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DIAGONALLY DOMINANT MATRICES

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Introduction. A. OSTROWSKI [5] has introduced the notion of an H-matrix as a complex-valued square matrix $A = (a_{ik})$ such that the class of all matrices $D^{-1}AD$ diagonally similar to A contains a matrix $C = (c_{ik})$ with the following property

$$|c_{ii}| > \sum_{k \neq i} |c_{ik}| \quad \text{for all } i .$$

For this class the name strongly diagonally dominant matrices has also been suggested.

In the present remark we intend to investigate a class of matrices which is, in a certain sense, dual to the class of H-matrices. It will be the class of matrices which are diagonally dominant in the following weaker sense. The class of all matrices $D^{-1}AD$ diagonally similar to A contains a matrix $C = (c_{ik})$ for which

$$|c_{ii}| > |c_{ik}| \quad \text{for all} \quad i, k, i \neq k .$$

These matrices, which we propose to call W-matrices, may be characterized, roughly speaking, by the fact that the operation of taking the Schur product of a matrix and a W-matrix transforms the class of H-matrices into itself. The precise formulation of this characterization is contained in Theorem (3.1) of the present remark. This theorem also contains generalizations of some results of LYNN [4].

1. Definitions and Notation. We shall denote by N the set $\{1, 2, \ldots, n\}$ where n is a fixed positive integer. A matrix is a complex-valued function on $N \times N$ or $N' \times N'$ where $N' \subset N$. The principal submatrix of a matrix A on $N \times N$ whose rows and columns correspond to indices from $N' \subset N$ is denoted by $A(N')$. Given a vector $a_1, \ldots, a_n$, we shall denote by diag $\{a_1, \ldots, a_n\}$ the diagonal matrix with a_i on the main diagonal.

Throughout the present paper we shall be using the properties of two classes of matrices, $\mathbf{K}$ and $\mathbf{K}_0$. We intend to list now the most important properties of these two classes; for the proofs, the reader is referred to [2].

The class $\mathbf{K}_0$ is defined as the class of those real-valued matrices $A = (a_{ik})$ which satisfy $a_{ik} \leq 0$ for $i \neq k$ and such that all principal minors of A are nonnegative. The class $\mathbf{K}$ is defined by the postulates that $\mathbf{K} \subset \mathbf{K}_0$ and the principal minors are required to be positive. It may be shown that a matrix $A \in \mathbf{K}_0$ which is nonsingular already belongs to $\mathbf{K}$. It follows from this that there exists, for each matrix $A \in \mathbf{K}_0$ a subset $F \subset N$ such that

1° $M \subset F$ implies $\det A(M) > 0$,

2° $M \supset F$, $M \neq F$ implies $\det A(M) = 0$.

Further, let us recall a well-known fact. Every matrix may be brought to the "irreducible block triangular form". The precise meaning of this statement is as follows: given a matrix B, there exists a permutation matrix P and a decomposition of N, $N = N_1 \cup \ldots \cup N_s$ such that the corresponding block decomposition of $C = PBP^*$ has the following properties:

1° the C_{ii} are irreducible,

2° $C_{ij} = 0$ for $j > i$.

The decomposition $N_1 \cup \ldots \cup N_s$ is uniquely determined. We shall say that a matrix A is generalized triangular if there exists a permutation matrix P such that PAP^{-1} is triangular.

Let us introduce the following abbreviations. With each matrix A we associate two real matrices $M(A)$ and $H(A)$ with elements m_{ik} and h_{ik} defined as follows

$$m_{ik} = |a_{ik}| \quad \text{for all} \quad i, k\,,$$
$$h_{ik} = |a_{ik}| \quad \text{if} \quad i = k \quad \text{and} \quad h_{ik} = -\,|a_{ik}| \quad \text{for} \quad i \neq k\,.$$

(1,1) *Let A and B be two matrices of the same order. Then*

$$M(AB) \geqq H(A)\, M(B)\,.$$

Proof. Denote by C the product AB. Let i and k be given. Then

$$|c_{ik}| = |a_{ii}b_{ik} + \sum_{j \neq i} a_{ij}b_{jk}| \geqq$$
$$\geqq |a_{ii}b_{ik}| - |\sum_{j \neq i} a_{ij}b_{jk}| \geqq |a_{ii}b_{ik}| - \sum_{j \neq i} |a_{ij}b_{jk}| =$$
$$= h_{ii}m_{ik} + \sum_{j \neq i} h_{ij}m_{jk}$$

which completes the proof.

We shall denote by $\mathbf{H}$ the class of all matrices A such that $H(A) \in \mathbf{K}$. Similarly the class $\mathbf{H}_0$ is defined by the postulate $A \in \mathbf{H}_0$ if and only if $H(A) \in \mathbf{K}_0$.

Further, we define $\mathbf{H}_*$ to be the class of those $A \in \mathbf{H}_0$ for which all $a_{ii} \neq 0$.

(1,2) *A matrix A belongs to $\mathbf{H}$ if and only if $B = D^{-1}AD$ satisfies $|b_{ii}| > \sum_{k \neq i} |b_{ik}|$ for each i where D is a suitable diagonal matrix with positive diagonal elements. If $A \in \mathbf{H}$ and D is a diagonal matrix with positive diagonal elements, then $D^{-1}AD \in \mathbf{H}$ as well.*

Proof. An immediate consequence of the properties of the class $\mathbf{K}$.

(1,3) *Suppose that the matrices A and B belong to* $\mathbf{K}$ *and that* $A \geqq B$. *Then*

1° $\det A \geqq \det B$,

2° *if* $\det A = \det B$ *and B is irreducible then* $A = B$.

Proof. The first statement is a well-known property of the class $\mathbf{K}$. A proof may be found in [2]. To prove the second statement, suppose that $A \neq B$. Then there exists a matrix C such that $A \geqq C \geqq B$ and the matrices C and B differ exactly in one entry, $c_{ik} \neq b_{ik}$. Since B is irreducible, we have $B^{-1} > 0$ and, consequently, $\operatorname{adj} B > 0$.

Further

$$\det C = \det B + (c_{ik} - b_{ik}) B_{ik} > \det B$$

since $c_{ik} - b_{ik} > 0$ and $B_{ik} > 0$. Since $A \geqq C$ it follows from the first statement of the present result that $\det A \geqq \det C$. Combining this with the inequality $\det C > > \det B$ just proved, we arrive at a contradiction.

(1,4) *Let $A \geqq B$ and $A, B \in \mathbf{K}$. Assume that B is written in the block form (B_{ij}) with irreducible B_{ii} and $B_{ij} = 0$ for $j > i$. Hence $A_{ij} = 0$ for $j > i$ as well. Then the following two statements are equivalent*:

1° $\det A = \det B$,

2° $A_{ii} = B_{ii}$ *for all i.*

Proof. It suffices to prove the implication 1° → 2° only. To prove that, let us recall that the inequality $A_{ii} \geqq B_{ii}$ implies $\det A_{ii} \geqq \det B_{ii}$ for each i by (1,3). Since $\det A$ and $\det B$ are equal to the products of the $\det A_{ii}$ and $\det B_{ii}$ respectively, the equation $\det A = \det B$ implies $\det A_{ii} = \det B_{ii}$ for each i. By (1,3) it follows that $A_{ii} = B_{ii}$.

2. Preliminary results. In this section, we collect some results which will be used in the proofs of the main theorems.

(2,1) *Let $A = (a_{ik}) \in \mathbf{H}_0$ and suppose that $a_{ii} = 0$ for some i. Then for any indices $k_1, k_2, \ldots, k_s$*

$$a_{ik_1} a_{k_1k_2} \ldots a_{k_{s-1}k_s} a_{k_si} = 0 .$$

Consequently, if S is a subset of N such that $i \in S$ then $\det A(S) = 0$.

Proof. Suppose that $a_{ik_1} a_{k_1k_2} \ldots a_{k_{s-1}k_s} a_{k_si} \neq 0$ for some indices $k_1, \ldots, k_s$ and that there is no shorter product of this type which is different from zero. Then the indices $i, k_1, \ldots, k_s$ are different from each other. Let N' be the set consisting of the indices $i, k_1, \ldots, k_s$. Since $A \in \mathbf{H}_0$, we have

$$0 \leqq \det H(A(N')) = \sum_{\pi} (-1)^{r(\pi)} \left|a_{p_1p_2} a_{p_2p_3} \ldots a_{p_{v_1}p_1}\right| \cdot$$

$$\cdot \left|a_{q_1q_2} a_{q_2q_3} \ldots a_{q_{v_2}q_1}\right| \ldots \left|a_{r_1r_2} a_{r_2r_3} \ldots a_{r_{v_k}r_1}\right|$$

where the sum is extended over all permutations π of indices in N' written as products of cycles

$$(p_1 \ldots p_{v_1})(q_1 \ldots q_{v_2}) \ldots (r_1 \ldots r_{v_k}),$$

$r(\pi)$ being the number of cycles with at least two elements. According to our assumption, all terms in this sum are equal to zero except possibly for those corresponding to exactly one cycle. All these terms are nonpositive since N' contains at least two elements and hence $r(\pi) = 1$. At least one of these terms being different from zero, we have a contradiction.

(2,2) *Let G be a finite directed graph (without loops and multiple edges) with vertices $1, 2, \ldots, n$, every edge $\overrightarrow{ik}$ of which is labelled with a real number p_{ik}. Then*

1° *p is subpotential, i.e. for some fixed real numbers $c_1, \ldots, c_n$ holds*

$$p_{ik} \leqq c_i - c_k$$

for each edge $\overrightarrow{ij} \in G$, if and only if the sum $\sum_{ik \in C} p_{ik} \leqq 0$ whenever the sum is extended over a cycle C of G.

2° *p is strongly subpotential, i.e. for some fixed real numbers $c_1, \ldots, c_n$ holds*

$$p_{ik} < c_i - c_k$$

for each edge $\overrightarrow{ik} \in G$, if and only if the sum $\sum_{ik \in C} p_{ik} < 0$ over any cycle $C \subset G$.

Proof. It suffices to prove the "if" part, the "only if" being obvious. Let us show this in the case 1°, first.

In G, there exists (C. Berge [1]) a basis B, i.e. a subset of the set of all vertices of G with the following two properties:

(i) if $i, j \in B$, $i \neq j$, then there exists no (directed) path in G from i to j;

(ii) whenever k is a vertex of G then there exists a vertex $i \in B$ such that there is a (directed) path from i to k in G.

Define for $k = 1, \ldots, n$

$$c_k = -\max_{P(B,k)} \sum_{\overrightarrow{rs} \in P(B,k)} p_{rs} \tag{*}$$

where $P(B, k)$ denotes a general (directed) path beginning in any vertex in B and ending in k. Paths with just one vertex are here included, the corresponding sum being zero. Since a general path with repeating vertices can be decomposed into cycles and a path without repeating vertices, it follows from the fact that the sum of p_{rs} on each cycle is nonpositive that the maximum in (*) exists. Let now $\overrightarrow{ik}$ be an edge in G. Then, if we join $\overrightarrow{ik}$ to any path $P(B, i)$ from B to i, we obtain a path from B to k.

Hence

$$\max_{P(B,k)} \sum_{\overrightarrow{rs} \in P(B,k)} p_{r,s} \geqq \max_{P(B,i)} \sum_{\overrightarrow{rs} \in P(B,i)} p_{rs} + p_{ik}$$

so that

$$p_{ik} \leqq c_i - c_k .$$

It remains to prove the analogous part in 2°. In C there is a finite number of cycles with distinct vertices. Thus, there exists a number $\varepsilon > 0$ such that the function $p_{ik} + \varepsilon$ still satisfies $\sum_{\overrightarrow{ik} \in G} (p_{ik} + \varepsilon) \leqq 0$ on every cycle C of G.

According to 1°, there exist real numbers $c_1, \ldots, c_n$ such that

$$p_{ik} + \varepsilon \leqq c_i - c_k$$

for each edge $\overrightarrow{ik} \in G$. Hence

$$p_{ik} < c_i - c_k$$

for each edge $\overrightarrow{ik} \in G$. The proof is complete.

3. Weakly diagonally dominant matrices. In this section, we intend to prove the equivalence of a certain number of properties of a matrix which are, in a certain sense, dual to the properties of the class $\boldsymbol{H}$.

(3,1) Theorem. *Let $n \geqq 2$. The following properties of a matrix $B = (b_{ik})$ are equivalent:*

1° *for any indices $k_1, \ldots, k_s$ different from each other*

$$|b_{k_1k_2} b_{k_2k_3} \ldots b_{k_{s-1}k_s} b_{k_sk_1}| < |b_{k_1k_1} b_{k_2k_2} \ldots b_{k_sk_s}|$$

2° *there exists a diagonal matrix D with positive diagonal elements such that the matrix $C = D^{-1}BD$ satisfies $|c_{ii}| > |c_{ik}|$ for all $i, k, i \neq k$.*

3° *whenever $A \in \boldsymbol{H}_*$ then $A \circ B \in \boldsymbol{H}$ and*

$$\det H(A \circ B) \geqq |b_{11}| \ldots |b_{nn}| \det H(A) ,$$

equality being attained (if and) only if A is generalized triangular.

4° *whenever $A \in \boldsymbol{H}$ then*

$$\det H(A \circ B) \geqq |b_{11}| \ldots |b_{nn}| \det H(A) ,$$

equality being attained (if and) only if A is generalized triangular.

5° *for any $A \in \boldsymbol{H}_*$ the matrix $A \circ B$ belongs to $\boldsymbol{H}$.*

6° *for any $A \in \boldsymbol{H}_*$ the matrix $A \circ B$ is nonsingular.*

7° *for any $A \in \boldsymbol{H}_*$ the determinant* $\det H(A \circ B) > 0$.

Proof. We intend to prove the following cycle of implications $1° \to 2° \to 3° \to \to 4° \to 5° \to 6° \to 7° \to 1°$.

$1° \to 2°$. Let $1°$ be satisfied. Denote by G the labelled graph with vertices $1, 2, \ldots, n$ and those edges $\vec{ik}(i \neq k)$ for which $b_{ik} \neq 0$, labelled with $p_{ik} = \log(|b_{ik}|/|b_{ii}|)$. This is possible since $|b_{ii}| \neq 0$ for all i. From $1°$ it follows that the function p on G is strongly subpotential, i.e. according to $2°$ of lemma (2,2), there exist real numbers $c_1, \ldots, c_n$ such that

$$p_{ik} < c_i - c_k$$

for each edge $\vec{ik} \in G$.

Put $d_i = \exp c_i$. Then $d_i > 0$ and $|b_{ik}|/|b_{ii}| = \exp p_{ik} < \exp(c_i - c_k) = d_i/d_k$. Hence

$$d_i^{-1}|b_{ii}|\, d_i > d_i^{-1}|b_{ik}|\, d_k \quad \text{for} \quad \vec{ik} \in G\,;$$

this is true for any $i, k, i \neq k$.

If we denote $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ and $C = D^{-1}BD$, the last inequality implies $|c_{ii}| > |c_{ik}|$ for all $i, k, i \neq k$.

$2° \to 3°$. Suppose there exists a diagonal matrix D with positive diagonal elements such that the matrix $C = D^{-1}BD$ satisfies $|c_{ii}| > |c_{ik}|$ for all $i \neq k$. Let $A \in \boldsymbol{H}_0$ and suppose that $|a_{ii}| > 0$ for all i. To prove that $A \circ B \in \boldsymbol{H}$ it suffices to show that $D_1^{-1}(A \circ B)\, D_2 \in \boldsymbol{H}$ for some diagonal matrices D_1 and D_2 with positive diagonal elements. There exists an $\alpha > 0$ such that

$$|c_{ii}| \geqq (1 + \alpha)\, |c_{ik}| \quad \text{for all} \quad i, k, i \neq k\,.$$

For any ε, $0 \leqq \varepsilon \leqq \alpha$, put

$$W(\varepsilon) = \operatorname{diag}\left\{\frac{1 + \varepsilon}{|c_{11}|}, \ldots, \frac{1 + \varepsilon}{|c_{nn}|}\right\}.$$

We have

$$W(\varepsilon)\, D^{-1}(A \circ B)\, D = A \circ W(\varepsilon)\, D^{-1}BD = A \circ W(\varepsilon)\, C\,,$$

whence

$$H(W(\varepsilon)\, D^{-1}(A \circ B)\, D) = H(A) \circ M(W(\varepsilon)\, C) \geqq H(A + \varepsilon \operatorname{diag}(|c_{11}| \ldots |c_{nn}|))$$

so that $H(W(\varepsilon)\, D^{-1}(A \circ B)\, D)$ belongs to $\boldsymbol{K}$ for any $\varepsilon > 0$. Hence $A \circ B \in \boldsymbol{H}$. It remains to prove the inequality for the determinants: Especially, for $\varepsilon = 0$, we obtain the inequality

$$H(W(0)\, D^{-1}(A \circ B)\, D) \geqq H(A)\,.$$

Let us distinguish two cases. If $\det H(A) = 0$, there is nothing to prove since $\det H(A \circ B) > 0$. The inequality $\det H(A) > 0$, together with $H(A) \in \boldsymbol{K}_0$, implies that $H(A) \in \mathbf{K}$. By (1,3) we obtain

$$\det H(W(0)\, D^{-1}(A \circ B)\, D) \geqq \det H(A)$$

whence

$$\det H(A \circ B) = \frac{\det H(W(0)\, D^{-1}(A \circ B)\, D)}{\det W(0)} \geqq \frac{\det H(A)}{\det W(0)} = |b_{11}| \dots |b_{nn}| \det H(A)\,.$$

Since $\det H(A \circ B) > 0$, equality can be attained only in the case that $\det H(A) > 0$. In this case we have by (1,4) that equality holds if and only if A is generalized triangular.

The implication $3° \to 4°$ is formal.

$4° \to 5°$. Suppose that 4° is satisfied and let us show first that all b_{ii} are different from zero. Without loss of generality, let us assume that $b_{11} = 0$. Choose $A = (a_{ik})$, $a_{ii} = 1$ for $i \in N$, $a_{12} = a_{21} = -1$. Since $A \in \mathbf{K}_0 \subset \mathbf{H}_0$ and A is not generalized triangular

$$-|b_{12}|\, |b_{21}|\, |b_{33}| \dots |b_{nn}| = \det H(A \circ B) > |b_{11}| \dots |b_{nn}| \det H(A) = 0$$

which is a contradiction.

Now let $A \in \mathbf{H}_0$ and suppose that $|a_{11}| > 0$ for all i. Let S be a subset of the set N. Let us define a matrix $\tilde{A}$ by the following requirements

$\tilde{a}_{ik} = a_{ik}$ if both i and k belong to S,
$\tilde{a}_{ik} = 0$ if $i \neq k$ and at least one of them does not belong to S,
$\tilde{a}_{ii} = 1$ for all i not in S.

It follows from 4° that $\det H(\tilde{A} \circ B) \geqq 0$. Suppose that $\det H(\tilde{A} \circ B) = 0$. Then we must have equality in

$$\det H(\tilde{A} \circ B) \geqq |b_{11}| \dots |b_{nn}| \det H(\tilde{A})\,.$$

It follows from 4° that $\tilde{A}$ is generalized triangular whence

$$\det H(\tilde{A} \circ B) = \prod_{i \in S} |a_{ii}| \prod_{j \in N} |b_{jj}|\ ;$$

this is a contradiction since all a_{ii} and b_{jj} are different from zero. It follows that $\det H(\tilde{A} \circ B) > 0$. If we denote by V the matrix $H(A \circ B)$, we have

$$\det V(S) \prod_{i \in N - S} |b_{jj}| = \det H(\tilde{A} \circ B) > 0$$

so that $\det V(S)$ is positive for any S. This proves that $A \circ B \in \mathbf{H}$.

The implication $5° \to 6°$ is immediate.

$6° \to 7°$. Suppose that 6° is satisfied and let us show first that this implies $b_{ii} \neq 0$ for all i. To see that, it suffices to take $A = E$. Suppose now that $\det H(A \circ B) \leqq 0$ for some $A \in \mathbf{H}_0$ with $|a_{ii}| > 0$. Put $Q = \operatorname{diag}\{a_{11}, \dots, a_{nn}\}$. Since $\det H(Q \circ B) > 0$, there exists a real α, $0 \leqq \alpha < 1$, such that $\det H((A + \alpha Q) \circ B) = 0$. Now $A + \alpha Q \in \boldsymbol{H}_0$ since $H(A + \alpha Q) = H(A) + \alpha H(Q)$. The diagonal elements of $A + \alpha Q$ being $(1 + \alpha) a_{ii} \neq 0$, we have a contradiction.

$7° \to 1°$. Since $E \circ B$ has to be nonsingular by 6°, the diagonal elements b_{ii} have to be $\neq 0$. Let $k_1, \ldots, k_s$ be a sequence of indices different from each other. Denote by C the matrix with elements c_{ik} defined by

$c_{ik} = 1$ if $i = k_j$, $k = k_{j+1}$ for some $j = 1, \ldots, s - 1$ or $i = k_s$, $k = k_1$,
$c_{ik} = 0$ otherwise.

Since $E - C \in \boldsymbol{H}_0$ and has positive diagonal elements, $\det H((E - C) \circ B) > 0$ whence

$$|b_{11}| |b_{22}| \ldots |b_{nn}| - \prod_{j \notin \{k_1, \ldots, k_s\}} |b_{jj}| (|b_{k_1k_2}| |b_{k_2k_3}| \ldots |b_{k_{s-1}k_s}| |b_{k_sk_1}|) > 0 .$$

This proves 1°. The proof is complete.

(3,2) Definition. Let $n \geqq 2$. We shall denote by $\boldsymbol{W}$ the class of all square matrices which satisfy one (and hence all) of the conditions of the preceding theorem; a matrix consisting of a single element belongs to $\boldsymbol{W}$ if and only if this element is different from zero.

(3,3) *If $A \in \boldsymbol{W}$ then the transpose of A belongs to $\boldsymbol{W}$ as well.*

Proof. Condition 1° of (3,1) does not change if we replace A by A^T.

(3,4) *If A is positive definite, then $A \in \boldsymbol{W}$.*

Proof. If $k_1 \neq k_2$, we have

$$|a_{k_1k_2}| < |a_{k_1k_1}|^{1/2} |a_{k_2k_2}|^{1/2} .$$

This immediately implies condition 1° of (3,1).

(3,5) If $A \in \boldsymbol{H}$ then A^{-1} exists and $A^{-1} \in \boldsymbol{W}$.

Proof. The existence of A^{-1} is a well-known property of the class $\boldsymbol{H}$. Since $A \in \boldsymbol{H}$ there exists a diagonal matrix D with positive diagonal elements such that $B = D^{-1}AD$ satisfies $|b_{ii}| > \sum_{k \neq i} |b_{ik}|$, $i \in N$. Since $A^{-1} = DB^{-1}D^{-1}$ it suffices by (3,1) to prove that $B^{-1} \in \boldsymbol{W}$. By (3,3), it suffices to prove that the transpose of B^{-1} belongs to $\boldsymbol{W}$. We intend to show that the elements β_{ik} of B^{-1} satisfy the inequalities $|\beta_{ii}| > |\beta_{ji}|$ whenever $i \neq j$, $i, j = 1, 2, \ldots, n$. To prove this, take a fixed index i and suppose that there exists a j different from i such that $|\beta_{ji}| \geqq |\beta_{si}|$ for all s. It follows that $|\beta_{ji}| > 0$. By (1,1), we have

$$E = M(E) = M(BB^{-1}) \geqq H(B) M(B^{-1}) .$$

Upon writing down the (j, i)-th elements of E and $H(B)\,M(B^{-1})$, we obtain the inequality

$$0 \geqq |b_{jj}|\,|\beta_{ji}| - \sum_{s \neq j} |b_{js}|\,|\beta_{si}|\,,$$

whence

$$|b_{jj}|\,|\beta_{ji}| \leqq \sum_{s \neq j} |b_{js}|\,|\beta_{si}| \leqq |\beta_{ji}| \sum_{s \neq j} |b_{js}| < |b_{jj}|\,|\beta_{ji}|\,.$$

This contradiction proves the theorem.

4. In this section we intend to study two classes of matrices containing the class **W** which are obtained by loosening some of the postulates: roughly speaking, they are obtained by replacing strict inequalities by $\geqq$.

(4,1) Theorem. *Let $n \geqq 2$. Then the following properties of an $n \times n$ matrix $B = (b_{ik})$ are equivalent:*

1° *For any indices $k_1, \ldots, k_s$*

$$|b_{k_1k_2}b_{k_2k_3} \ldots b_{k_{s-1}k_s}b_{k_sk_1}| \leqq |b_{k_1k_1}b_{k_2k_2} \ldots b_{k_sk_s}| \neq 0\,;$$

2° *there exists a diagonal matrix D with positive diagonal elements such that the matrix $C = D^{-1}BD$ satisfies $0 \neq |c_{ii}| \geqq |c_{ik}|$ for all $i, k \in N$;*

3° *whenever $A \in$ **H** then $A \circ B \in$ **H** and*

$$\det H(A \circ B) \geqq |b_{11}| \ldots |b_{nn}| \det H(A)\,;$$

4° *whenever $A \in$ **H** then*

$$\det H(A \circ B) \geqq |b_{11}| \ldots |b_{nn}| \det H(A) \neq 0\,;$$

5° *whenever $A \in$ **H** then $A \circ B \in$ **H**;*

6° *whenever $A \in$ **H** then $A \circ B$ is nonsingular;*

7° *whenever $A \in$ **H** then* $\det H(A \circ B) > 0$;

8° *whenever $C \in$ **W** then $B \circ C \in$ **W**;*

9° *$B \circ (U + \varepsilon E) \in$ **W** for all $\varepsilon > 0$ where $u_{ik} = 1$ for all i, k;*

10° *$b_{ii} \neq 0$ for each i and whenever $A \in$ $\mathbf{H}_0$ then* $\det H(A \circ B) \geqq 0$;

11° *whenever $A \in \mathbf{H}_*$ then $A \circ B \in \mathbf{H}_*$.*

Proof. 1° → 2°. Let 1° be fulfilled. Let G be the labelled graph with vertices $1, 2, \ldots$ $\ldots, n$ and those edges $\vec{ik}$, $i \neq k$, for which $b_{ik} \neq 0$, labelled with $p_{ik} = \log(|b_{ik}|/|b_{ii}|)$. From 1° it follows that the function p on G is subpotential, i.e. according to 1° of lemma (2,2) there exist real numbers $c_1, \ldots, c_n$ such that

$$p_{ik} \leqq c_i - c_k$$

for each edge $\vec{ik} \in G$.

If we put $d_i = \exp c_i$, we have $d_i > 0$ and

$$|b_{ik}|/|b_{ii}| = \exp p_{ik} \leqq d_i/d_k .$$

Hence

$$d_i^{-1}|b_{ii}|\, d_i \geqq d_i^{-1}|b_{ik}|\, d_k$$

for $\vec{ik} \in G$; but this holds for any i, k.

It follows that $C = D^{-1}BD$ where $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ has the property

$$0 \neq |c_{ii}| \geqq |c_{ik}|$$

for all $i, k \in N$, $i \neq k$.

$2° \to 3°$. Let $C = D^{-1}BD$ satisfy $0 \neq |c_{ii}| \geqq |c_{ik}|$ for all $i, k \in N$. Then $|b_{ii}| \neq 0$. Let $A = (a_{ik})$ belong to $\mathbf{H}$. Define $W = \operatorname{diag}\{1/|c_{11}|, \ldots, 1|/c_{nn}|\}$ so that the diagonal elements of WC are equal to 1 in modulus and the off-diagonal elements less or equal to 1 in modulus.

We have

$$WD^{-1}(A \circ B)\, D = A \circ WD^{-1}BD = A \circ WC$$

whence

$$H(WD^{-1}(A \circ B)\, D) = H(A \circ WC) = H(A) \circ M(WC) \geqq H(A) .$$

It follows that $H(WD^{-1}(A \circ B)\, D)$ belongs to $\mathbf{K}$ and therefore $A \circ B \in \mathbf{H}$. By (1,3),

$$\det H(WD^{-1}(A \circ B)\, D) \geqq \det H(A)$$

so that

$$\det H(A \circ B) = \frac{\det H(WD^{-1}(A \circ B)\, D)}{\det W} \geqq \frac{\det H(A)}{\det W} = |b_{11}| \ldots |b_{nn}| \det H(A) .$$

$3° \to 4°$ being immediate, let us show that

$4° \to 5°$: Observe that 4° implies that all b_{ii} are different from zero. Let thus $A \in \mathbf{H}$ so that $|a_{ii}| > 0$ for all i. Let S be a non-void subset of the set N. Define a matrix $\tilde{A} = (\tilde{a}_{ik})$ by

$\tilde{a}_{ik} = a_{ik}$ if both i and k belong to S;
$\tilde{a}_{ik} = 0$ if $i \neq k$ and at least one of them does not belong to S;
$\tilde{a}_{ii} = 1$ for all i not in S.

Since $\tilde{A} \in \mathbf{H}$, it follows from 4° that $\det H(\tilde{A} \circ B) > 0$. If we denote by V the matrix $H(A \circ B)$, we have

$$(\det V(S)) \prod_{j \in N - S} |b_{jj}| = \det H(\tilde{A} \circ B) > 0$$

so that $\det V(S)$ is positive for any S. This proves that $A \circ B \in \mathbf{H}$.

The implication $5° \to 6°$ is immediate.

$6° \to 7°$. Let $6°$ be satisfied. If we choose $A = E$ we obtain that $b_{ii} \neq 0$ for all i. Suppose now that $\det H(A \circ B) < 0$ for some $A \in \mathbf{H}$.

Put $Q = \operatorname{diag}\{a_{11}, \ldots, a_{nn}\}$. Since $\det H(Q \circ B) > 0$, there exists a real α, $0 < \alpha < 1$, such that $\det H((A + \alpha Q) \circ B) = 0$. This is a contradiction since $A + \alpha Q \in \mathbf{H}$.

$7° \to 8°$. Let us suppose that B satisfies $7°$ and let C be a matrix from $\mathbf{W}$. This means according to $5°$ of Theorem (3,1) that for any matrix $A \in \mathbf{H}_*$ the matrix $A \circ C$ belongs to $\mathbf{H}$ whence $\det H((A \circ C) \circ B) > 0$ according to $7°$. But $(A \circ C) \circ B = A \circ (B \circ C)$ so that, according to $7°$ of Theorem (3,1), we obtain that $B \circ C \in \mathbf{W}$.

Since $8° \to 9°$ follows easily from the fact that $U + \varepsilon E \in \mathbf{W}$ for $\varepsilon > 0$, let us show that $9° \to 10°$. First, $b_{ii}(1 + \varepsilon) \neq 0$ for all i and all $\varepsilon > 0$ so that we have $b_{ii} \neq 0$. Let now $A \in \mathbf{H}_0$. If $a_{ii} = 0$ for some i, $\det H(A \circ B) = 0$ according to (2,1). Thus, assume that $a_{ii} \neq 0$ for all i. According to $7°$ of Theorem (3,1), $\det H(A \circ (B \circ (U + \varepsilon E))) > 0$ for all $\varepsilon > 0$ so that $\det H(A \circ B) \geqq 0$.

$10° \to 11°$. Let $A \in \mathbf{H}^*$ and denote $C = A \circ B$. Since $b_{ii} \neq 0$ for each i, we have $c_{ii} \neq 0$ as well. Let $M \subset N$. Define a matrix Q in the following manner: if both i, k belong to M then $q_{ik} = a_{ik}$, if $i \in N - M$ then $q_{ii} = 1$, $q_{ik} = 0$ for all other pairs of indices. Hence $Q \in \mathbf{H}_0$. According to $10°$ it follows $\det H(Q \circ B) \geqq 0$ so that

$$0 \leqq \det H(Q \circ B) = \det H(C(M)) \,.\, p$$

where p is the product of the elements $|b_{ii}|$ for $i \in N - M$. Since $H(C(M)) = H(C)(M)$, it follows that $H(C) \in \mathbf{K}_0$ so that $C \in \mathbf{H}_*$.

It remains to prove the implication $11° \to 1°$. It is easy to see that it suffices to prove

$$|b_{k_1k_2}b_{k_2k_3} \ldots b_{k_sk_1}| \leqq |b_{k_1k_1}b_{k_2k_2} \ldots b_{k_sk_s}|$$

for any mutually disjoint indices $k_1, \ldots, k_s$ since $b_{ii} \neq 0$ and these inequalities imply the remaining ones.

Denote by C the matrix (c_{ij}) with $c_{k_1k_2} = c_{k_2k_3} = \ldots = c_{k_sk_1} = 1$, $c_{ij} = 0$ otherwise. The matrix $\tilde{A} = E - C$ belongs to $\mathbf{H}_*$ so that $\tilde{A} \circ B \in \mathbf{H}_*$ as well. Hence the principal minor of $H(\tilde{A} \circ B)$ with indices $k_1, \ldots, k_s$ is nonnegative. But an easy computation shows that this minor is equal to

$$|b_{k_1k_1}|\,|b_{k_2k_2}| \ldots |b_{k_sk_s}| - |b_{k_1k_2}|\,|b_{k_2k_3}| \ldots |b_{k_sk_1}| \,.$$

This implies the required inequality. The proof is complete.

(4,2) Definition. Let $n \geqq 2$. We shall denote by $\mathbf{W}_*$ the class of all n by n matrices which satisfy one (and hence all) of the conditions of the preceding theorem. A matrix consisting of a single element belongs to $\mathbf{W}_*$ if and only if this element is different from zero.

(4,3) Theorem. *Let $n \geqq 2$. Then the following properties of an n by n matrix $B = (b_{ik})$ are equivalent:*

1° *For any indices $k_1, \ldots, k_s$*

$$|b_{k_1k_2}b_{k_2k_3} \ldots b_{k_{s-1}k_s}b_{k_sk_1}| \leqq |b_{k_1k_1}b_{k_2k_2} \ldots b_{k_sk_s}| ;$$

2° *whenever D is a diagonal matrix with positive diagonal elements then $M(B) + D \in \mathbf{W}$;*

3° *whenever $\varepsilon > 0$ then $M(B) + \varepsilon E \in \mathbf{W}$;*

4° *whenever $A \in \mathbf{H}_0$ then $A \circ B \in \mathbf{H}_0$ and*

$$\det H(A(S) \circ B(S)) \geqq \prod_{i \in S} |b_{ii}| \det H(A(S))$$

where S is the set of all indices i for which $b_{ii} = 0$;

5° *if $b_{ii} = 0$ then $b_{ik_1}b_{k_1k_2} \ldots b_{k_si} \neq 0$ for every nonvoid set of indices $k_1, \ldots, k_s$ and, for each $A \in \mathbf{H}_0$*

$$\det H(A(S) \circ B(S)) \geqq \prod_{i \in S} |b_{ii}| \det H(A(S))$$

where S is the set of all indices i for which $b_{ii} \neq 0$;

6° *whenever $A \in \mathbf{H}_0$ then $A \circ B \in \mathbf{H}_0$;*

7° *if S is the set of all indices i for which $b_{ii} \neq 0$ then $B(S) \in \mathbf{W}_*$ and $b_{ii} = 0$ implies*

$$b_{ik_1}\, b_{k_1k_2} \ldots b_{k_si} = 0$$

for every nonvoid set of indices $k_1, \ldots, k_s$.

Proof. 1° → 2°. Let $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ where $d_1, \ldots, d_n$ are positive numbers. Let $k_1, \ldots, k_s$ be indices different from each other. Since

$$|b_{k_1k_2}|\,|b_{k_2k_3}| \ldots |b_{k_{s-1}k_s}|\,|b_{k_sk_1}| \leqq |b_{k_1k_1}|\,|b_{k_2k_2}| \ldots |b_{k_sk_s}| ,$$

it follows that

$$|b_{k_1k_2}|\,|b_{k_2k_3}| \ldots |b_{k_sk_1}| < (|b_{k_1k_1}| + d_1)(|b_{k_2k_2} + d_2) \ldots (|b_{k_sk_s}| + d_s)$$

whence $M(B) + D \in \mathbf{W}$ according to 1° of Theorem (3,1).

The implication 2° → 3° is immediate. To prove 3° → 4°, choose $A \in \mathbf{H}_0$ and denote by S_1 the set of all indices for which $|a_{ii}| > 0$. Since $C_\varepsilon = M(B) + \varepsilon E \in \mathbf{W}$ for any $\varepsilon > 0$, it follows from 3° of (3,1) that $A(S_1) \circ C_\varepsilon(S_1) \in \mathbf{H}$ whence $A(S_1) \circ C_0(S_1) \in \mathbf{H}_0$. Hence $A(S_1) \circ B(S_1) \in \mathbf{H}_0$. Moreover, if $\tilde{S}$ is any subset of N which is not contained in S_1 then an easy application of (2,1) shows that $\det(A(\tilde{S}) \circ B(\tilde{S})) = 0$. Hence $A \circ B \in \mathbf{H}_0$.

If S is not contained in S_1, the inequality $\det H(A(S) \circ B(S)) \geqq \Pi |b_{ii}| \det H(A(S))$ is fulfilled in a trivial manner since both sides are equal to zero. Let now $S \subset S_1$. Since $A(S) \in \mathbf{H}_0$ and $M(B(S) + \varepsilon E(S)) \in \mathbf{W}$ for any $\varepsilon > 0$, we have according to 4° of Theorem (3,1)

$$\det H(A(S) \circ (M(B(S) + \varepsilon E(S))) \geqq \prod_{i \in S} (|b_{ii}| + \varepsilon) \det H(A(S))$$

so that

$$\det H(A(S) \circ B(S)) \geqq \prod_{i \in S} |b_{ii}| \det H(A(S)) .$$

4° → 5°. The first part follows from (2,1) and from the fact that there exists a matrix A in $\mathbf{H}_0$ with all elements different from zero. The second part is an immediate consequence of 4°.

5° → 6°. Let $A \in \mathbf{H}_0$ and denote $V = H(A \circ B)$. We have to show that for any non-void subset $S_1 \subset N = \{1, \ldots, n\}$ the inequality $\det V(S_1) \geqq 0$ is satisfied.

Let first $S_1 \not\subset S$, $i \in S_1$, $i \notin S$, so that $b_{ii} = 0$.
Then, any term of $\det V(S_1)$ is of the form

$$(-1)^{\sigma} |a_{ik_1}| |a_{k_1k_2}| \ldots |a_{k_si}| |b_{ik_1}| |b_{k_1k_2}| \ldots |b_{k_si}| \prod_{(r,s)}' |a_{rs}| \prod_{(r,s)}' |b_{rs}|$$

where Π' contains the remaining products. By 5°, $\det V(S_1) = 0$.

On the other hand, $B(S) \in \mathbf{W}_*$ according to 4° of Theorem (4,1). It follows easily from 10° in Theorem (4,1) that if $S_1 \subset S$ and $A \in \mathbf{H}_0$ then $\det V(S_1) \geqq 0$. This proves 6°.

6° → 7°. Observe first that 6° implies that whenever $A(S) \in \mathbf{H}_0$ then $A(S) \circ B(S) \in \mathbf{H}_0$. Since $b_{ii} \neq 0$ for $i \in S$, it follows from 10° of Theorem (4,1) that $B(S) \in \mathbf{W}_*$. Since there exists a matrix $A \in \mathbf{H}_0$ whose all elements are different from zero, the second part in 7° follows immediately from (2,1).

To prove the implication 7° → 1°, let us distinguish two cases. If in 1° at least one of the indices $k_1, \ldots, k_s$ does not belong to S, both sides in the inequality in 1° are equal to zero. If all indices $k_1, \ldots, k_s$ in 1° belong to S, the inequality in 1° follows from 1° in Theorem (4,1) since $B(S) \in \mathbf{W}_*$.

The proof is complete.

(4,4) Definition. We shall denote by $\mathbf{W}_0$ the class of all square matrices which satisfy one (and hence all) of the conditions of Theorem (4,3). All matrices consisting of a single element belong also to $\mathbf{W}_0$.

(4,5) *If A is positive semidefinite, then $A \in \mathbf{W}_0$.*

Proof. Analogous to the proof of (3,4).

Let us conclude with some remarks.

Since evidently $\boldsymbol{H} \subset \boldsymbol{W} \subset \boldsymbol{W}_*$, 5° of Theorem (4,1) generalizes a theorem of M. S., Lynn [4] which stated that $A \in \boldsymbol{H}$, $B \in \boldsymbol{H}$, implies $A \circ B \in \boldsymbol{H}$. In the same paper, Lynn has proved that then, in our notation,

$$\det H(A \circ B) \geqq \max \{\prod_i |b_{ii}| \det H(A), \prod_i |a_{ii}| \det H(B)\} .$$

Koteljanskij [3] has proved that if $C \in \boldsymbol{H}$ then $|\det C| \geqq \det H(C)$. This, together with 4° of Theorem (3,1), weakens the assumptions in one part of Lynn's inequality: If $A \in \boldsymbol{H}$, $B \in \boldsymbol{W}_*$ then

$$|\det (A \circ B)| \geqq \prod_i |b_{ii}| \det H(A) .$$

The proof of lemma (2,1) is analogous to the proof of a related result of A. Ostrowski [5], Satz 9.

Further, the classes $\boldsymbol{W}$, $\boldsymbol{W}_*$ and $\boldsymbol{W}_0$ satisfy several inclusions which are easy to prove such as

$$\boldsymbol{H} \subset \boldsymbol{W} \subset \boldsymbol{W}_* \subset \boldsymbol{W}_0 , \quad \boldsymbol{H}_0 \subset \boldsymbol{W}_0 , \quad \boldsymbol{W} \circ \boldsymbol{W}_* = \boldsymbol{W} , \quad \boldsymbol{W} \circ \boldsymbol{W} = \boldsymbol{W}$$

and similar obvious properties.

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