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POLARITY FOR A SIMPLEX[1])

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The main purpose of this paper is to relate the first polar, for a simplex S in an n-space, of a point with a locus (Theorems 10, 11, 14, 15) and that of a hyperplane with an envelope (Theorems 8, 9, 13) associated with a pair of harmonic systems of quadrics respectively inscribed to an S-configuration ((S – C) and circumscribed to its dual or reciprocal (R.S – C), S being self polar for quadrics of both systems (Theorem 12) as the common diagonal simplex of the (S – C) and (R.S – C). Their association with isotomic and isogonal transformations too is observed. The relation of S with the polar quadric of a point as well as of a hyperplane for S is also considered (Theorems 2, 3).

1. INTRODUCTION

Let S be a simplex in a projective space of n dimensions, or briefly in an n-space, denoted by $[n]$.

We may consider S to be a *degenerate primal of order* $n + 1$ represented by the joint equation $x_0 \dots x_n = 0$ of its $n + 1$ primes $a_i \equiv x_i = 0$ or *of class* $n + 1$ represented tangentially by the joint equation $u_0 \dots u_n = 0$ of its $n + 1$ vertices $A_i \equiv u_i = 0$. The *first polar* ([18], p. 11) of a point $Z(Z_0, \dots, Z_n)$ for S is then found to be the primal $\sum(Z_i/x_i) = 0$, of order n, circumscribed to S; ...; *the* $(n - 1)$*th polar* is the quadric $(Z) \equiv \sum(x_i x_j/Z_i Z_j) = 0$ $(i \neq j)$, circumscribed to S, called the *polar quadric of* Z *for* S; the *nth polar* is the hyperplane $z \equiv \sum(x_i/Z_i) = 0$, called the *polar of* Z *for* S. The $n + 1$ *tangential coordinates* of z are then $u_i = 1/Z_i$.

Tangentially, the first polar (cf. [17], p. 118) of z for S may be defined to be similarly the primal $\sum(z_i/u_i) = 0$, of class n, inscribed to S; ...; the $(n - 1)$th polar as its polar quadric $(z) \equiv \sum(u_i u_j/z_i z_j) = 0$, inscribed to S; the nth polar as its *pole* $Z \equiv \sum(u_i/z_i) = 0$.

As an immediate consequence we have the following

[1]) Proceedings of the 48th Session of the Indian Science Congress Association held at Roorkee in the first week of January 1961, p. 8.

Theorem 1. *The polar z of Z or the pole Z of z for a simplex S is the same for its polar primals, w.r.t. S, of all orders, in particular for its first polar as well as for its polar quadric* (cf. [17], p. 51).

2. POLAR QUADRIC

The tangent hyperplane of the polar quadric (Z), of the point Z (§ 1) for the simplex S, at its vertex A_j is given by

$$\sum(x_i/Z_i) - x_j/Z_j = 0 .$$

It is evidently coaxal with the prime $x_j = 0$ and the polar prime $\sum(x_i/Z_i) = 0$ of Z for S or (Z).

Again the point of contact of the polar quadric (z), of the hyperplane z (§ 1) for the simplex S, with its prime a_j is given by

$$\sum(u_i/z_i) - u_j/z_j = 0 .$$

It is obviously collinear with the vertex $u_j = 0$ of S and the pole $\sum(u_i/z_i) = 0$ of z for S, or (z). Thus we have the following

Theorem 2. *The polar reciprocal S'' of a simplex S in $[n]$ w.r.t. the polar quadric (Z) of a point Z for S is the "anticevian simplex" of S for Z, and that, say S', w.r.t. the polar quadric (z) of a hyperplane z for S is the "cevian simplex" of S for z. If Z, z be the pole and polar for S, they are also pole and polar for both (Z) and (z) as the centre and hyperplane of perspectivity of the 3 simplexes S, S', S''. Hence the vertices of S and Z form a "self-conjugate" $(n + 2)$ad of points for both (Z) and (z) such that the join of any two of them is conjugate to the hyperplane of the other n points for both (Z) and (z). Dually the primes of S and z form a self-conjugate $(n + 2)$ad of hyperplanes for both (z) and (Z) such that the polar hyperplanes for (z) and (Z), of the point common to any n of them pass through the common $[n - 2]$ of the other two hyperplanes of the $(n - 2)$ad.* (cf. [6], p. 97, ex. 5; 11; 13).

3. ISODYNAMIC AND ISOGONIC SIMPLEXES

We may define an *isodynamic simplex S* as one whose solid faces all form *isodynamic tetrahedra* such that the tangent hyperplanes of its circumhypersphere (S) at its vertices form a simplex S'' perspective with S. Consequently S'' is said to be *isogonic* such that the simplex S formed of the points of contact of its inscribed hypersphere (S) is perspective with it. The centre L of perspectivity of S, S'' may be then referred to as the *Lemoine point* of S, *Gergonne point* following COXETER ([7])

or *Fermat point* ([15]) of S'', and the hyperplane l of their perspectivity as the *Lemoine prime* of S in analogy with those of such tetrahedra ([5]; [14]). Now follows from Theorem 2 the following

Theorem 3. *The polar quadric of a point* L (*hyperplane* l) *for a simplex* S *is a hypersphere* (S), *if and only if* S *is isodynamic* (*isogonic*) *such that* L, l *are pole and polar for both* S *and* (S).

4. POLARITY FOR A QUADRIC

Following COURT ([1]) we can prove the following

Theorem 4. *If* Y, z *be pole and polar for a quadric* Q, *and* S, S'' *be polar reciprocal simplexes for* Q, *the pole* Z *of* z *and the polar* y *of* Y *for* S *and* S'' *respectively are also pole and polar for* Q.

Theorem 5. *If* z *is a tangent hyperplane for a quadric* Q, *at a point* Y *on it, the pole* Z *of* z *and the polar* y *of* Y *for a simplex* S, *self-polar for* Q, *are the pole and polar for* Q.

5. AN S-CONFIGURATION

a. If P_{ij} be the trace of a hyperplane z (§ 1) on an edge A_iA_j of a simplex S, and Q_{ij} be its harmonic conjugate on this edge w.r.t. A_i, A_j, the points P_{ij}, Q_{ij} then lie by $\binom{n+1}{2}$s in the 2^n hyperplanes, like z, of an *S-configuration* (S−C) as the $\binom{n+1}{2}$ pairs of its opposite vertices, S being the diagonal simplex ([8]). The equations of these hyperplanes referred to S are

$$x_0/Z_0 \pm x_1/Z_1 \pm \dots \pm x_n/Z_n = 0$$

b. The 2^n points $(Z_0, \pm Z_1, \dots, \pm Z_n)$ form an *associated* [(10)] or *closed set* ([8]; [10]) as the vertices of the *dual* or *reciprocal* (R.S−C) of the (S−C) such that all the quadrics, for which their common diagonal simplex S is self-polar, passing through one of them, pass through all of them. Conversely we can show the following

Theorem 6. *The quadrics circumscribed to an* (R.S−C) *form a system* s *such that its diagonal simplex* S *is self-polar for all of them.*

They are therefore represented by the equations

$$\sum k_i x_i^2 = 0 = \sum k_i Z_i^2 , \quad \text{referred to } S .$$

Dually we shall have the following

Theorem 7. *The quadrics inscribed to an* (S – C) *form a tangential system* s' *such that its diagonal simplex* S *is self-polar for all of them.*

They are therefore represented tangentially by the equations referred to S,

$$\sum p_i u_i^2 = 0 = \sum p_i z_i^2 = 0\,,$$

the tangential coordinates of the hyperplanes of (S – C) being $(z_0, \pm z_1, \ldots, \pm z_n)$.

6. SYSTEMS OF QUADRICS

a. The point of contact Y of a quadric Q' of the tangential system s' (Theorem 7) inscribed in the (S – C) with its hyperplane z (§§ 1, 5) is given by $\sum p_i u_i z_i = 0 = \sum p_i z_i^2$. The tangential coordinates of the polar y of Y for the diagonal simplex S of (S – C) are then $u_i = 1/p_i z_i$ where $\sum p_i z_i^2 = 0$. Thus the envelope of y, as Q' varies is the primal $\sum z_i/u_i = 0$ (§ 1).

Again by the Theorem 4 or otherwise by § 1, y is the polar prime for Q', of the pole Z of z for S. Hence we have the following

Theorem 8. *The polars of the point of contact of the quadrics* Q' *of a tangential system* s' *inscribed in an* (S – C) *with one of its* 2^n *hyperplanes, say* z, *for its diagonal simplex* S *or the polars, for* Q', *of the pole* Z *of* z *for* S *envelope a primal of class* n *no other than the first polar of* z *for* S.

Theorem 9. *The envelope of the polar of a variable point of a given hyperplane* z *for a simplex* S *coincides with the first polar of* z *for* S. (cf. [6], p. 97, ex. 3)

b. Dually thus we have the following

Theorem 10. *The poles of the tangent hyperplanes of the quadrics* Q *of a system* s *circumscribed to an* (R.S – C) *at one of its vertices, say* Z, *for its diagonal simplex* S *or the poles, for* Q, *of the polar of* Z *for* S *describe a primal of order* n *no other than the first polar of* Z *for* S.

Theorem 11. *The locus of the pole of a variable hyperplane through a given point* Z *for a simplex* S *coincides with the first polar of* Z *for* S (cf. [6], p. 97).

c. The 2 systems s, s' (Theorems 6 – 8, 10) of quadrics are evidently related dually and may be said to be *harmonically associated* (cf. [1]), if the vertices of the (R.S – C) inscribed to s are the poles of the hyperplanes of the (S – C) circumscribed to s' w.r.t. their common diagonal simplex S. In fact, there exists a quadric W for which

s, s' are polar reciprocal, viz., $W \equiv \sum z_i^2 x_i^2 = 0$ (which is the same as $\sum Z_i^2 u_i^2 = 0$ tangentially, where $Z_i z_i = 1$ (§ 1)). For in this polarity, the quadric $Q \equiv \sum k_i x_i^2 = \sum k_i Z_i^2$ of s corresponds to $Q' \equiv \sum p_i u_i^2 = 0 = \sum p_i z_i^2$ of s' for $p_i = k_i Z_i^4$. Thus we have

Theorem 12. *The harmonically associated systems of quadrics have a common self-polar simplex S, and their relation is reciprocal such that one reciprocates into other w.r.t. a quadric W for which too S is self-polar.*

7. ISOTOMIC TRANSFORMATION

a. A pair of points Z_{ij}, Z'_{ij} on an edge A_iA_j of a simplex S equidistant from its midpoint M_{ij} are said to be *isotomic conjugates* w.r.t. A_iA_j ([4]; [19]). If Z_{ij} be the feet of the *bicevians* of S through a point Z (secants through Z to its edges and the respectively opposite $[n-2]$s) on its edges, it is shown in [13] that their isotomic conjugates Z'_{ij} thereat are also the feet of the bicevians of S through another point Z'. The pair of points like Z, Z' are said to be *isotomic conjugates for* S, and their polars z, z' for S are consequently called *isotomically conjugate hyperplanes* for S. Thus: *The* 2^n *vertices of the* (R.S$-$C) *with one vertex at the centroid G of its diagonal simplex S and their polar hyperplanes for S, that of G being at infinity, are all isotomically self-conjugate for S.*

b. It can be shown that if G be taken as unit point $(1, 1, \ldots, 1)$ of S, and the coordinates of Z be as before (§ 1), those of its isotomic conjugate point Z' are proportional to their reciprocals respectively. That is, $Z_i Z'_i = k$ (say). Similarly, therefore, are related the tangential coordinates of z, z' too. That is, $z_i z'_i = 1/k = k'$ (say), for $Z_i z_i = 1 = Z'_i z'_i$ (§ 1). Thus $Z_i = k z'_i$, $z_i = k' Z'_i$.

c. Consider a variable hyperplane u' $(u'_0, \ldots, u'_n)$ through the given point Z', and its isotomic conjugate hyperplane u $(u_0, \ldots, u_n)$ for the simplex S. Then $\sum Z'_i u'_i = 0$, $u_i u'_i = k'$. Hence, as u' varies through Z', u envelope the primal $\sum Z'_i/u_i = 0$ which is no other than the first polar (§ 1) of the hyperplane z for S. For its tangential coordinates are $z_i = k' Z'_i$. Thus we have the following

Theorem 13. *The envelope of the isotomic conjugates of the hyperplanes through a given point Z' for a simplex S in an* $[n]$ *is a primal of class n no other than the first polar, for S, of the isotomic conjugate hyperplane z of the polar z' of Z' for S.*

d. Dually thus we have the following

Theorem 14. *The locus of the isotomic conjugate of a variable point in a given hyperplane z' for a simplex S in an* $[n]$ *is a primal of order n no other than the first polar, for S, of the isotomic conjugate point Z of the pole Z' of z' for S* (cf. [3]).

8. ISOGONAL TRANSFORMATION

The coordinates of a pair of *isogonal conjugate points* (cf. [6], [9], [16], [19]) Z, Z' for a simplex S with unit point at its incentre I are also seen to be related reciprocally. That is, $Z_i Z'_i$ is a constant. Hence following the argument of the preceding section we have

Theorem 15. *The locus of the isogonal conjugate of a variable point in a given hyperplane z' for a simplex S in an $[n]$ is a primal of order n other than the first polar, for S, of the isogonal conjugate point Z of the pole Z' of z' for S.*

9. WHEN $n = 2$

The first polar and the polar quadric of a point Z in the plane of a triangle t for t is the polar conic ([4]) of Z for t. The harmonically associated system (§ 5) s, s' of quadrics become respectively a pencil and a range of conics for which t is self-polar. For detail of these particular cases of evident interest reference be made to Court ([1]; [2]; [3]) who has also discussed the isotomic conics of the 4 isotomically self-conjugate transversals of t one being the line at infinity in the plane of t.

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References

[1] *N. A. Court:* Pencils of conics. Bol. Mat., *28* (1955), 7—15.

[2] *N. A. Court:* Three hyperbolas associated with a triangle. Am. Math. Mon., *64* (1957), 241—247.

[3] *N. A. Court:* Sur la transformation isotomique. Mathesis, *66* (1957), 291—297.

[4] *N. A. Court:* Sur la cubique gauche, application de la transformation harmonique. Mathesis, *68* (1959), 110—127.

[5] *N. A. Court:* Modern Pure Solid Geometry. New York, 1964.

[6] *H. S. M. Coxeter:* The real projective plane. Cambridge, (1955).

[7] *H. S. M. Coxeter:* Review A 1253. Math. Reviews *23* (1962), 224.

[8] *S. R. Mandan:* An *S*-configuration in Euclidean and elliptic *n*-space. Can. J. Math., *10* (1958), 489—501.

[9] *S. R. Mandan:* Semi-orthocentric and orthogonal simplexes in a 4-space. Bul. Cal. Math. Soc. *50* (Supp. 1958), 21—29.

[10] *S. R. Mandan:* Harmonic Inversion. Math. Mag. *32* (1959), 13—20.

[11] *S. R. Mandan:* Polarity for a quadric in an *n*-space. Rev. Fac. Soc. Uni. Istanbul, A *24* (1959), 21—40.

[12] *S. R. Mandan:* A closed set of 2^n points in an *n*-space. Math. Sem. *1* (1960), 43—44.

[13] *S. R. Mandan:* Cevian simplexes. Proc. Amer. Math. Soc. *11* (1960), 837—845.

[14] *S. R. Mandan:* Semi-isodynamic and isogonic tetrahedra. Rend. Math. (3—4), *19* (1960), 401—415.

[15] *S. R. Mandan:* Isodynamic and isogonal simplexes. Ann. Mat. (4) *53* (1961), 45—55, (To Enricho Bompiani on his Scientific Jubilee).
[16] *S. R. Mandan:* Altitudes of a simplex in an *n*-space. Jour. Australian Math. Soc., *2* (1962), 403—424.
[17] *G. Salmon:* Higher plane curves. (New York, 1934).
[18] *J. G. Semple* and *L. Roth:* Algebraic Geometry. (Oxford 1949).
[19] *D. M. Y. Sommerville:* Analytical conics. (London, 1949).

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Резюме

ПОЛЯРНЫЕ СООТНОШЕНИЯ ДЛЯ СИМПЛЕКСА

САГИБ РАМ МАНДАН (Sahib Ram Mandan), Харагпур (Индия)

Изучаются полярные соотношения относительно симплекса и их связ с изотомическим и изогональним соотношениями.