$\lambda\bar{x}^2\bar{y} + k\lambda\bar{x}\bar{y}^2 - \lambda^2\bar{x}^2 - k\lambda^2\bar{x}\bar{y} - k^2\lambda^2\bar{y}^2 + k^2\lambda^4 = 0 , \tag{59}$$
$$\bar{x} = x - x_H , \quad \bar{y} = y - y_H .$$

Let us denote by $F(x, y)$ the left side of the equation (58) and by $G(x, y)$ the left side of the equation (59). The condition of coincidence $\varkappa(\mathscr{P}) \equiv \varkappa_0$ is given by the relation $G(x, y) = \lambda F(x, y)$ and hence

$$\frac{\partial^2 G(0, 0)}{(\partial x)^2} = 0 , \quad \frac{\partial^2 G(0, 0)}{(\partial y)^2} = 0 , \quad \frac{\partial^2 G(0, 0)}{\partial x \, \partial y} = q\lambda .$$

From here we obtain

$$y_H + \lambda = 0 , \quad x_H + k\lambda = 0 , \quad k\lambda + q = 0 .$$

These equations together with the preceding ones uniquely determine $\mathscr{P}$. Taking into account the assertions $a)-d)$ proved above we can see that Proposition 14 defines the domain of values of the object $\varkappa(\mathscr{P})$ on the manifold $\mathfrak{N}_2(-1)$.

Theorem 11. *The object $\varkappa(\mathscr{P})$ is a representing frame on the manifold $\mathfrak{N}_2(-1)$. The manifold $\mathfrak{N}_2(-1)$ is an orbit of dimension* 6. *The isotropy group under* **G** *at each point $\mathscr{P} \in \mathfrak{N}_2(-1)$ is a finite cyclic group of order* 3.

Proof is obvious.

Note. The last assertion of Theorem 11 admits an inversion: *the only blocks $\mathscr{P} \in \Gamma_4$ whose isotropy groups are cyclic of order* 3 *are the elements of the submanifold* $\mathfrak{N}_2(-1)$. In order to prove it, we have to perform the complete classification of orbits of Γ_4 beforehand.

Let $\mathscr{P} \in \mathfrak{N}_2(0)$. Choose a coordinate system $\mathfrak{R}^\alpha$ with the origin $H(\mathscr{P})$ such that $\vec{y} \equiv k_H(\mathscr{P})$. Then $D_2 \neq 0$, $D_1 = D_3 = 0$, and from the equations (26), (25′) follows $M = S = L = 0$, $D_2 = 2N^2$; thus $N \neq 0$. The equation (56) has the form

$$\varkappa(\mathscr{P}) \equiv Tx^3 - 3N^2y - 3N^2x^2 = 0 . \tag{56c}$$

Thus $\varkappa(\mathscr{P})$ decomposes into the double line $x^2 = 0$ and into another line

$$r(\mathscr{P}) \equiv Tx - 3Ny - 3N^2 = 0\,, \quad N \neq 0\,. \tag{60}$$

Obviously $r(\mathscr{P}) \nparallel k_H(\mathscr{P})$, $H(\mathscr{P}) \notin r(\mathscr{P})$.

Proposition 15. *Let two real non-parallel lines r_0, k_0 and a point H_0 be given such that $H_0 \in k_0$, $H_0 \notin r_0$. Then there is exactly one block $\mathscr{P} \in \mathfrak{N}_2(0)$ such that $H(\mathscr{P}) \equiv H_0$, $k_H(\mathscr{P}) \equiv k_0$, $r(\mathscr{P}) \equiv r_0$.*

Proof. Choose a coordinate system $\mathfrak{R}^\alpha$ with the origin H_0 such that $\vec{y} \equiv k_0$, $\vec{x} \parallel r_0$. From the conditions of coincidence we obtain easily $M = S = L = 0$. If $r_0 \equiv y - y_0 = 0$, $y_0 \neq 0$, them $T = 0$, $N = -y_0$, $R = -2y_0$ and $D_2 = 2y_0^2 \neq 0$. Hence we obtain our assertion and also the following Theorem:

Theorem 12. *The equivariant objects $H(\mathscr{P})$, $k_H(\mathscr{P})$, $r(\mathscr{P})$ form a representing frame on the manifold $\mathfrak{N}_2(0)$. The manifold $\mathfrak{N}_2(0)$ is an orbit of dimension* 5.

Let us consider the case $\mathscr{P} \in \mathfrak{N}_1$. Choose the origin of the coordinates at the point $H(\mathscr{P})$. According to $D_1 = D_2 = D_3 = 0$ the equation (56) implies

$$\varkappa(\mathscr{P}) \equiv Tx^3 - 3Nx^2y - 3Sxy^2 + My^3 = 0\,. \tag{56d}$$

The relations $D_1 = D_2 = D_3 = 0$ can be re-written as $TM = NS$, $N^2 = -ST$, $S^2 = -MN$, and hence we can deduce easily that $\varkappa(\mathscr{P})$ is the threefold line $(Tx + My)^3 = 0$.

Let a point H_0 and a real line $q_0 \ni H_0$ be given; put $\mathfrak{N}(H_0, q_0) = \{\mathscr{P} \in \mathfrak{N}_1 \mid H(\mathscr{P}) \equiv H_0, \varkappa(\mathscr{P}) \equiv q_0^3\}$. If we choose a coordinate system $\mathfrak{R}^\alpha$ with the origin H_0 such that $\vec{x} \parallel q_0$, then for any $\mathscr{P} \in \mathfrak{N}(H_0, q_0)$ we obtain $T = N = S = 0$, $L = R = 0$, $u_1 = v_4 = 0$. The blocks $\mathscr{P} \in \mathfrak{N}(H_0, q_0)$ are thus uniquely determined by the parameter $M \neq 0$. We find easily that a dilatation from the origin H_0, $x' = \lambda x$, $y' = \lambda y$, preserves the set $\mathfrak{N}(H_0, q_0)$, and the parameter M is subjected to the transformation $M' = \lambda M$. From here we deduce that $\mathfrak{N}(H_0, q_0)$ is an orbit of dimension 1 under G. From the preceding we obtain

Theorem 13. *The manifold $\mathfrak{N}_1$ is an orbit of dimension* 4.

The proof of the following Theorem will be left to the reader:

Theorem 14. *The point $H(\mathscr{P})$ is a representing frame on the manifold $\mathfrak{N}_0$. The manifold $\mathfrak{N}_0$ is an orbit of dimension 2, its elements are subalgebras of $\mathfrak{ga}(2)$. Each subalgebra $\mathscr{P} \in \mathfrak{N}_0$ determines an isotropy group with respect to the action of G in the plane A^2, namely the isotropy group of the point $H(\mathscr{P})$.*

(To be continued)