Jozef Gruska

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Czechoslovak Mathematical Journal, Vol. 16 (1966), No. 1, 76–90

Persistent URL: http://dml.cz/dmlcz/100713

ISOLABLE AND WEAKLY ISOLABLE SETS

JOZEF GRUSKA, Bratislava

(Received January 26, 1965)

1. INTRODUCTION AND SUMMARY

The undecidability of the ambiguity problem for Chomsky's context-free grammars is well known [1, 3]. Of course this does not mean that it is impossible to devise methods which may be useful to decide, at least for some languages, whether or not they are structurally ambiguous (s.a.). Some methods of this kind have been recently investigated by Fabian [2], for slightly more general languages.

In paper [2] it has been proved that if a set $\mathscr{A}$ of non-terminal symbols is isolable (as to the definition of a language, structural unambiguity, isolable set and so on, see paper [2]), in a language $\mathscr{L}$, then $\mathscr{L}$ is structurally unambiguous (s.u.) if and only if so is $\mathscr{L}_0$, where the language $\mathscr{L}_0$ is constructed from $\mathscr{L}$ as follows: in the meta-texts of $\mathscr{L}$ (i.e. in such texts by which non-terminal symbols may be replaced) the symbols from $\mathscr{A}$ are replaced by new terminal symbols in such a way that different symbols from $\mathscr{A}$ are replaced by different terminal symbols.

A concept of weakly isolable set is introduced in Section 5. If a set $\mathscr{A}$ is weakly isolable in the language $\mathscr{L}$, then $\mathscr{L}$ is s.u. if and only if so is the language $\mathscr{L}_1$ which is constructed similarly as the language $\mathscr{L}_0$ but with the difference that two symbols A_1, A_2 from $\mathscr{A}$ are replaced by the same terminal symbol if and only if there is a text derivable from both A_1 and A_2.

If a set $\mathscr{A}$ is isolable (weakly isolable), then in order to investigate the structural unambiguity of $\mathscr{L}$, it is sufficient to investigate the structural unambiguity of the language $\mathscr{L}_0$ ($\mathscr{L}_1$), which is simpler than $\mathscr{L}$.

In this paper some sufficient conditions for a set $\mathscr{A}$ to be isolable or weakly isolable, respectively, are given.

As to sufficient conditions for the existence of isolable set they are similar to those given in paper [2] but a little different approach makes it possible to obtain simpler and more useful conditions.

The results obtained in this paper have proved to be very useful in the investigation of structural unambiguity of a language (see paper [5]), which is a slight modification of ALGOL 60.

The present paper uses notations and definitions of [2]. The reader should be familiar with sections 1 to 9.

2. PRELIMINARIES

In this paper we shall consider only non-cyclic languages $\mathscr{L}$ (i.e. languages in which $t \to t$ for no text t), such that $\mathbf{d}\mathscr{L}$ and $\{\alpha; A \in \mathbf{d}\mathscr{L}, \alpha \in \mathscr{L}A - \boldsymbol{\sigma}_t\mathscr{L}\}$ are finite sets. Denote $\mathscr{C}_1$ the class of such languages.

2.1. Notations. In this paper we shall use the symbol $\mathscr{L}$ only for denoting a language from $\mathscr{C}_1$. If $g \in \mathbf{g}\mathscr{L}$, then by $S_{\mathscr{L}}g$ ($\bar{S}_{\mathscr{L}}g$) we shall denote the set of all structures $[\alpha, \tau]$ (such that $\alpha \neq [g1]$[1])) of g in $\mathscr{L}$. Moreover, by Qg we shall denote the set $\{g_1; [\alpha, \tau] \in \bar{S}g, i \in \mathbf{d}\alpha, [\alpha i] \to \tau i, g_1 = [\alpha i, \tau i]\}$. The set of all s.u. (s.a.) grammatical elements of a language $\mathscr{L}$ will be denoted by $\mathbf{g}_u\mathscr{L}$ ($\mathbf{g}_a\mathscr{L}$).

Directly from Definition 9·7, [2] it follows:

2.2. Theorem. *A non-empty subset $\mathscr{A} \subset \mathbf{d}\mathscr{L}$ is isolable if and only if a reducing transformation ϱ exists such that $g_1 1 \notin \mathscr{A}$ if $\varrho g = g$ and $g_1 \in Qg$.*

In the next section we shall use the following lemmas and theorems, the proof of which is given in paper [4].

2.3. Lemma. *If $g \in \mathbf{g}\mathscr{L}$, then*

$$\mu g = \max\{\lambda\sigma; \sigma \textit{ is a } [g1]\textit{-derivation of } g2\}$$

is a finite number.

2.4. Lemma. *Let $A \in \mathbf{d}\mathscr{L}$, $[A] \to t_1 \to t_2$, τ be a t_1-decomposition of t_2, $i \in \mathbf{d}t_1$, $[t_1 i] \to \tau i$. Then $\mu[t_1 i, \tau i] < \mu[A, t_2]$.*

2.5. Theorem. (*Structural induction*). *Let $M \subset \mathbf{g}\mathscr{L}$, let*

$$(1) \qquad g \in M \quad \textit{if} \quad Qg \subset M\,.$$

Then $M = \mathbf{g}\mathscr{L}$.

2.6. Theorem. *Let $M \subset \mathbf{g}\mathscr{L}$ and f_0, f_1, v be transformations defined on $N \supset M$ Let $f_0 g \in \mathbf{d}g2$, $f_1 g \in \mathbf{d}g2$, $f_0 g \leqq f_1 g$ for every $g \in M$ and let one of the following conditions be satisfied for every $[\alpha, \tau] \in Sg$, $x = \iota\tau$, $xi \leqq f_0 g < x(i+1)$:*

$$(1) \qquad f_0 g = xi\,, \quad f_1 g = x(i+1) - 1\,, \quad [\alpha i] \rightrightarrows [vg] \to \tau i\,,$$

$$(2) \qquad [g1] \neq \alpha\,, \quad [\alpha i. \tau i] \in M\,, \quad vg = v[\alpha i, \tau i]\,, \quad f_s g = f_s[\alpha i, \tau i] + xi - 1\,, \quad s = 0, 1.$$

[1]) Note that g is a sequence, $\lambda g = 2$ and therefore if $g = [A, t]$ then $g1 = A$ and $g2 = t$.

Further let

$$(3) \qquad vg = g1 \text{ if } [g1] \Rightarrow g2\,, \quad g \in M\,.$$

Let us define the transformations V, R and ϱ as follows: if $g \in \mathbf{g}\mathscr{L} - M$, then $Vg = g2$, $Rg = \delta_p g2$, $\varrho g = g$; if $g = [A, t] \in M$, then we put $\varrho g = [A, Vg]$ where

$$(4) \qquad Vg = t^{(1, f_0 g - 1)} \times [vg] \times t^{(f_1 g + 1, \lambda t)}\,.$$

$$(5) \qquad Rg = \delta_p(t^{(1, f_0 g - 1)} \times [t^{(f_0 g, f_1 g)}] \times \delta_p(t^{(f_1 g + 1, \lambda t)})\,.$$

Then ϱ is a (5)-reducing transformation.

3. RECOGNIZABLE SETS.

This section contains definitions and lemmas needed in Section 4 and Section 5 Definition 3.1 and 3.2 is a modification of Definition 10.3, [2] and 10·5, [2], respectively. (In our definitions the set ft contains always at most one element.) This modification has proved to be very useful. Lemma 3.5 is a generalization of Lemma 10.7, [2].

3.1. Definition. A subset $\mathscr{A} \subset \mathbf{d}\mathscr{L}$ is said to be recognizable (f-recognizable) if a function f exists such that $\mathbf{d}f \subset \mathbf{g}\mathscr{L}$, $1 \leqq f[A, t] \leqq \lambda t$ for each $[A, t] \in \mathbf{d}f$ and the following conditions are satisfied:

(1) if $A \in \mathscr{A}$, $[A] \to t$, then $[A, t] \in \mathbf{d}f$; if $[A] \Rightarrow t$, $[A, t] \in \mathbf{d}f$ then $A \in \mathscr{A}$.

(2) If

(2a) $[A, t] \in \mathbf{d}f$, $[\alpha, \tau] \in \bar{S}[A, t]$, $x = \iota\tau$, $xj \leqq f[A, t] < x(j + 1)$,

then

(2b) $[\alpha j] = \tau j$ implies: $f[A, \alpha] = j$ and $[\alpha j_0, \tau j_0] \in \mathbf{d}f$ for no $j_0 \in \mathbf{d}\alpha$,

(2c) $[\alpha j] \to \tau j$ implies $f[\alpha j, \tau j] = f[A, t] - xj + 1$.

(3) If $g \in \mathbf{g}\mathscr{L}$ and $Qg \wedge \mathbf{d}f \neq \Lambda$, then $g \in \mathbf{d}f$.

3.2. Definition. Let $\mathscr{A}$ be an f-recognizable subset of $\mathbf{d}\mathscr{L}$. We shall say that the functions f_0 and f_1 indicate the beginning and the end for f if $\mathbf{d}f_0 = \mathbf{d}f = \mathbf{d}f_1$, $1 \leqq f_0 g \leqq fg \leqq f_1 g \leqq \lambda g2$ for each $g \in \mathbf{d}f$ and the following conditions are satisfied:

(1) if $[A] \Rightarrow t$, then $f_0[A, t] = 1$, $f_1[A, t] = \lambda t$.

(2) If

(2a) $[A] \to t$, $[\alpha, \tau] \in \bar{S}[A, t]$, $x = \iota\tau$, $xj \leqq f[A, t] < x(j + 1)$

then

(2b) $[\alpha j] = \tau j$ implies $f_0[A, t] = 1, f_1[A, t] = \lambda t$

(2c) $[\alpha j] \to \tau j$ implies $f_s[A, t] = f_s[\alpha j, \tau j] + xj - 1$ for $s = 0, 1$.

3.3. Definition. A subset $\mathscr{A} \subset \mathbf{d}\mathscr{L}$ is said to be strongly recognizable (or (f_0, f, f_1)-recognizable) if functions f_0, f and f_1 exist such that $\mathscr{A}$ is f-recognizable and the functions f_0 and f_1 indicate the beginning and the end for f.

3.4. Lemma. *Let $\mathscr{A}$ be an (f_0, f, f_1)-recognizable subset of $\mathbf{d}\mathscr{L}$. If $g = [A, t] \in \in \mathbf{g}\mathscr{L}$, $fg = i$, $f_s g = i_s$ for $s = 0, 1$, then an $A_0 \in \mathscr{A}$ exists such that*

(1) $[A] \stackrel{\rightarrow}{=} t^{(1, i_0 - 1)} \times [A_0] \times t^{(i_1 + 1, \lambda t)} \to t$

and

(2) $[A_0] \to t^{(i_0, i_1)}$.

Proof. Denote M the set of all grammatical elements for which Lemma holds. Let $g = [A, t]$ and $Qg \subset M$. Obviously $g \in M$ if $g \notin \mathbf{d}f$. Now suppose that $g \in \mathbf{d}f$. If $Qg = \Lambda$, then $[A] \Rightarrow t$ and, by (3.1.1) and (3.2.1), $[A, t] \in M$. Let $Qg \neq \Lambda$. Then an $[\alpha, \tau] \in \bar{S}_{\mathscr{L}} g$ exists. Put $x = \iota\tau$ and let j be such that $xj \leqq i < x(j + 1)$. If $[\alpha j] = = \tau j$, then, by (3.1.2b) and (3.2.2b), conditions (1) and (2) hold with $A_0 = A$, $i_0 = 1$, $i_1 = \lambda t$ and hence $[A, t] \in M$. If $[\alpha j] \to \tau j$, then $[\alpha j, \tau j] \in M$ and from (3.1.2c) and (3.2.2c) it follows $g \in M$ again. Application of Theorem 2.5 yields that $M = \mathbf{g}\mathscr{L}$

3.5. Lemma. *Let $\mathscr{A}$ be an f-recognizable subset of $\mathbf{d}\mathscr{L}$. Put $\mathscr{R} = \{\alpha i;\ i = f[A, \alpha], [A] \Rightarrow \alpha\}$, $\mathscr{L}_1 = \mathscr{L}_{\mathbf{d}\mathscr{L} - \mathscr{A}}$ and, for each $q \in \mathscr{R}$, $\mathscr{B}_q = \{\alpha;\ [A] \Rightarrow \alpha,\ i = f[A, \alpha], \alpha i = q\}$, $\mathscr{A}_q = \{A;\ [A] \Rightarrow \alpha \in \mathscr{B}_q\}$, $B_q = \{u;\ A \in \mathscr{A}, [A] \Rightarrow \alpha, f[A, \alpha] = j, \alpha j = q, \mathscr{L}_1\colon \alpha^{(1,j)} \stackrel{\rightarrow}{=} u\}$, $E_q = \{u;\ A \in \mathscr{A}, [A] \Rightarrow \alpha, f[A, \alpha] = j, \alpha j = q, \mathscr{L}_1\colon \alpha^{(j, \lambda\alpha)} \stackrel{\rightarrow}{=} u\}$. Let $\mathscr{R} \subset \mathbf{a}_t\mathscr{L}$ and let the following conditions be satisfied for every $q \in \mathscr{R}$:*

(1) *if $u_1, u_2 \in E_q$ and $u_2 = u_1 \times t$, $t \neq \Lambda$, then $t1 \notin \mathbf{rdel}\ \mathscr{A}_q$;*

(2) *if $u_1, u_2 \in B_q$ and $u_2 = t \times u_1$, $t \neq \Lambda$, then $t\lambda t \notin \mathbf{ldel}\ \mathscr{A}_q$.*

Then $\mathscr{A}$ is strongly recognizable.

Proof. Let us state two assumptions:

(3a) $g = [A, t] \in \mathbf{d}f$, $fg = i$, $q = ti$;

(3b) condition (3a) holds, $[\alpha, \tau] \in \bar{S}_{\mathscr{L}}[A, t]$, $x = \iota\tau$, $xj \leqq i < x(j + 1)$, $p = xj - 1$.

First we shall prove the assertion:

(4) if (3a) holds, then $q \in \mathscr{R}$.

If $Qg = \Lambda$, then (4) follows from (3.1.1). Now suppose that $Qg \neq \Lambda$ and that (4) holds for all $g_1 \in Qg$. Then (3b) holds with suitably chosen x, j, p. If $[\alpha j] = \tau j$ ($\mathscr{L}$: $[\alpha j] \to \tau j$), then from (3.1.2b) ((3.1.2c)) it follows that (4) holds for our q, too. According to Theorem 2.5 the assertion (4) is proved for all $g \in \mathbf{g}\mathscr{L}$.

Further let $g = [A, t] \in \mathbf{d}f$ and $fg = i$. Define $f_0 g$ ($f_1 g$) as the largest (smallest) integer for which $f_0 g \leqq i$ ($f_1 g \geqq i$) and either $f_0 g = 1$ ($f_1 g = \lambda t$) or there is satisfied condition (5), ((6)), where

(5) $t^{(f_0 g, i)} \in B_q$, $t(f_0 g - 1) \in \mathbf{ldel}\, \mathscr{A}_q$.

(6) $t^{(i, f_1 g)} \in E_q$, $t(f_1 g + 1) \in \mathbf{rdel}\, \mathscr{A}_q$.

As the next step we shall prove:

(7) if $\mathscr{L}$: $[A] \to t$, $g = [A, t] \notin \mathbf{d}f$, then $\mathscr{L}_1$: $[A] \to t$.

By (3.1.1), the assertion (7) is satisfied if $Qg = \Lambda$. Now suppose that $Qg \neq \Lambda$ and that (7) holds for all $g_1 \in Qg$. Then an $[\alpha, \tau] \in \bar{S}_{\mathscr{L}} g$ exists. If $i \in \mathbf{d}\alpha$ and $\mathscr{L}$: $[\alpha i] \to \tau i$, then (3.1.3) implies $[\alpha i, \tau i] \notin \mathbf{d}f$ and hence, by inductive assumption, $\mathscr{L}_1$: $[\alpha i] \to \tau i$. Moreover, according to (3.1.1) we have $[A, \alpha] \in \mathbf{g}\mathscr{L}_1$ and hence $[A, t] \in \mathbf{g}\mathscr{L}_1$. Application of Theorem 2.5 yields that (7) holds for all $g \in \mathbf{g}\mathscr{L}$. Now we shall prove the following four properties:

(8a) if (3a) holds and $f_0 g = 1$ then there is a β such that

$$\mathscr{L}: [A] \overset{\to}{=} \beta \overset{\to}{=} t\,, \quad \beta 1 \in \mathscr{A}_q\,, \quad t^{(1,i)} \in B_q\,;$$

(8b) if (3a) holds and $f_1 g = \lambda t$ then there is a β such that

$$\mathscr{L}: [A] \overset{\to}{=} \beta \overset{\to}{=} t\,, \quad \beta \lambda \beta \in \mathscr{A}_q\,, \quad t^{(i, \lambda t)} \in E_q\,;$$

(9) if (3b) holds and $[\alpha j] = \tau j$, then $f_0 g = 1$, $f_1 g = \lambda t$;

(10) if (3b) holds and $[\alpha j] \neq \tau j$, then $xj \leqq f_0 g \leqq f_1 g < x(j + 1)$.

In proving (8a) to (10) we shall use structural induction. If $Qg = \Lambda$ then (9) and (10) are satisfied trivially and, by (3.1.1) and (3.2.1), (8a) and (8b) hold with $\beta = [A]$. Further suppose that $Qg \neq \Lambda$ and that (8a) to (10) hold for all $g_1 \in Qg$. If (3a) holds then there exists an $[\alpha, \tau] \in \bar{S}_{\mathscr{L}} g$ and (3b) holds with suitably chosen x, j, p. Hence in proving (8a) and (8b) we may assume that not only (3a) but also (3b) holds.

First let $[\alpha j] \neq \tau j$. (9) holds trivially and (3.1.2c) implies $f[\alpha j, \tau j] = i - \mathrm{p}$. If $f_0[\alpha j, \tau j] > 1$ then directly from the definition of f_0 we conclude $f_0 g > xj$ and hence (8a) and the first inequality in (10) hold. Let $f_0[\alpha j, \tau j] = 1$. Since $[\alpha j, \tau j] \in Qg$, there exists by inductive assumption, a β_0 such that $\mathscr{L}$: $[\alpha j] \overset{\to}{=} \beta_0 \overset{\to}{=} \tau j$, $\beta_0 1 \in \mathscr{A}_q$, $\tau^{(1, i-p)} \in B_q$. If $xj = 1$ then (8a) is satisfied with $\beta = \beta_0 \times \prod \tau^{(j+1, \lambda \tau)}$ and the first inequality in (10) holds. Let $xj > 1$. Since $\beta_0 1 \in \mathscr{A}_q$, we have $t(xj - 1) \in \mathbf{ldel}\, \mathscr{A}_q$

and $\tau^{(1,i-\rho)} \in B_q$. From this and from the definition of f_0 it follows that the first inequality in (10) holds, too. (8a) is satisfied trivially. (8b) and the second inequality in (10) can be proved similarly.

Secondly, let $[\alpha j] = \tau j$. If $j_0 \neq j$, then (3.1.2b) implies $[\alpha j_0, \tau j_0] \notin \mathbf{d}f$ and hence, by (7), $\mathscr{L}_1 : [\alpha j_0] \overset{\rightarrow}{\underset{=}{\leftarrow}} \tau j_0$. Thus, $t^{(1,xj)} \in B_q$ and $t^{(xj,\lambda t)} \in E_q$. By this and by (1), (2), (5) and (6) we get that (8a) to (10) hold.

Now we see that all the conditions of Definition 3.2 are satisfied: (3.2.1) follows from the definition of f_0 and f_1. (3.2.2b) follows from (9) and (3.2.2c) from (10).

4. ISOLABLE SETS

In this section we shall prove two sufficient conditions for a strongly recognizable set to be isolable.

At first we shall prove two lemmas which will be used in further part of this section and in Section 5.

4.1. Lemma. *Let $\mathscr{A}$ be an (f_0, f, f_1)-recognizable subset of $\mathbf{d}\mathscr{L}$. If $g \in \mathbf{g}\mathscr{L}$, $Qg \wedge \wedge \mathbf{d}f \neq \Lambda$, then $g \in \mathbf{d}f$ and either $g1 \notin \mathscr{A}$ or $f_0 g \neq 1$ or $f_1 g \neq \lambda g2$ or $A_1, A_2 \in \mathscr{A}$ exist such that $[A_1] \rightarrow [A_2]$.*

Proof. Let $g = [A, t] \in \mathbf{g}\mathscr{L}$ and $Qg \wedge \mathbf{d}f \neq \Lambda$. According to (3.1.3) this implies $g \in \mathbf{d}f$. Now suppose that $A \in \mathscr{A}$, $f_0 g = 1$, $f_1 g = \lambda t$. Since $Qg \neq \Lambda$, an $[\alpha, \tau] \in \bar{S}g$ exists such that $[\alpha i, \tau i] \in Qg$ for some $i \in \mathbf{d}\alpha$. Put $x = \imath\tau$ and let j be such that $xj \leqq fg < x(j+1)$. By (3.1.2b) we have $[\alpha j] \neq \tau j$ and, moreover, (3.1.2c) and (3.2.2c) imply: $[\alpha j, \tau j] \in \mathbf{d}f$, $f_0[\alpha j, \tau j] = 1$, $f_1[\alpha j, \tau j] = \lambda \tau j = \lambda t$, $\tau j = t$, $i = j$. By Lemma 3.4 a $B \in \mathscr{A}$ exists such that $[\alpha i] \overset{\rightarrow}{\underset{=}{\leftarrow}} [B] \rightarrow \tau j$ and hence $[A] \Rightarrow \alpha \overset{\rightarrow}{\underset{=}{\leftarrow}} [\alpha i] \overset{\rightarrow}{\underset{=}{\leftarrow}} [B] \rightarrow \tau i = t$. Thus, $[A] \rightarrow [B]$ and Lemma is proved.

4.2. Lemma. *Let $\mathscr{A}$ be an (f_0, f, f_1)-recognizable subset of $\mathbf{d}\mathscr{L}$. Let V, R and ϱ be transformations defined on $\mathbf{g}\mathscr{L}$ in the following way: If*

(1) *$g = [A, t] \in \mathbf{d}f$ and there exists just one $vg \in \mathscr{A}$ such that*

$$[A] \rightarrow t^{(1, f_0 g - 1)} \times [vg] \times t^{(f_1 g + 1, \lambda t)} = t_0 \rightarrow t\,, \quad [vg] \rightarrow t^{(f_0 g, f_1 g)},$$

then

(1a) $Vg = t_0$, $Rg = \delta_p t^{(1, f_0 g - 1)} \times [t^{(f_0 g, f_1 g)}] \times \delta_p t^{(f_1 g + 1, \lambda t)}$;

(2) *If (1) does not hold, then*

(2a) $Vg = t$, $Rg = \delta_p t$.

Let $\varrho g = [A, Vg]$ for all $g \in \mathbf{g}\mathscr{L}$. Then ϱ is a (5)-reducing transformation.

Proof. Denote M the set of all grammatical elements g for which (1) holds. It is easy to see that

(3) $[A, t] \notin M$ if $g = [A, t] \in \mathbf{d}f$, $A \in \mathscr{A}$, $f_0 g = 1$, $f_1 g = \lambda t$.

Indeed, if $[A, t] \in M$ and $g = [A, t] \in \mathbf{d}f$, $A \in \mathscr{A}$, $f_0 g = 1$, $f_1 g = \lambda t$, then, by (1), $[A] \to [A]$ which contradicts the assumption $\mathscr{L} \in \mathscr{C}_1$. Now for the proof of Lemma we shall use Theorem 2.6. Let $g = [A, t] \in \mathbf{g}\mathscr{L}$. If $[A] \Rightarrow t$ and $[A, t] \notin \mathbf{d}f$, then obviously $[A, t] \notin M$. If $[A, t] \in \mathbf{d}f$, then, by (3.1.1) and (3.2.1), $A \in \mathscr{A}$, $f_0 g = 1$, $f_1 g = \lambda t$ and hence, by (3), again $[A, t] \notin M$. Thus condition (2.6.3) holds. Now suppose that $[A, t] \in M$, $[\alpha, \tau] \in S_{\mathscr{L}}[A, t]$, $x = \iota\tau$, $xi \leqq fg < x(i + 1)$. Then $[\alpha, \tau] \in \bar{S}_{\mathscr{L}}[A, t]$ and $[\alpha i] \neq \tau i$. Indeed, if $[\alpha, \tau] = [[A], [t]]$, ($[\alpha, \tau] \in \bar{S}[A, t]$ and $[\alpha i] = \tau i$), then from (3.1.1) and (3.2.1) (from (3.1.2b) and (3.2.2b)) it follows $A \in \mathscr{A}$, $f_0[A, t] = 1$, $f_1[A, t] = \lambda t$ which contradicts (3). According to (3.2.2c) this implies $xi \leqq f_0[A, t] \leqq f_1[A, t] < x(i + 1) - 1$. Let us consider two cases:

Case I. $f_0[A, t] = xi$ and $f_1[A, t] = x(i + 1) - 1$. In this case (3.2.2c) implies $f_0[\alpha i, \tau i] = 1$, $f_1[\alpha i, \tau i] = \lambda \tau i$. According to Lemma 3.4 a $\beta \in \mathscr{A}$ exists such that $[\alpha i] \stackrel{\rightarrow}{=} [B] \to \tau i$. By this and by unambiguity of vg we have $\beta = vg$ and (2.6.1) holds.

Case II. Either $xi < f_0[A, t]$ or $f_1[A, t] < x(i + 1) - 1$. In this case (3.2.2c) implies $[\alpha j, \tau i] \in M$ and $f_s g = f_s[\alpha i, \tau i] + xi - 1$ for $s = 0, 1$. Moreover, $vg = v[\alpha i, \tau i]$ according to (1) and Lemma 3.4 and hence (2.6.2) holds. The transformation ϱ is defined similarly as in Theorem 2.6 and therefore ϱ is a (5)-reducing transformation.

The following two theorems give sufficient conditions for a strongly recognizable set to be isolable.

4.3. Theorem. *Let $\mathscr{A}$ be an (f_0, f, f_1)-recognizable subset of $\mathbf{d}\mathscr{L}$. Let*

(1) *for each $g \in \mathbf{d}f$, $g = [A, t]$, a unique $vg \in \mathscr{A}$ exist such that*

$$[A] \stackrel{\rightarrow}{=} t^{(1, f_0 g - 1)} \times [vg] \times t^{(f_1 g + 1, \lambda t)}, \quad [vg] \to t^{(f_0 g, f_1 g)} .$$

Then $\mathscr{A}$ is an isolable set.

Proof. If V, R and ϱ are transformations defined as in the preceding lemma, then ϱ is a (5)-reducing transformation and, according to (1), (3.2.1) and (3.2.3),

(2) $\varrho g \neq g$ if and only if $g \in N = \{\bar{g}; \bar{g} \in \mathbf{d}f$ and either $\bar{g}1 \notin \mathscr{A}$ or $f_0 \bar{g} \neq 1$ or $f_1 \bar{g} \neq \lambda \bar{g}2\}$.

Now let $g \in \mathbf{g}\mathscr{L}$, $g_1 \in Qg$ and $\varrho g_1 \neq g_1$. Then $g_1 \in \mathbf{d}f$ and hence $Qg \wedge \mathbf{d}f \neq \Lambda$. According to Lemma 3.1 this implies that either $g \in N$ or $[A_1] \to [A_2]$ for some $A_1, A_2 \in \mathscr{A}$. In the former case $\varrho g \neq g$ according to (2) and the second case is impossible according to (1). Hence (9.1.6), [2] holds and ϱ is a reducing transformation.

Moreover, if $g \in \mathbf{g}\mathscr{L}$, $g_1 \in Qg$, $g_1 1 \in \mathscr{A}$ then $Qg \wedge \mathbf{d}f \neq \Lambda$ and similarly as above we conclude that $\varrho g \neq g$. By Theorem 2.2, $\mathscr{A}$ is a ϱ-isolable set. This completes the proof.

4.4. Theorem. Let $\mathscr{A}$ be a strongly recognizable subset of $\mathbf{d}\mathscr{L}$. Let, for every $A_1, A_2 \in \mathscr{A}$, $A_1 \neq A_2$ imply $\mathbf{t}(\mathscr{L}, A_1) \wedge \mathbf{t}(\mathscr{L}, A_2) = \Lambda$. Then $\mathscr{A}$ is an isolable set.

Proof. By Lemma 3.4 and Theorem 4.3.

5. WEAKLY ISOLABLE SETS

In this section relations are studied between structural unambiguity of a given language $\mathscr{L}$ and structural unambiguity of a language $\mathscr{L}_0$, which is constructed from $\mathscr{L}$ in such a way that in metatexts of $\mathscr{L}$ all symbols from a set $\mathscr{A} \subset \mathbf{d}\mathscr{L}$ are replaced by new terminal symbols. Especially, the case is investigated that in the construction of $\mathscr{L}_0$, two symbols A_1, A_2 from $\mathscr{A}$ are replaced by the same terminal symbol if and only if $\mathbf{t}(\mathscr{L}, A_1) \wedge \mathbf{t}(\mathscr{L}, A_2) \not\subset \{\Lambda\}$. If, in this last case $\mathscr{L}$ is s.u. if and only if so is $\mathscr{L}_0$, then we shall say that $\mathscr{A}$ is a weakly isolable set. We shall prove some sufficient conditions for a set $\mathscr{A}$ to be weakly isolable.

5.1. Definition. Let $\mathscr{L}$ be a language, $\Lambda \neq \mathscr{A} \subset \mathbf{d}\mathscr{L}$ and φ be a transformation defined on $\mathbf{a}\mathscr{L}$ in the following manner:

(1) $\varphi a = a$ if $a \notin \mathscr{A}$ and $\varphi a \notin \mathbf{a}\mathscr{L}$ if $a \in \mathscr{A}$.

Denote $\mathscr{L}^{\varphi}_{\mathscr{A}}$ and $\overline{\mathscr{L}}^{\varphi}_{\mathscr{A}}$ the languages defined as follows:

$$\mathbf{d}\mathscr{L}^{\varphi}_{\mathscr{A}} = \mathbf{d}\mathscr{L}\,, \quad \mathscr{L}^{\varphi}_{\mathscr{A}}A = \{\bar{\varphi}\alpha;\, \alpha \in \mathscr{L}A\}\,, \quad \mathbf{d}\overline{\mathscr{L}}^{\varphi}_{\mathscr{A}} = \mathbf{d}\mathscr{L} \cup \{\varphi A;\, A \in \mathscr{A}\}$$

where

$$\bar{\varphi}t = \prod_{i=1}^{\lambda t} \varphi t i \quad \text{for any} \quad t \in \boldsymbol{\sigma}\mathscr{L}$$

and

$$\overline{\mathscr{L}}^{\varphi}_{\mathscr{A}}A = \mathscr{L}A \text{ if } A \in \mathbf{d}\mathscr{L} - \mathscr{A};\ \overline{\mathscr{L}}^{\varphi}_{\mathscr{A}}A = \{[\varphi A]\} \text{ if } A \in \mathscr{A},$$

$$\overline{\mathscr{L}}^{\varphi}_{\mathscr{A}}A = \bigcup\{\mathscr{L}B;\, \varphi B = \mathrm{A}\} \text{ if } A \in \{\varphi B;\, B \in \mathscr{A}\}\,.$$

A set $\mathscr{A}$ is said to be φ-reducible if either both languages $\mathscr{L}^{\varphi}_{\mathscr{A}}$ and $\mathscr{L}$ are s.u. or both are not s.u. If a set $\mathscr{A}$ is φ-reducible for some φ satisfying

(2) $A_1, A_2 \in \mathscr{A}$, $\varphi A_1 = \varphi A_2$ if and only if $\mathbf{t}(\mathscr{L}_1, A_1) \wedge \mathbf{t}(\mathscr{L}, A_2) \not\subset \{\Lambda\}$

then we say that $\mathscr{A}$ is weakly isolable.

5.2. Remark. If a set $\mathscr{A}$ is isolable then it is φ-reducible for any one-to-one transformation φ such that (5.1.1) holds. (According to Def. 9.7, [2] and Theorem 9.11, [2].).

5.3. Theorem. *If*

(1) $\mathbf{t}(\mathscr{L}, A_1) \wedge \mathbf{t}(\mathscr{L}, A_2) \subset \{A\}$ *for no* $A_1, A_2 \in \mathscr{A}$ *such that* $\varphi A_1 = \varphi A_2$,

and the language $\mathscr{L}$ *is s.u. then so is* $\mathscr{L}^{\varphi}_{\mathscr{A}}$.

Proof. Put $\mathscr{L}_0 = \mathscr{L}^{\varphi}_{\mathscr{A}}$. First we shall prove the following assertion:

(2) If $g = [A, t] \in \mathbf{g}\mathscr{L}_0$, $[\alpha, \tau] \in S_{\mathscr{L}_0}g$, then there exist β, ξ and u such that $t = \bar{\varphi}u$, $\tau = \tilde{\varphi}\xi$ [2]), $[A, u] \in \mathbf{g}\mathscr{L}$, $[\beta, \xi] \in S_{\mathscr{L}}[A, u]$ and either $\alpha = [A] = \beta$ or $[A] \neq \alpha = \bar{\varphi}\beta$, $\beta \in \mathscr{L}A$.

If $[[A], [t]] \in S_{\mathscr{L}_0}[A, t]$, then a $u \in \mathscr{L}A$ exists such that $t = \bar{\varphi}u$ and [2] holds with $\beta = [A]$, $\xi = [u]$. Thus, the assertion (2) holds if $Qg = A$. Now let g be such that (2) holds for all $g_1 \in Qg$. By the above it is sufficient to suppose $Qg \neq A$ and $[\alpha, \tau] \in \bar{S}_{\mathscr{L}_0}g$. If $i \in \mathbf{d}\alpha$ and $\mathscr{L}_0$: $[\alpha i] \to \tau i$, then by inductive assumption a t_i exists such that $\mathscr{L}$: $[\alpha i] \to t_i$, $\bar{\varphi}t_i = \tau i$. For these i we have $\alpha i \notin \{\varphi A; A \in \mathscr{A}\}$ and hence choosing $\beta \in \mathscr{L}A$ in such a way that $\bar{\varphi}\beta = \alpha$ we have $\mathscr{L}$: $[\beta i] \to t_i$. Putting $\xi i = t_i$ if $[\alpha i] \neq \tau i$; $\xi i = \beta i$ for all other $i \in \mathbf{d}\alpha$ and $u = \prod \xi$ we get $\bar{\varphi}u = t$, $\tilde{\varphi}\xi = \tau$ and $[\beta, \xi] \in \bar{S}_{\mathscr{L}}[A, u]$. Application of Theorem 2.5 yields that (2) holds for all $g \in \mathbf{g}\mathscr{L}$.

Now suppose that there is an $[A, t] \in \mathbf{g}\mathscr{L}_0$ with two different structures $[\alpha_i, \tau_i]$ in $\mathscr{L}_0$. By (2) there are β_i, ξ_i and u_i such that $t = \bar{\varphi}u_i$, $\tau_i = \tilde{\varphi}\xi_i$, $[\beta_i, \xi_i] \in S_{\mathscr{L}}[A, u_i]$ and either $\alpha_i = [A] = \beta_i$ or $[A] \neq \alpha_i = \bar{\varphi}\beta_i$, $\beta_i \in \mathscr{L}A$. Obviously $\beta_i = [A]$ is not true for both i. Since $[\alpha_1, \tau_1] \neq [\alpha_2, \tau_2]$, we conclude from (2) that either $\beta_1 \neq \beta_2$ or $\iota\xi_1 \neq \iota\xi_2$. Thus, $[A, u_i] \in \mathbf{g}_a\mathscr{L}$ if $u_1 = u_2$. Now suppose that $u_1 \neq u_2$. For each j such that $u_1 j \neq u_2 j$ we have $u_1 j \in \mathscr{A}$, $\varphi u_1 j = \varphi u_2 j$ and we may choose a non-empty text t_j such that $\mathscr{L}$: $u_i j \to t_j$. Put $\zeta j = t_j$ for such j and $\zeta j = [u_i j]$ otherwise; $u = \prod \zeta$. Clearly $\xi_i \otimes \zeta$ is an β_i-decomposition of u in $\mathscr{L}$ and $[A, u] \in \mathbf{g}\mathscr{L}$. If $\beta_i \neq [A]$, then $[\beta_i, \xi_i \otimes \zeta] \in S_{\mathscr{L}}[A, u]$; otherwise $[u_i, \zeta] \in S_{\mathscr{L}}[A, u]$. If $\beta_i \neq [A]$ for both $i = 1, 2$, then $[\beta_1, \xi_1 \otimes \zeta] \neq [\beta_2, \xi_2 \otimes \zeta]$. If $\beta_1 = [A]$, $\beta_2 \neq [A]$ and $[A, u] \in \mathbf{g}_u\mathscr{L}$, then $[u_1, \zeta] = [\beta_2, \xi_2 \otimes \zeta]$. This implies $t = \bar{\varphi}u_1 = \bar{\varphi}\beta_2 = \alpha_2$, and since $\beta_2 \neq [A]$, $\mathscr{L}_0$: $[A] \Rightarrow \alpha_2 \to t = \alpha_2$. But this is impossible as $\mathscr{L}_0$ is an non-cyclic language. (The last assertion is a consequence of the fact that $\mathscr{L}_0$: $[B] \to t \in \boldsymbol{\sigma}\mathscr{L}$ implies $\mathscr{L}$: $[B] \to t$.) Similarly, the case $\beta_1 \neq [A]$, $\beta_2 = [A]$ is impossible. Hence $[A, u] \in \mathbf{g}_a\mathscr{L}$ and Theorem is proved.

5.4. Theorem. *If the set* $\{\varphi A; A \in \mathscr{A}\}$ *is isolable in* $\overline{\mathscr{L}^{\varphi}_{\mathscr{A}}}$ *and*

(1) $\bar{\varphi}\alpha_1 \neq \bar{\varphi}\alpha_2$ *if* $A \in \mathbf{d}\mathscr{L}$, $\alpha_1, \alpha_2 \in \mathscr{L}A$, $\alpha_1 \neq \alpha_2$,

then $\mathscr{L}$ *is s.u. if so is* $\mathscr{L}^{\varphi}_{\mathscr{A}}$.

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[2]) If τ is a decomposition then $\tilde{\varphi}\tau$ is defined as follows: $\mathbf{d}\tilde{\varphi}\tau = \mathbf{d}\tau$, $(\tilde{\varphi}\tau)\, i = \varphi\tau i$ for each $i \in \mathbf{d}\tilde{\varphi}\tau$.

Proof. Put $\mathscr{L}_1 = \overline{\mathscr{L}^{\varphi}_{\mathscr{A}}}$, $\mathscr{L}_0 = \mathscr{L}^{\varphi}_{\mathscr{A}}$, $N = \{\varphi A; A \in \mathscr{A}\}$. Let ϱ be the reducing transformation for $\mathscr{L}_1$ which guarantees the isolability of N, i.e., by Theorem 2.2, $\varrho g \neq g$ if $g \in \mathbf{g}\mathscr{L}_1$, $g_1 \in Qg$, $g_1 1 \in N$. Note that $\mathscr{L}: t_1 \to t_2$ implies $\mathscr{L}_1: t_1 \to t_2$ and $\mathscr{L}: [A] \Rightarrow \alpha$ implies $\mathscr{L}_1: [A] \Rightarrow \alpha$ if $A \notin \mathscr{A}$; if $A \in \mathscr{A}$, then $\mathscr{L}: [A] \Rightarrow \alpha$ implies $\mathscr{L}_1: [\varphi A] \Rightarrow \alpha$.

Let $g \in \mathbf{g}\mathscr{L}$ have two different structures $[\alpha_1, \tau_1]$ and $[\alpha_2, \tau_2]$ in $\mathscr{L}$. Put

$$\bar{g} = \begin{cases} g = [A, t] & \text{if } A \notin \mathscr{A} \\ [\varphi A, t] & \text{if } A \in \mathscr{A} \end{cases}$$

and

$$\bar{\alpha}_i = \begin{cases} [\varphi A] & \text{if } \alpha_i = [A],\ A \in \mathscr{A} \\ \alpha_i & \text{otherwise.} \end{cases}$$

Then $[\bar{\alpha}_i, \tau_i]$ are two different structures of $\bar{g}$ in $\mathscr{L}_1$. They are diffferent because evidently $\bar{\alpha}_1 = \bar{\alpha}_2$ only if $\alpha_1 = \alpha_2$.

Now suppose that $\bar{g}$ is ϱ-invariant. By Theorem 4.3, [4] (or Lemmas 9.9 and 9.10, [2]), we get, for every i, that either $\bar{\alpha}_i = [\varphi A]$ or $\mathscr{L}_2: [\alpha_i j] \overset{\rightarrow}{=} \tau_i j$ for each $j \in \mathbf{d}\alpha_i$ and $\mathscr{L}_2 = \mathscr{L}_{1_{\mathbf{d}\mathscr{L}_1 - N}}$. The language $\mathscr{L}_2$ is thus defined on $\mathbf{d}\mathscr{L}$; the only text derivable in $\mathscr{L}_2$ from an $A \in \mathscr{A}$ is $[\varphi A]$. Hence and by definition of $\mathscr{L}_0$ we get

(2) $\mathscr{L}_2: [A] \to t_2$ implies $\mathscr{L}_0: [\varphi A] \overset{\rightarrow}{=} \bar{\varphi} t_2$. [3])

Thus, if $\bar{\alpha}_i \neq [\varphi A]$, then

(3) $\mathscr{L}_0: [\varphi \alpha_i j] \overset{\rightarrow}{=} \bar{\varphi} \tau_i j$ for all $j \in \mathbf{d}\alpha_i$.

Put $g_0 = [A, \bar{\varphi} t]$. Fix an i. If $\bar{\alpha}_i = [\bar{A}]$, then $\alpha_i = [A]$, $t \in \mathscr{L}A$, $\bar{\varphi} t \in \mathscr{L}_0 A$ and g_0 has in $\mathscr{L}_0$ the structure $[[A], [\bar{\varphi} t]]$. If $\bar{\alpha}_i \neq [\bar{A}]$, then $\bar{\alpha}_i = \alpha_i \in \mathscr{L}A$ and, by (3), $[\bar{\varphi}\alpha_i, \tilde{\varphi}\tau_i] \in S_{\mathscr{L}_0} g_0$ and $\bar{\varphi}\alpha_i \neq [A]$. Of course we have not $\bar{\alpha}_i = [\bar{A}]$ for both i and either $g_0 \in \mathbf{g}_a\mathscr{L}_0$ or $\alpha_1, \alpha_2 \in \mathscr{L}A$, $\bar{\varphi}\alpha_1 = \bar{\varphi}\alpha_2$, $\tilde{\varphi}\tau_1 = \tilde{\varphi}\tau_2$. If $\tau_1 \neq \tau_2$, then $\tilde{\varphi}\tau_1 \neq \tilde{\varphi}\tau_2$ and $g_0 \in \mathbf{g}_a\mathscr{L}_0$. If $\tau_1 = \tau_2$, then $\alpha_1 \neq \alpha_2$ and hence, by (1), $\bar{\varphi}\alpha_1 \neq \bar{\varphi}\alpha_2$. Thus $g_0 \in \mathbf{g}_a\mathscr{L}_0$ in all cases.

However it remains the case in which $\bar{g}$ is not ϱ-invariant. Since $\bar{g} \in \mathbf{g}_a\mathscr{L}_1$, we have, by (9.1.3), [2] and by Lemma 9.3, [2], $t \notin \mathscr{L}_1\bar{A}$ and therefore $\mathscr{L}_1: [\bar{A}] \Rightarrow \bar{\alpha}_i = \alpha_i$. Let k be the smallest integer such that $\varrho^k\bar{g}$ is a ϱ-invariant grammatical element. Such k exists by (9.1.5), [2] and by Lemma 9.3, [2]. Similarly as in the proof of Lemma 9·3, [2] we can deduce, for $i = 1, 2, \ldots, k - 1$, that $\varrho^i\bar{g}$ has, in $\mathscr{L}_1$, two different structures

(4) $[\alpha_1, \xi_1^i], [\alpha_2, \xi_2^i]$

[3]) We put $\varphi A = A$ if $A \in N$.

and the grammatical element $\varrho^k \bar{g}$ has two different structures either with form (4) or one of the following forms:

(5) $[[\bar{A}], [(\varrho^k \bar{g})\, 2]], [\alpha_1, \xi_1^k], \alpha_2 = (\varrho^k \bar{g})\, 2$,
(6) $[[\bar{A}], [(\varrho^k \bar{g})\, 2]], [\alpha_2, \xi_2^k], \alpha_1 = (\varrho^k \bar{g})\, 2$.

Moreover $(\varrho^k \bar{g})\, 1 = A$. Similarly as in the case that $\bar{g}$ is ϱ-invariant we conclude that $[A, \bar{\varphi}(\varrho^k \bar{g})\, 2] \in \mathbf{g}_a \mathscr{L}_0$. This completes the proof of the Theorem .

5.5. Corollary. *If the set $\{\varphi A; A \in \mathscr{A}\}$ is isolable in $\overline{\mathscr{L}}{}^{\varphi}_{\mathscr{A}}$ and conditions (5.3.1) and (5.4.1) are satisfied, then $\mathscr{A}$ is a φ-reducible set.*

5.6. Theorem. *Let $\mathscr{A}$ be an (f_0, f, f_1)-recognizable subset of $\mathbf{d}\mathscr{L}$. Let φ be a transformation satisfying* (5.1.1). *Let* (5.4.1) *and the following three conditions hold:*

(1) *$[A_1] \to [A_2]$ for no $A_1, A_2 \in \mathscr{A}$.*
(2) *If $g_1 = [A, t_1]$, $g_2 = [A, t_2]$, $g_1 \in \mathbf{d}f$, $\bar{\varphi} t_1 = \bar{\varphi} t_2$, then $fg_1 = fg_2$ and $f_s g_1 = f_s g_2$ for $s = 0,1$.*
(3) *If $A_1, A_2 \in \mathscr{A}$, then $\varphi A_1 = \varphi A_2$ if and only if $\mathbf{t}(\mathscr{L}, A_1) \wedge \mathbf{t}(\mathscr{L}, A_2) \notin \{A\}$.*

Then $\mathscr{A}$ is a weakly isolable set.

Proof. Put $\mathscr{L}_0 = \mathscr{L}^{\varphi}_{\mathscr{A}}$. For $g = [A, t] \in \mathbf{d}f$ denote vg the set $\{g_1$; there exists a $B \in \mathscr{A}$ such that

$$[A] = [g_1 1] \underset{=}{\overset{\to}{}} t^{(1, f_0 g - 1)} \times [B] \times t^{(f_1 g + 1, \lambda t)} = g_1 2 \to t, [B] \to t^{(f_0 g, f_1 g)} .$$

By Lemma 3.4 we have $vg \neq A$ if $g \in \mathbf{d}f$. Moreover it is obvious that $vg \subset \mathbf{g}\mathscr{L}$ if $g \in \mathbf{d}f$ and either $A \notin \mathscr{A}$ or $f_0 g \neq 1$ or $f_1 g \neq \lambda t$.

Let us state two assumptions:

(4) $g = [A, t] \in \mathbf{g}\mathscr{L}$ and either $g \notin \mathbf{d}f$ or: $A \in \mathscr{A}$, $f_0 g = 1$, $f_1 g = \lambda t$.
(5) $g_1 = [A, t_1]$, $g_2 = [A, t_2]$, $g_1 \neq g_2$, $\bar{\varphi} t_1 = \bar{\varphi} t_2$.

Now we shall prove the following assertion:

(6) If (4) holds, then $\bar{g} = [A, \bar{\varphi} t] \in \mathbf{g}\mathscr{L}_0$; moreover, if $[\alpha, \tau] \in S_{\mathscr{L}} g$, then $[\alpha, \tau] = [[A], [t]]$ implies $[[A], [\bar{\varphi} t]] \in S_{\mathscr{L}_0} g$ and $[\alpha, \tau] \in \bar{S}_{\mathscr{L}} g$ implies $[\bar{\varphi}\alpha, \tilde{\varphi}\tau] \in \bar{S}_{\mathscr{L}_0} g$.

Proof of the assertion (6). Let (4) hold and let g be such that (6) holds for all $g_1 \in Qg$. Directly from the definition of $\mathscr{L}_0$ it follows that (6) holds if $[\alpha, \tau] = [[A], [t]]$. Now let $[\alpha, \tau] \in \bar{S}_{\mathscr{L}} g$. If $\mathscr{L}$: $[\alpha i] \to \tau i$ and $i \in \mathbf{d}\alpha$, then, by (1) and Lemma 4.1, we have $[\alpha i, \tau i] \notin \mathbf{d}f$. Hence and by inductive assumption $[\alpha i, \bar{\varphi}\tau i] \in \mathbf{g}\mathscr{L}_0$, Since $\alpha i \notin \mathscr{A}$, we have $\varphi \alpha i = \alpha i$ and hence $[\varphi \alpha i, \bar{\varphi}\tau i] \in \mathbf{g}\mathscr{L}_0$, too. Moreover $[A, \bar{\varphi}\alpha] \in \mathbf{g}\mathscr{L}_0$ and hence $[A, \bar{\varphi} t] \in \mathbf{g}\mathscr{L}_0$, $[\bar{\varphi}\alpha, \tilde{\varphi}\tau] \in \bar{S}_{\mathscr{L}_0}[A, \bar{\varphi} t]$. According to Theorem 2.5 this completes the proof of (6).

As the next step we shall prove the assertion:

(7) If (5) holds, if $g_1 \in \mathbf{d}f$ and either $A \notin \mathscr{A}$ or $f_0 g_1 \neq 1$ or $f_1 g_1 \neq \lambda t_1$, then there exist $g_3 = [B, t_3]$ and $g_4 = [B, t_4]$ such that $g_3, g_4 \in \mathbf{g}\mathscr{L}$, $g_3 \neq g_4$, $\bar{\varphi} t_3 = \bar{\varphi} t_4$, $\mu g_3 < \mu g_1$ and $\mu g_4 < \mu g_2$.

Proof of the assertion (7). If the assumptions of this assertion are satisfied, then, by (2), $g_2 \in \mathbf{d}f$ and either $A \notin \mathscr{A}$ or $f_0 g_2 \neq 1$ or $f_1 g_2 \neq \lambda t_2$. Thus νg_1 and νg_2 are non-empty subsets of $\mathbf{g}\mathscr{L}$ and therefore $\bar{g}_1$ and $\bar{g}_2$ exist such that $\bar{g}_1 \in \nu g_1$, $\bar{g}_2 \in \nu g_2$ and
$\bar{g}_1 = [A, t_1^{(1, f_0 g_1 - 1)} \times [C_1] \times t_1^{(f_1 g_1 + 1, \lambda t_1)}]$, $\bar{g}_2 = [A, t_2^{(1, f_0 g_2 - 1)} \times [C_2] \times t_2^{(f_1 g_1 + 1, \lambda t_2)}]$, where $C_1, C_2 \in \mathscr{A}$, $[C_i] \to u_i = t^{(f_0 g_1, f_1 g_1)}$. Obviously $\bar{\varphi} u_1 = \bar{\varphi} u_2$. If $u_1 j \neq u_2 j$ for some $j \in \mathbf{d} u_1$, then $u_i j \in \mathscr{A}$, $\varphi u_1 j = \varphi u_2 j$. According to (3), a $\bar{t}_j$ exists such that $\Lambda \neq \bar{t}_j$ and $\mathscr{L}: [u_i j] \to \bar{t}_j$. Let us define the decomposition ξ as follows: $\xi j = \bar{t}_j$ if $u_1 j \neq u_2 j$; $\xi j = [u_i j]$ otherwise. Put $u = \Pi \xi$. Obviously $\mathscr{L}: u_i \stackrel{\rightarrow}{=} u$ and we have $\varphi C_1 = \varphi C_2$ according to (3). Hence the assertion (7) is satisfied with $g_3 = \bar{g}_1$ and $g_4 = \bar{g}_2$ if $\bar{g}_1 \neq \bar{g}_2$ and with $g_3 = [C_1, u_1]$ and $g_4 = [C_2, u_2]$ if $\bar{g}_1 = \bar{g}_2$. This completes the proof of (7).

Now we shall prove that $\mathbf{g}_a \mathscr{L} \neq \Lambda$ implies $\mathbf{g}_a \mathscr{L}_0 \neq \Lambda$. Suppose that $\mathbf{g}_a \mathscr{L} \neq \Lambda$. Then the smallest integer n exists such that one of the following conditions holds:

(8) There exists a $g = [A, t] \in \mathbf{g}_a \mathscr{L}$ such that $\mu g < n$.

(9) There exist $g_1, g_2 \in \mathbf{g}\mathscr{L}$ such that (5) holds and $\max \{\mu g_1, \mu g_2\} = n$.

First suppose that (8) holds. Then (4) holds, too. Indeed, suppose that (4) does not hold. Then $\Lambda \neq \nu g \subset \mathbf{g}\mathscr{L}$ and the set νg contains either exactly one element, for example g_1, or at least two elements, for example g_1 and g_2. In the first case $\mu g_1 < \mu g$ and, moreover, by Lemma 4.2 and Theorem 9.3, [2], $g_1 \in \mathbf{g}_a \mathscr{L}$ which contradicts the choice of n. In the second case $\max \{\mu g_1, \mu g_2\} < n$, g_1 and g_2 satisfy condition (5) and we have again a contradiction with the choice of n. Hence (4) holds and we have $\mathbf{g}_a \mathscr{L}_0 \neq \Lambda$ according to (6) and (5.4.1).

Secondly suppose that (9) holds. Since n is the smallest integer such that (9) holds, we get, by (7), that either $g_1 \notin \mathbf{d}f$ or $A \in \mathscr{A}, f_0 g_1 = 1, f_1 g_1 = \lambda t_1$. In the former case $g_2 \notin \mathbf{d}f$ and in the latter one $A \in \mathscr{A}, f_0 g_2 = 1, f_1 g_2 = \lambda t_2$. If we denote $\bar{g} = [A, \bar{\varphi} t_1]$, then (6) implies $\bar{g} \in \mathbf{g}\mathscr{L}_0$. Now we shall prove that $\bar{g} \in \mathbf{g}_a \mathscr{L}_0$. According to (5.4.1) we have that either $\mathscr{L}: [A] \not\Rightarrow t_1$ or $\mathscr{L}: [A] \not\Rightarrow t_2$. If either $\mathscr{L}: [A] \Rightarrow t_1$ and $\mathscr{L}: [A] \not\Rightarrow t_2$ or $\mathscr{L}: [A] \not\Rightarrow t_1$ and $\mathscr{L}: [A] \Rightarrow t_2$, then $\bar{g} \in \mathbf{g}_a \mathscr{L}_0$ according to (6). Now suppose that $[\alpha_1, \tau_1] \in \bar{S}_{\mathscr{L}} g_1$ and $[\alpha_2, \tau_2] \in \bar{S}_{\mathscr{L}} g_2$. Then $[\bar{\varphi}\alpha_1, \tilde{\varphi}\tau_1] \in S_{\mathscr{L}_0} g_0$ and $[\bar{\varphi}\alpha_2, \tilde{\varphi}\tau_2] \in S_{\mathscr{L}_0} g_0$ according to (6). If either $\alpha_1 \neq \alpha_2$ or $\iota\tau_1 \neq \iota\tau_2$ then $[\bar{\varphi}\alpha_1, \tilde{\varphi}\tau_1] \neq [\bar{\varphi}\alpha_2, \tilde{\varphi}\tau_2]$ according to (5.4.1) and we have $g_0 \in \mathbf{g}_a \mathscr{L}_0$. If $\alpha_1 = \alpha_2$ and $\iota\tau_1 = \iota\tau_2$, then an $i_0 \in \mathbf{d}\alpha_1$ exists such that $\tau_1 i_0 \neq \tau_2 i_0$, $\bar{\varphi}\tau_1 i_0 = \bar{\varphi}\tau_2 i_0$. According to (1) we have $\mathscr{L}: [\alpha_1 i_0] \to \tau_1 i_0$ and $\mathscr{L}: [\alpha_2 i_0] \to \tau_2 i_0$. Obviously $\max \{\mu[\alpha_1 i_0, \tau_1 i_0], \mu[\alpha_2 i_0, \tau_2 i_0]\} < n$ which contradicts the choice of n, Thus, the case $\alpha_1 = \alpha_2$, $\iota\tau_1 = \iota\tau_2$ is impossible.

This finishes the proof of the assertion: $\mathbf{g}_a\mathscr{L} \neq \Lambda$ implies $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$. From Theorem 5.3 we conclude that $\mathbf{g}_a\mathscr{L}_0 \neq \Lambda$ implies $\mathbf{g}_a\mathscr{L} \neq \Lambda$. This completes the proof of the Theorem.

The following Lemma, very often used in the paper [5], gives sufficient conditions for a set $\mathscr{A}$ to be strongly recognizable (weakly isolable) in such a case that, roughly speaking, the beginnings of all texts from $\mathbf{t}(\mathscr{L}, A)$, $A \in \mathscr{A}$ are characterized by special terminal symbols.

5.7. Lemma. *Let* $\Lambda \neq \mathscr{A} \subset \mathbf{d}\mathscr{L}$, $\Lambda \notin \mathscr{L}A$ *for* $A \in \mathscr{A}$, $Q = \{\alpha 1, \alpha \in \mathscr{L}A, A \in \mathscr{A}\} \subset \subset \mathbf{a}_t\mathscr{L}$. Let

(1) $A \in \mathbf{d}\mathscr{L}$, $[A] \Rightarrow \alpha$, $\alpha i = q \in Q$ *implies* $A \in \mathscr{A}$, $i = 1$.
For each $q \in Q$ *let* $\mathscr{B}_q = \{\alpha; [A] \Rightarrow \alpha,\ \alpha 1 = q\}$, $\mathscr{A}_q = \{A; [A] \Rightarrow \alpha \in \mathscr{B}_q\}$, $\mathscr{L}_1 = \mathscr{L}_{\mathbf{d}\mathscr{L} - \mathscr{A}}$.

If for each $q \in Q$ *there holds at least one of the conditions*:

(2) $\mathbf{symb}\, \{u^{(1,\lambda u - 1)}; \mathscr{L}_1 \colon \alpha \overset{\rightarrow}{\underset{=}{\supset}} u, \alpha \in \mathscr{B}_g\} \wedge \mathbf{symb}_e\, \{u; \mathscr{L}_1 \colon \alpha \overset{\rightarrow}{\underset{=}{\supset}} u, \alpha \in \mathscr{B}_q\} = \Lambda$,

(3) $\mathbf{symb}\, \{u^{(2,\lambda u)}; \mathscr{L}_1 \colon \alpha \overset{\rightarrow}{\underset{=}{\supset}} u, \alpha \in \mathscr{B}_g\} \wedge \mathbf{rdel}\, \mathscr{A}_q = \Lambda$

then $\mathscr{A}$ *is a strongly recognizable set.*

If for each $q \in Q$ *either conditions* (2), (4) *and* (6) *or conditions* (3), (5) *and* (6) *hold, where*

(4) $\mathbf{symb}_e\, \{u; \mathscr{L}_1 \colon \alpha \overset{\rightarrow}{\underset{=}{\supset}} u, \alpha \in \mathscr{B}_q\} \subset \mathbf{a}_t\mathscr{L}$;

(5) *if* $A_1, A_2 \in \mathscr{A}$, $\mathbf{t}(\mathscr{L}, A_1) \wedge \mathbf{t}(\mathscr{L}, A_2) \neq \Lambda$, $A_1 \in \mathbf{rdel}\, \mathscr{A}_q$, *then* $A_2 \notin \mathbf{symb}\, \{u^{(2,\lambda u)}; \mathscr{L}_1 \colon \alpha \overset{\rightarrow}{\underset{=}{\supset}} u, \alpha \in \mathscr{B}_q\}$;

(6) *if* $B \in \mathbf{d}\mathscr{L}$, $\alpha_1, \alpha_2 \in \mathscr{L}B$, $\alpha_1 \neq \alpha_2$, $\lambda\alpha_1 = \lambda\alpha_2$ *then an* $i \in \mathbf{d}\alpha_1$ *exists such that* $\alpha_1 i \neq \alpha_2 i$ *and either* $\{\alpha_1 i, \alpha_2 i\} \not\subset \mathscr{A}$ or $\mathbf{t}(\mathscr{L}, \alpha_1 i) \wedge \mathbf{t}(\mathscr{L}, \alpha_2 i) = \Lambda$,

then $\mathscr{A}$ *is a weakly isolable set.*

Proof. Let f be the function defined as follows:

(7) $\mathbf{d}f = \{[A, t]; i \in \mathbf{d}t, ti \in Q\}$, $f[A, t] = \max\{i, ti \in Q\}$.

We shall show that $\mathscr{A}$ is f-recognizable. (3.1.1) follows from (1). Now suppose that (3.1.2a) holds. If $[\alpha j] = \tau j$, then (1) implies $j = 1 = f[A, \alpha]$ and (3.1.2b) holds. If $[\alpha j] \neq \tau j$, then (3.1.2c) follows directly from the definition of f. If $g_1 \in Qg$, $g \in \mathbf{g}\mathscr{L}$ and $g_1 \in \mathbf{d}f$, then an $i \in \mathbf{d}t$ exists such that $ti \in Q$; therefore $g \in \mathbf{d}f$ and (3.1.3) holds, too. Thus, $\mathscr{A}$ is an f-recognizable set. In order to prove that $\mathscr{A}$ is a strongly recognizable set we shall use Lemma 3.5. (3.5.2) holds trivially as $B_q = \{[q]\}$ for each $g \in Q$ Now suppose that there are q, u_1 u_2 and t such that

(8) $u_1, u_2 \in E_q$, $u_2 = u_1 \times t$, $t \neq \Lambda$, $t1 \in \mathbf{rdel}\, \mathscr{A}_q$.

Then $u_2(\lambda u_1) \in \mathbf{symb}_e \{u; \mathscr{L}_1: \alpha \overset{\to}{\geqq} u, \alpha \in \mathscr{B}_q\}$ and $u_2(\lambda u_1 + 1) \in \mathbf{rdel}\, \mathscr{A}_q$. But this is not possible since $u_2(\lambda u_1) \notin \mathbf{symb}_e\{u; \mathscr{L}_1: \alpha \overset{\to}{\geqq} u, \alpha \in \mathscr{B}_q\}$ if (2) holds for our q and $u_2(\lambda u_1 + 1) \notin \mathbf{rdel}\, \mathscr{A}_q$ if (3) holds. Thus, (3.5.1) is satisfied and, by Lemma 3.5, $\mathscr{A}$ is an (f_0, f, f_1)-recognizable set if the functions f_0 and f_1 are defined in such a way as in the proof of Lemma 3.5, i.e., by (3.5.5) and (3.5.6), $f_0 = f$ and

(9) $f_1 g = \min \{i, i \geqq fg$ and either $i = \lambda g2$ or $(g2)\, i \in \mathbf{symb}_e \{u; \mathscr{L}_1: \alpha \overset{\to}{\geqq} u, \alpha \in \mathscr{B}_q\}\}$

if (2) holds for $(g2)\, fg$ and

(10) $f_1 g = \min \{i; i \geqq fg$ and either $i = \lambda g2$ or $(g2)\,(i + 1) \in \mathbf{rdel}\, \mathscr{A}_q\}$

if (3) holds for $(g2)\, fg$.

In proving the second assertion of Lemma we shall use Theorem 5.6. By the preceding it is obvious that $\mathscr{A}$ is an (f_0, f, f_1)-recognizable subset of $\mathbf{d}\mathscr{L}$ where the functions f_0, f and f_1 are defined as in the proof of first assertion of Lemma. Let φ be any transformation satisfying (5.1.1) and (5.1.2). Then (5.4.1) follows from (6). Since $\alpha 1 \in \mathbf{a}_t\mathscr{L}$ for each $A \in \mathscr{A}$, $\alpha \in \mathscr{L}A$ we have $[A_1] \nleftrightarrow [A_2]$ if $A_1, A_2 \in \mathscr{A}$ and (5.6.1) holds. (5.6.3) follows from the choice of φ. Now we shall prove that the condition (5.6.2) is also satisfied. Let $g_1 = [A, t_1]$, $g_2 = [A, t_2]$, $\bar{\varphi} t_1 = \bar{\varphi} t_2$. From the definition of f it is easy to see that $fg_1 = fg_2$ if $g_1 \in \mathbf{d}f$. Since $f_0 = f$ we have also $f_0 g_1 = f_0 g_2$. If for $q = (g_1 2) fg_1$ conditions (2) and (4) [(3) and (5)] hold, then $f_1 g_1 = f_1 g_2$ according to (9) ((10)). This completes the proof of the second assertion of lemma.

Similarly we can prove the lemma which gives sufficient conditions for a set $\mathscr{A}$ to be strongly recognizable (weakly isolable) in such a case that the ends of all texts from $\mathbf{t}(\mathscr{L}, A)$, $A \in \mathscr{A}$ are characterized by special terminal symbols.

6. PARANTHESIZED SETS

6.1 Definition. A set $\mathscr{A} \subset \mathbf{d}\mathscr{L}$ is said to be paranthesized if there are two sets R and $L \subset \mathbf{a}_t\mathscr{L}$ such that $A \in \mathbf{d}\mathscr{L}$, $[A] \Rightarrow \alpha$, $\alpha i \in L(\in R)$ if and only if $A \in \mathscr{A}$, $i = 1$ $(= \lambda\alpha)$.

6.2 Theorem. *Let $\mathscr{A}$ be a paranthesized subset of $\mathbf{d}\mathscr{L}$ and let condition (5.7.6) be satisfied. Then $\mathscr{A}$ is weakly isolable.*

Proof. It is easy to see that conditions (5.7.1), (5.7.2) and (5.7.4) are satisfied for each $q \in Q = R$.

Acknowledgement: The author wishes to acknowledge VÁCLAV FABIAN, who made many suggestions which improved the presentation.

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Author's address: Bratislava—Patrónka, Dúbravská cesta, ČSSR (Ústav technickej kybernetiky SAV).

Резюме

ИЗОЛИРУЕМЫЕ И СЛАБО ИЗОЛИРУЕМЫЕ МНОЖЕСТВА

ЙОСЕФ ГРУСКА, (Jozef Gruska), Братислава

В работе изучаются формальные языка, определенные в работе [2] В. Фабиана, которые являются обобщением Хомского грамматик типа 2. Выводится ряд достаточных условий для того чтобы данный язык $\mathscr{L}$ был структурно однозначен тогда и только тогда если структурно однозначен другой язык $\mathscr{L}_0$, который получается из $\mathscr{L}$ таким образом, что во всех метатекстах языка $\mathscr{L}$ (в тех текстах, которыми можно непосредственно заменить нетерминальный символ) все символы из некоторого множества $\mathscr{A}$ нетерминальных символов заменяются новыми терминальными символами.