Czechoslovak Mathematical Journal

Miloš Dostál

On Fourier image of the singular support of a distribution

Czechoslovak Mathematical Journal, Vol. 16 (1966), No. 2, 231–237

Persistent URL: http://dml.cz/dmlcz/100725

Чехословацкий математический журнал, т. 16 (91) 1966, Прага

ON FOURIER IMAGE OF THE SINGULAR SUPPORT OF A DISTRIBUTION

MILOŠ DOSTÁL, Praha

(Received April 4, 1965)

0° Let $\Phi \in \mathscr{E}'(R^n)$ be a distribution with compact support in R^n. If $p \in R^n$ we say that $\Phi \in C^\infty(p)$ if there exists a neighbourhood Ω of p and $f(x) \in C^\infty(R^n)$ such that for every $\varphi \in \mathscr{D}(\Omega)$

$$\Phi(\varphi) = \int_{R^n} f(x)\, \varphi(x)\, \mathrm{d}x\,.$$

Then we put

$$\operatorname{sing\,supp} \Phi = \operatorname{supp} \Phi \setminus \{p : \Phi \in C^\infty(p)\},$$

where supp Φ means the support of Φ. In the present article we will give a precise description of the convex hull of sing supp Φ by means of the Fourier transform $\hat{\Phi}$ of Φ.

First of all we remember an analogous situation with the set supp Φ. The well known Paley-Wiener theorem in Schwartz's modification says – roughly spoken – that the radius of the least sphere with center at 0 containing supp Φ is equal to the type of the entire function $\hat{\Phi}$. A more precise theorem has been proved by PLANCHEREL and PÓLYA ([1]). They introduced a notion of P-indicator of an entire function and proved that for $\Phi \in L^2(R^n)$ the corresponding P-indicator of $\hat{\Phi}$ is equal to the supporting function of the set $H(\operatorname{supp} \Phi)$, where H means the convex hull. On the other side a theorem of Paley-Wiener's type is valid for the singular supports ([2]). We will find a corresponding variant of the Plancherel-Pólya theorem for the case of singular support i.e. we shall define a kind of indicator of $\hat{\Phi}$ and using it we shall completely describe the set $H(\operatorname{sing\,supp} \Phi)$. After some notations we begin with a generalization of Paley-Wiener theorem in the Plancherel-Pólya direction (theorem 1) which will be necessary later (comp. [3], p. 130, Remarque 4°) .

1° Let R^n ev. C^n be the real ev. complex n-dimensional space. We write the elements of C^n in the form $\zeta = \xi + i\eta = (\xi_1 + i\eta_1, \ldots, \xi_n + i\eta_n)$, $\xi \in R^n_\xi$, $\eta \in R^n_\eta$. Let C_α denotes an arbitrary direction in R^n, i.e. $C_\alpha = (\cos \alpha_1, \ldots, \cos \alpha_n)$ and $\sum \cos^2 \alpha_i = 1$. If

$H(M)$ is a closed convex hull of some $M \subset R^n$, denote by $\mathscr{K}_M(\alpha)$ the supporting function of the set $H(M)$ so that

$$\mathscr{K}_M(\alpha) = \sup_{x \in M} \langle x, C_\alpha \rangle$$

where $\langle x, C_\alpha \rangle = \sum x_k \cos \alpha_k$. For $\Phi \in \mathscr{E}'(R^n)$ we write $\mathscr{K}_\Phi(\alpha)$ instead of $\mathscr{K}_{\text{supp}\Phi}(\alpha)$. Conversely if $k(\alpha)$ is a function defined on 1-sphere such that for some $M \subset R^n$ we have $k(\alpha) \equiv \mathscr{K}_M(\alpha)$, we call $k(\alpha)$ to be a t.c. function (trigonometrically convex). If $\Phi \in \mathscr{E}'(R^n)$ the Fourier transform $\hat{\Phi}$ can be defined by $\hat{\Phi}(\zeta) = \Phi_x(e^{-i\langle x, \zeta \rangle})$.

Theorem 1. a) *Let $\Phi \in \mathscr{E}'(R^n)$ and $k(\alpha)$ be a t.c. function for which $k(\alpha) > \mathscr{K}_\Phi(\alpha)$ for every α. Then the following assertion holds*:

($\mathscr{PW}$) *There exist constants $C > 0$, N (N integer depending only on Φ is the order of Φ and C depends on Φ and k) so that for every α, $r > 0$ and $\xi \in R^n$*

$$(1) \qquad |\hat{\Phi}(\xi + iC_\alpha r)| \leqq C(1 + |\xi|)^N e^{rk(\alpha)}$$

Conversely: Let F be an entire function and $k(\alpha)$ a bounded t. c. function such that for suitable C and N the condition ($\mathscr{PW}$) is satisfied, then $F = \hat{\Phi}$ for some $\Phi \in \mathscr{E}'(R^n)$ and $\mathscr{K}_\Phi(\alpha) \leqq k(\alpha)$.

b) *Let $\Phi \in \mathscr{D}(R^n)$ and let $k(\alpha)$ be a t.c. function for which $k(\alpha) \geqq \mathscr{K}_\Phi(\alpha)$ then*

($\mathscr{PW}^\infty$) *For every integer N there exists $C_N > 0$ depending only on N and Φ such that for every α, $r > 0$, $\xi \in R^n$*

$$(2) \qquad |\hat{\Phi}(\xi + iC_\alpha r)| \leqq C_N(1 + |\xi|)^{-N} e^{rk(\alpha)}$$

and conversely as above.

The proof is a slight modification of Hörmander's proof of Paley-Wiener theorem ([2]) but we shall reproduce it only with the aim for comparing his main idea – translation of the integration domain from R^n_ξ to C^n – with a similar one in the case of singular supports, where the integration domain R^n_ξ has to be deformed in a more complicated way.

First we shall prove the necessity of ($\mathscr{PW}$). If $\Phi \in \mathscr{E}'(R^n)$ then there exist constants C_1, N such that for every $\varphi \in \mathscr{D}(R^n)$

$$(3) \qquad |\Phi(\varphi)| \leqq C_1 \sum_{|\kappa| \leqq N} \sup_x |D^\kappa \varphi(x)|$$

Furthermore for every $\chi \in \mathscr{E}'(R^n)$ such that $\chi \equiv 1$ in some neighbourhood of supp Φ we have

$$(4) \qquad \hat{\Phi}(\zeta) = \Phi(e^{-i\langle x, \zeta \rangle} \chi(x))$$

Supoose C_α to be fixed. Put $\chi(x) = \psi(|\zeta| (\langle x, C_\alpha\rangle - \mathscr{K}_\Phi(\alpha)))$, where $\psi \in C^\infty(R^n)$, $\psi(t) = 1$ for $t \leqq 2^{-1}$ and $\psi(t) = 0$ for $t \geqq 1$. Therefore we have

$$(5) \qquad |\hat{\Phi}(\zeta)| \leqq C_1 \sum_{|\kappa| \leqq N} \sup_x |D_x^\kappa(e^{-i\langle x,\zeta\rangle} \chi(x))|$$

It is easy to obtain from (5) the estimate

$$(6) \qquad |\hat{\Phi}(\zeta)| \leqq C_2(1 + |\zeta|)^N e^{r\mathscr{K}_\Phi(\alpha)}$$

where C_2 is independent on α, and combining (6) with the obvious inequality $(1 + |\zeta|)^N \leqq C_3(\varepsilon)(1 + |\xi|)^N e^{\varepsilon r}$, where $0 < \varepsilon < \inf(k(\alpha) - \mathscr{K}_\Phi(\alpha))$ we obtain (1).

Taking $\Phi \in \mathscr{D}(R^n)$ and applying (1) with $N = 0$ on $(1 + \Delta)^M \Phi$ we obtain $(\mathscr{PW}^\infty)$. Conversely if for some bounded t.c. function $k(\alpha)$ and an entire function F $(\mathscr{PW}^\infty)$ is valid, then $\Phi(x) = (2\pi)^{-n} \int F(\xi)\, e^{i\langle x,\xi\rangle}\, d\xi$ lies in $\mathscr{S}$ (using the usual Paley-Wiener theorem and the boundedness of k, we see that $\Phi \in \mathscr{D}(R^n)$). Take an arbitrary C_α and $r > 0$. Then for $\eta = C_\alpha r$ we have by the Cauchy-Poincaré formula

$$(7) \qquad \Phi(x) = (2\pi)^{-n} \int F(\xi + i\eta)\, e^{i\langle x,\xi+i\eta\rangle}\, d\xi$$

so that for a great N we obtain from (2) and (7)

$$(8) \qquad |\Phi(x)| \leqq (2\pi)^{-n}\, C_N e^{r(k(\alpha) - \langle x,C_\alpha\rangle)} \int_{R^n} (1 + |\xi|)^{-N}\, d\xi$$

and leting $r \to \infty$ we see that in the case of $x \in \operatorname{supp} \Phi$ necessarily $k(\alpha) \geqq \langle x, C_\alpha\rangle$ i.e. $\mathscr{K}_\Phi(\alpha) \leqq k(\alpha)$.

By means of a reguralization it is easy to obtain the sufficiency of $(\mathscr{PW})$.

Now we recall briefly the notion of P-indicator:

Definition. *Let F be an entire function. Then for any direction C^α in R^n_η we put*

$$(9) \qquad \mathscr{H}_F(\alpha) = \sup_{\xi\in R^n} \overline{\lim_{r\to\infty}}\, r^{-1} \ln |F(\xi + iC_\alpha r)|$$

and call this function P-indicator of F.

Theorem 2. *For $\Phi \in \mathscr{E}'(R^n)$ holds: $\mathscr{K}_\Phi(\alpha) \equiv \mathscr{H}_{\hat{\Phi}}(\alpha)$.*

This is an easy consequence of the classical Plancherel-Pólya theorem and Theorem 1. Indeed, Theorem 1. gives $\mathscr{H}_{\hat{\Phi}}(\alpha) \leqq \mathscr{K}_\Phi(\alpha)$ and for a regularization function $\varphi_\varepsilon \in \mathscr{D}(R^n)$ with $\operatorname{supp} \varphi_\varepsilon = \{x : |x| \leqq \varepsilon\}$ we have $\mathscr{K}_\Phi + \varepsilon = \mathscr{K}_{\Phi_\varepsilon} = \mathscr{H}_{\hat{\Phi}_\varepsilon} \leqq \mathscr{H}_{\hat{\Phi}} + \mathscr{H}_{\hat{\varphi}_\varepsilon} = \mathscr{H}_{\hat{\Phi}} + \varepsilon$ where $\Phi_\varepsilon = \Phi * \varphi_\varepsilon$ and Plancherel-Pólya theorem, the theorem on supports and subadditivity of P-indicator were used.

Corrolary. *For every $\Phi \in \mathscr{E}'(R^n)$ there exists an integer N ($=$ order of Φ) so that for every $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that for all α, $r > 0$, $\xi \in R^n$ we have*

$$|\hat{\Phi}(\xi + iC_\alpha r)| \leqq C_\varepsilon(1 + |\xi|)^N e^{r(\mathscr{H}_{\hat{\Phi}}(\alpha) + \varepsilon)} \tag{10}$$

Let us remark that this corrolary cannot be obtained only from Theorem 1 and that on the other side it says more than Theorem 2.

2° Now take a function $\mathscr{H}_F(\alpha)$ in the case $n = 1$. Then there are obviously only two directions $C_\alpha = \pm 1$ in R^1_η corresponding to $\alpha = 0, \pi$ respectively. It is not hard to show ([1]) that then for every $\xi \in R^1_\xi$ we have $\mathscr{H}_F(0) = \lim_{r \to \infty} r^{-1} \ln |F(\xi + ir)|$ and so $\mathscr{H}_F(0) = h_F(\frac{1}{2}\pi)$ where $h_F(\gamma)$ denotes the well known Phragmén-Lindelöf indicator of F (similar in the case $C_\alpha = -1$) Using the continuity of $h_F(\gamma)$ we obtain

$$\mathscr{H}_F(0) = \overline{\lim_{t \to \infty}}\ \overline{\lim_{\xi \to \infty}} \frac{\ln |F(\xi + i\xi t)|}{\xi t} \tag{11}$$

It is a special case of the following general situation. Take a fixed positive function $\eta(\xi, t)$ defined on $R^n_\xi \times (0, +\infty)$ which tends to infinity in every variable separately. For every $t > 0$ and α the function $\eta(\xi, t)$ determines an n-dimensional complex manifold $\Gamma_{t,\alpha}$ in $C^n = R^{2n} : \Gamma_{t,\alpha} = \{\zeta \in C^n : \zeta = \xi + iC_\alpha \eta(\xi, t)\}$. Now let α be fixed. The limit $\lim_{\zeta \in \Gamma_{t,\alpha},\, |\xi| \to \infty} (\ln |F(\zeta)|)/(|\eta(\xi, t)|)$ describes the growth of F on $\Gamma_{t,\alpha}$ at infinity and so the double limit $\overline{\lim_{t \to \infty}}\ \overline{\lim_{\xi \to \infty}} (\ln |F(\xi + iC_\alpha \eta(\xi, t))|)/(|\eta(\xi, t)|) = \mathfrak{K}_F(\alpha)$ gives us a simultaneous estimate of behaviour of F at infinity on the one parametrical system Γ_t in the direction C_α. It is to be expected that for some suitable choice of function $\eta(\xi, t)$ determining the system Γ_t we obtain an indicator $\mathfrak{K}_F$ which will indicate a certain property of F.

We put now $\eta(\xi, t) = t \log (1 + |\xi|)$ and the corresponding function $\mathfrak{K}_F(\alpha)$ denote $\mathscr{H}^s_F(\alpha)$ and call it the singular indicator of F. We shall consider $\mathscr{H}^s_F$ only for $F = \hat{\Phi}$, $\Phi \in \mathscr{E}'(R^n)$. Further put $\mathscr{K}^s_\Phi(\alpha) = \mathscr{K}_{\operatorname{singsupp}\Phi}(\alpha)$

Theorem 3. *Let $\Phi \in \mathscr{E}'(R^n)$ and $k(\alpha)$ be a t.c. function so that $k(\alpha) > \mathscr{K}^s_\Phi(\alpha)$. Then*

($\mathscr{P}\mathscr{P}$) There exist an integer N and a positive function $C(t)$ such that for every α, $t > 0$, $\xi \in R^n$ we have

$$|\hat{\Phi}(\xi + iC_\alpha \eta(\xi, t))| \leqq C(t)\,(1 + |\xi|)^N e^{k(\alpha)\eta(\xi, t)} \tag{12}$$

Conversely if for a $\hat{\Phi} \in \hat{\mathscr{E}}'(R^n)$ the condition ($\mathscr{P}\mathscr{P}$) holds with some t.c. function k, then $\mathscr{K}^s_\Phi(\alpha) \leqq k(\alpha)$ for all α.

We shall prove this theorem together with the following.

Theorem 4. *For every* $\Phi \in \mathscr{E}'(R^n)$ *we have* $\mathscr{H}^s_{\hat{\Phi}}(\alpha) \equiv \mathscr{K}^s_{\Phi}\Phi(\alpha)$.

Proof: The main idea is due to L. Ehrenpreis (see [4]). First of all we prove the necessity of $(\mathscr{P}\mathscr{P})$. For $0 < \varepsilon < \inf_\alpha (k(\alpha) - \mathscr{K}^s_{\Phi}(\alpha))$ we can find by means of a suitable partition of unity the distributions $\Phi_i \in \mathscr{E}'(R^n)$ $(i = 1, 2)$ such that

$$\Phi = \Phi_1 + \Phi_2\,, \quad \Phi_2 \in \mathscr{D}(R^n)\,, \quad \mathscr{K}_{\Phi_1}(\alpha) = \mathscr{K}^s_{\Phi}(\alpha) + \varepsilon/2 \tag{13}$$

From Theorem 1 we obtain for all α, $r > 0$, $\xi \in R^n$

$$|\hat{\Phi}_1(\xi + iC_\alpha r)| \leqq C(1 + |\xi|)^N e^{rk(\alpha)} \tag{14}$$

and further we obtain that there exists a constant $R > 0$ such that for every integer M we have fore some C_M

$$|\hat{\Phi}_2(\zeta)| \leqq C_M(1 + |\xi|)^{-M} e^{Rr} \quad (r = |\text{Im}\,\zeta|) \tag{15}$$

for all $\zeta \in C^n$. Take a fixed t and α and chose $M \geqq (R - R_0)\,t - N$, where $R_0 = \inf_\alpha k(\alpha)$ (evidently $R_0 > -\infty$). Then for $\eta = \eta(\xi, t)$ we have

$$|\hat{\Phi}_2(\zeta)| \leqq \tilde{C}_t(1 + |\xi|)^N e^{k(\alpha)\eta(\xi,t)} \tag{16}$$

and (16) together with (14) gives (12). From (12) we obtain immediately

$$\mathscr{H}^s_{\hat{\Phi}}(\alpha) \leqq \mathscr{K}^s_{\Phi}(\alpha) \tag{17}$$

Now let us prove the inverse inequality. It is sufficient to prove the following assertion: for every $\varepsilon > 0$ we have: if α is an arbitrary but fixed, then for every $x \in R^n$ such that $\langle x, C_\alpha\rangle > \mathscr{H}^s_{\hat{\Phi}}(\alpha) + \varepsilon$ holds $\Phi \in C^\infty(x)$, that is $\mathscr{K}^s_{\Phi}(\alpha) \leqq \mathscr{H}^s_{\hat{\Phi}}(\alpha) + \varepsilon$ for every $\varepsilon > 0$, α. Now take an arbitrary integer $j > 0$ and put $T_\delta = \{x : \langle x, C_\alpha\rangle > \mathscr{H}^s_{\hat{\Phi}}(\alpha) + \varepsilon + \delta\}$ for $\delta > 0$. It is sufficient to prove that $\Phi \in C^j(T_\delta)$ for every $\delta > 0$. Take an arbitrary $\delta > 0$. From the definition of $\mathscr{H}^s_{\hat{\Phi}}$ follows that there exists $t_0 > 0$ so that for $t \geqq t_0$

$$\overline{\lim_{|\xi|\to\infty}} \frac{\log|\hat{\Phi}(\zeta)|}{\eta(\xi, t)} < \mathscr{H}^s_{\hat{\Phi}}(\alpha) + \varepsilon \tag{18}$$

Put now

$$t_1 = \max\left(t_0, (n + j + 1)\,\delta^{-1}\right). \tag{19}$$

Further for some $\xi_0(t_1) > 0$ we have

$$|\hat{\Phi}(\xi + iC_\alpha t_1 \log(1 + |\xi|)| < \exp\{\eta(\xi, t_1)\,(\mathscr{H}^s_{\hat{\Phi}}(\alpha) + \varepsilon)\} \tag{20}$$

for all $|\xi| \geqq \xi_0(t_1)$ so that for some constant $C_1(t_1)$ holds

$$|\hat{\Phi}(\xi, t_1)| \leqq C_1(t_1)\,(1 + |\xi|)^{t_1(\mathscr{H}^s_{\hat{\Phi}}(\alpha)+\varepsilon)} \tag{21}$$

for all $\xi \in R^n_\xi$. We put

$$(22)\quad \chi(x) = \int_{\Gamma_{t_1,\alpha}} e^{i\langle x,\zeta\rangle}\, \hat{\Phi}(\zeta)\, d\zeta_1 \wedge \ldots \wedge d\zeta_n = \int_{R_\xi^n} e^{i\langle x,\zeta\rangle}\, \hat{\Phi}(\xi, t_1) \frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(\xi_1, \ldots, \xi_n)}\, d\xi$$

and if we take ι to be an arbitrary multiindex with the lenght $|\iota| \leqq j$ then estimating the ι-th derivative of the last integrand absolutely and uniformly in T_δ by a summable function we shall prove that $\chi(x) \in C^j(T_\delta)$ and so $\chi(x) \in C^j(T_0)$.

Evidently we have for some $C_i(t_1)$ $(i = 2, 3)$

$$(23)\qquad |D^\iota_x(e^{i\langle x,\zeta\rangle})| \leqq C_2(t_1)\,(1 + |\xi|)^{j+t_1\langle x,C_\alpha\rangle}\,; \quad x \in T_\delta,\ \xi \in R^n_\xi$$

$$(24)\qquad \left|\frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(\xi_1, \ldots, \xi_n)}\right| \leqq C_3(t_1)$$

Combining (21), (23), (24) we see that with regard to (19)

$$(25)\quad \int_{R_\xi^n} \left| D^\iota_x(e^{i\langle x,\zeta\rangle})\, \hat{\Phi}(\xi, t_1) \frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(\xi_1, \ldots, \xi_n)}\right| d\xi \leqq C_4(t_1) \int_R \frac{d\xi}{(1 + |\xi|)^{n+1}}$$

so $D^\iota_x(\chi(x)) = \int_\Gamma D^\iota_x(e^{i\langle x,\zeta\rangle})\, \hat{\Phi}(\zeta)\, d\zeta$ and $\chi(x) \in C^j(T)$. Especially from our conslusions follows for every $\varphi(x) \in \mathscr{D}(T)$:

$$(26)\qquad \int_{R_x^n}\int_{R_\xi^n} \left| e^{i\langle x,\zeta\rangle}\, \hat{\Phi}(\xi, t_1) \frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(\xi_1, \ldots, \xi_n)}\, \varphi(x)\right| dx\, d\xi < +\infty$$

so that we can change the order of integrations and using the formulas of Plancherel and of Cauchy-Poincaré[1]) we obtain finally

$$(27)\quad \Phi(\varphi)\,(2\pi)^{-n} = \int_{R_\xi^n} \hat{\Phi}(\xi)\, \hat{\varphi}(-\xi)\, d\xi = \int_\Gamma \hat{\Phi}(\zeta)\, \hat{\varphi}(-\zeta)\, d\zeta =$$

$$= \int_{R_x^n}\int_{R_x^n} \hat{\Phi}(\xi, t_1)\, e^{i\langle x,\zeta\rangle}\, \varphi(x) \frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(\xi_1, \ldots, \xi_n)}\, d\xi\, dx =$$

$$= \int_{R_x^n} \left(\int_\Gamma \hat{\Phi}(\xi, t_1)\, e^{i\langle x,\zeta\rangle}\, d\zeta\right) \varphi\, dx = \int_{R_x^n} \chi(x)\, \varphi(x)\, dx$$

which means that $\Phi = \chi$ on T and so $\Phi \in C^j(T)$ and Theorem 4 is therefore proved. If we suppose $(\mathscr{P}\mathscr{P})$ to be valid for some $k(\alpha)$, then we have obviously $\mathscr{H}^s_{\hat{\Phi}}(\alpha) \leqq k(\alpha)$ which is by the Theorem 4 the same as $\mathscr{K}^s_\Phi(\alpha) \leqq k(\alpha)$.

---

[1]) $\hat{\varphi}$ decreases very fast, see $(\mathscr{P}\mathscr{W}^\infty)$.

Added in the proofs: Theorem 2 is proved in [5].

References

[1] *Plancherel-Pólya:* Fonctions entières et intégrales de Fourier multiples. Comm. Math. Helvet, *9*, 10 (1936—1938).
[2] *L. Hörmander:* Linear partial differential operators, N. York—Berlin 1963.
[3] *L. Schwartz:* Théorie des distributions II, Paris 1951.
[4] *L. Ehrenpreis:* Solutions of some problems of division. IV, Amer. J. Math., *82* (1960).
[5] *T. Kakita*: Two theorems on Fourier transforms. Proc. Japan Acad., 34 (1958).

Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).

Резюме

ОБ ОБРАЗЕ ФУРЬЕ НОСИТЕЛЯ СИНГУЛЯРНОСТЕЙ ОБОБЩЕННОЙ ФУНКЦИИ

МИЛОШ ДОСТАЛ, (Miloš Dostál), Прага

Пусть $\Phi \in \mathscr{E}'(R^n)$ является обобщенной функцией с компактным носителем. Обозначим через sing supp Φ ее носитель сингулярностей. Для целой функции $\hat{\Phi}$ (преобразование Фурье Φ) мы положим

$$\mathscr{H}^s_{\hat{\Phi}}(\alpha) = \overline{\lim_{t\to\infty}}\ \overline{\lim_{|\xi|\to\infty}} \left(\ln \left|\hat{\Phi}(\xi + iC_\alpha \eta(\xi, t))\right|\right) . \left|\eta(\xi, t)\right|^{-1},$$

где $\eta(\xi, t) = t \log(1 + |\xi|)$ и $C_\alpha = (\cos\alpha_1, \ldots, \cos\alpha_n)$ — единичный вектор в n-мерном вещественном пространстве R^n. Функция $\mathscr{H}^s_{\hat{\Phi}}(\alpha)$ описывает рост функции $\hat{\Phi}$ в направлении C_α на однопараметрической системе $(\Gamma_t)_{t>0}$ многообразий $\Gamma_t = \{\zeta \in C^n : \zeta = \xi + iC_\alpha\eta(\xi, t)\}$ (C^n n-мерное комплексное пространство). Если мы теперь положим

$$\mathscr{K}^s_\Phi(\alpha) = \sup_{x \in \operatorname{sing\,supp}\Phi} \langle x, C_\alpha \rangle,$$

то имеет место следующая теорема:

Теорема 4. *Для каждого* $\Phi \in \mathscr{E}'(R^n)$ *имеем* $\mathscr{H}^s_{\hat{\Phi}}(\alpha) \equiv \mathscr{K}^s_{\Phi}(\alpha)$. Теорема 3 приводит другую формулировку этого утверждения при помощи неравенств. Для доказательств этих теорем нужна теорема Пэйли Винер-Шварца, сформулированная более точным образом. Это Теорема 1, которая аналогична теореме 3. Теорема 2, аналогом которой является теорема 4, представляет обобщение теоремы Планшерель-Пойа на случай обобщенных функций.