Josef Král

The Fredholm radius of an operator in potential theory. [Continuation]

Czechoslovak Mathematical Journal, Vol. 15 (1965), No. 4, 565–588

Persistent URL: http://dml.cz/dmlcz/100695

THE FREDHOLM RADIUS OF AN OPERATOR IN POTENTIAL THEORY

JOSEF KRÁL, Praha

(Received December 12, 1965)

(Continuation)

§ 2

Throughout this paragraph we shall keep the notation and the assumptions introduced in § 1. The distance between two sets $A, B \subset E_2$ will be denoted by $\operatorname{dist}(A, B)$ $(= \inf\{|z - \zeta|; z \in A, \zeta \in B\})$. For $z \in E_2$ we write $\operatorname{dist}(z, B)$ instead of $\operatorname{dist}(\{z\}, B)$. Let us start with a simple lemma which will be used later.

2.1. Lemma. *Let $A \subset E_2$ and suppose that* $\operatorname{dist}(A, K) > 0$. *Then all the functions*

$$\int_K \frac{F(\zeta)}{\zeta - z}\, d\zeta = \Lambda F(z)$$

with $F \in C(K)$, $\|F\| \leqq 1$ are uniformly bounded and equicontinuous (more precisely, they fulfil the Lipschitz condition with the same constant) in A.

Proof. Let $d = \operatorname{dist}(A, K)$. It is easily seen that

$$(F \in C(K),\ z \in A) \Rightarrow |\Lambda F(z)| \leqq d^{-1} . \lambda K . \|F\| ,$$

where λK is the length of K. Further we have for every $F \in C(K)$ and any pair z_1, z_2 of points in A

$$|\Lambda F(z_1) - \Lambda F(z_2)| \leqq \|F\| \max_{\zeta \in K} |(\zeta - z_1)^{-1} - (\zeta - z_2)^{-1}|\, \lambda K \leqq$$
$$\leqq \|F\| . |z_1 - z_2| . d^{-2} . \lambda K ,$$

which concludes the proof.

2.2. Notation. Given $M \subset K$ and a real-valued function F on M we put $N = \psi^{-1}(M) \cap \langle 0, 2k \rangle$ and define

$$\int_M \frac{F(\zeta)}{|\zeta - z|}\, d|\zeta - z| = \int_N \frac{F(\psi(t))}{|\psi(t) - z|}\, d|\psi(t) - z|$$

provided the Lebesgue-Stieltjes integral on the right-hand side exists.

Remark. The integral

$$\int_K \frac{F(\zeta)}{|\zeta - z|} \, d|\zeta - z| = \operatorname{Re} \int_K \frac{F(\zeta)}{\zeta - z} \, d\zeta \tag{59}$$

was called by N. I. MUSKHELISHVILI the modified logarithmic potential of the single distribution (cf. [4], chap. I, § 12). In this paragraph we shall examine some of its properties which will enable us in § 3 to establish the uniqueness and existence of solution of the modified Dirichlet problem for a sufficiently wide class of domains. Our main objective here is the proof of the following theorem which reminds of the PRIWALOW fundamental lemma as presented in [5], chap. III, § 2.

2.3. Theorem. *Let $\mathscr{F}K < \pi$ (cf. 1.15 for notation). Then, for $F \in C(K)$, the following conditions (A), (B) are equivalent to each other:*

(A) *The integral (59), considered as a function of the variable z, is uniformly continuous in a complementary domain of K.*

(B) *The integral*

$$\text{V.p.} \int_K \frac{F(\zeta)}{|\zeta - \eta|} \, d|\zeta - \eta| = \lim_{r \to 0+} \int_{K_r(\eta)} \frac{F(\zeta)}{|\zeta - \eta|} \, d|\zeta - \eta|, \tag{60}$$

where $K_r(\eta) = \{\zeta; \zeta \in K, |\zeta - \eta| > 2r\}$, converges uniformly in $\eta \in K$.

If (B) holds then the function MF defined by

$$MF(z) = \int_K \frac{F(\zeta)}{|\zeta - z|} \, d|\zeta - z|, \quad z \in E_2 - K, \tag{61}$$

$$MF(\eta) = \text{V.p.} \int_K \frac{F(\zeta)}{|\zeta - \eta|} \, d|\zeta - \eta|, \quad \eta \in K,$$

is continuous on the whole of E_2.

The proof of 2.3 is based on the following auxiliary results 2.4 and 2.5 whose proofs will be presented in 2.8 and 2.14.

2.4. Proposition. *Let $\mathscr{F}K < \pi$ and let D be a complementary domain of K. Then, for every $a \in K$, there is a $b(a) \in D$ such that $|b(a) - a|$ is constant on K and that, for the set H of all the couples $[a, b]$ with $a \in K$ and $b = a + \xi(b(a) - a)$, $0 < \xi \leqq$ $\leqq 1$, the following condition*

$$\inf \left\{ \frac{\text{dist}(b, K)}{|b - a|}; \; [a, b] \in H \right\} > 0 \tag{62}$$

is fulfilled.

2.5. Proposition. *Suppose that $\mathscr{F}K < \pi$ and let H be any set of couples $[a, b]$*

with $a \in K$ *and* $b \notin K$ *such that* (62) *holds. Given* $F \in C(K)$ *and* $[a, b] \in H$ *put* $K_{ab} = \{\zeta; \zeta \in K, |\zeta - a| > 2|b - a|\}$ *and define*

$$\Phi(a, b, F) = \int_K \frac{F(\zeta)}{|\zeta - b|} \, \mathrm{d}|\zeta - b| - \int_{K_{ab}} \frac{F(\zeta)}{|\zeta - a|} \, \mathrm{d}|\zeta - a| \, .$$

Then, for every $F \in C(K)$, $\Phi(a, b, F) \to 0$ as $|b - a| \to 0$, $[a, b] \in H$. Propositions 2.4 and 2.5 having been proved we can establish theorem 2.3 as follows:

Suppose that MF (with $F \in C(K)$; cf. (61)) is uniformly continuous in D, a complementary domain of K, and denote by H the set from 2.4. Let $|b(a) - a| = r$ for every $a \in K$. By 2.5 we conclude that

$$0 = \lim_{\xi \to 0+} \Phi(a, a + \xi(b(a) - a), F) =$$
$$= \lim_{\xi \to 0+} \left(MF(a + \xi(b(a) - a)) - \int_{K_{r\xi}(a)} \frac{F(\zeta)}{|\zeta - a|} \, \mathrm{d}|\zeta - a|\right)$$

uniformly in $a \in K$. In view of the uniform continuity of MF on D, $MF(a + \xi(b(a) - - a))$ must tend to a limit (as $\xi \to 0+$) uniformly in $a \in K$. Consequently, also

$$\text{V.p.} \int_K \frac{F(\zeta)}{|\zeta - a|} \, \mathrm{d}|\zeta - a| = \lim_{\xi \to 0+} \int_{K_{r\xi}(a)} \frac{F(\zeta)}{|\zeta - a|} \, \mathrm{d}|\zeta - a|$$

converges uniformly in $a \in K$ and the implication (A) $\Rightarrow$ (B) is verified.

Conversely, suppose that (B) is fulfilled ($F \in C(K)$) and associate with every $b \in E_2 - K$ an $a(b) \in K$ such that $|a(b) - b| = \operatorname{dist}(b, K)$. Denoting by H the set of all $[a(b), b]$ ($b \in E_2 - K$) we observe at once that (62) holds. Employing the remark following 1.11 we obtain that

$$\int_{K_r(\eta)} \frac{F(\zeta)}{|\zeta - \eta|} \, \mathrm{d}|\zeta - \eta| \quad (= \operatorname{Re} \tilde{w}_{2r} F(\psi(s)), \ s \in E_1, \ \psi(s) = \eta)$$

is a continuous function of the variable $\eta \in K$ provided $2r \in (0, +\infty) - \mathcal{R}$ (cf. also 1.11). $\mathcal{R}$ being at most countable (see 1.10) we conclude that, in view of the uniform convergence of (60), $MF|_K$ is continuous on K. To prove that MF is continuous on E_2 it is therefore sufficient to verify that $MF(b_n) \to MF(a)$ for every sequence of points $b_n \in E_2 - K$ tending to $a \in K$ as $n \to \infty$. Fix such a sequence $\{b_n\}$ and put $a_n = a(b_n)$, $r_n = |a_n - b_n|$. Clearly, $r_n \to 0$ and $a_n \to a$ as $n \to \infty$. Let us write $K_n = K_{r_n}(a_n)$. Taking into account 2.5 and the uniform convergence of (60), we see that

$$MF(b_n) - MF(a) = \left(MF(b_n) - \int_{K_n} \frac{F(\zeta)}{|\zeta - a_n|} \, \mathrm{d}|\zeta - a_n|\right) +$$
$$+ \left(\int_{K_n} \frac{F(\zeta)}{|\zeta - a_n|} \, \mathrm{d}|\zeta - a_n| - MF(a_n)\right) + (MF(a_n) - MF(a)) \to 0$$

as $n \to \infty$.

In order to make the proof of theorem 2.3 complete we have thus to prove propositions 2.4 and 2.5.

Before going into the proof of 2.4 we shall establish two simple lemmas.

2.6. Lemma. Let $\psi(t_0) = \zeta$, $t_0 - k \leqq \alpha < t_0 < \beta \leqq t_0 + k$, $0 < \delta < \pi$ and suppose that var $[\vartheta_\zeta; (\alpha, \beta)] \leqq \pi - \delta$ (cf. 1.1 for notation). Then there is a $\gamma \in E_1$ such that

$$\tfrac{1}{2}\delta \leqq \vartheta_\zeta(t_0) - \gamma \leqq \pi - \tfrac{1}{2}\delta \,, \tag{63}$$

$$\tfrac{1}{2}\delta \leqq \vartheta_\zeta(t_0-) - \gamma \leqq \pi - \tfrac{1}{2}\delta \,, \tag{64}$$

$$x \in E_1 \Rightarrow \text{dist}\,(\zeta + x \exp i\gamma;\ \psi(\alpha, \beta)) \geqq |x| \sin \tfrac{1}{2}\delta \,. \tag{65}$$

Proof. We may assume that $\vartheta_\zeta(t)$ is an argument of $\psi(t) - \zeta$ in $(t_0, t_0 + 2k)$ (cf. 1.1). Let $a_1 \leqq b_1$ be the end-points of the interval $\vartheta_\zeta((\alpha, t_0))$ and let $a_2 \leqq b_2$ be the end-points of the interval $\vartheta_\zeta((t_0, \beta))$. Clearly,

$$b_1 - a_1 + |\vartheta_\zeta(t_0) - \vartheta_\zeta(t_0-)| + b_2 - a_2 \leqq \text{var}\,[\vartheta_\zeta;\ (\alpha, \beta)] \leqq \pi - \delta \,, \tag{66}$$

$$a_2 \leqq \vartheta_\zeta(t_0) = \vartheta_\zeta(t_0+) \leqq b_2 \,, \tag{67}$$

$$a_1 \leqq \vartheta_\zeta(t_0-) \leqq b_1 \,. \tag{68}$$

Since $|\vartheta_\zeta(t_0) - \vartheta_\zeta(t_0-)| \geqq \vartheta_\zeta(t_0-) - \vartheta_\zeta(t_0) \geqq a_1 - b_2$ we obtain from (66) $\pi - \delta \geqq b_1 - a_1 + a_1 - b_2 + b_2 - a_2 = b_1 - a_2$,
so that

$$b_1 \leqq a_2 + \pi - \delta \,. \tag{69}$$

In a similar way $|\vartheta_\zeta(t_0) - \vartheta_\zeta(t_0-)| \geqq \vartheta_\zeta(t_0) - \vartheta_\zeta(t_0-) \geqq a_2 - b_1$ together with (66) implies

$$b_2 \leqq a_1 + \pi - \delta \,. \tag{70}$$

Put $\tilde{b} = \max\,(b_1 - \pi,\ b_2 - \pi)$. Noting that, by (66), $b_2 - a_2 \leqq \pi - \delta$, we obtain from (69)

$$\tilde{b} \leqq a_2 - \delta \,. \tag{71}$$

In a similar way (70) and $b_1 - a_1 \leqq \pi - \delta$ (compare (66)) imply

$$\tilde{b} \leqq a_1 - \delta \,. \tag{72}$$

Defining $\tilde{a} = \min\,(a_1, a_2)$ we have by (71), (72)

$$\tilde{b} \leqq \tilde{a} - \delta \,. \tag{73}$$

Put $\gamma = \frac{1}{2}(\tilde{a} + \tilde{b})$. Given $u \in \langle a_1 - \pi, b_1 - \pi \rangle$ we have by (73)

$$\gamma \geqq \tilde{b} + \tfrac{1}{2}\delta \geqq b_1 - \pi + \tfrac{1}{2}\delta \geqq u + \tfrac{1}{2}\delta\,,$$
$$\gamma \leqq \tilde{a} - \tfrac{1}{2}\delta \leqq a_1 - \tfrac{1}{2}\delta \leqq u + \pi - \tfrac{1}{2}\delta\,,$$

so that

(74) $$u \in \langle a_1 - \pi,\ b_1 - \pi \rangle \Rightarrow \tfrac{1}{2}\delta \leqq \gamma - u \leqq \pi - \tfrac{1}{2}\delta\,.$$

In a similar way $u \in \langle a_2, b_2 \rangle$ implies

$$\gamma \geqq \tilde{b} + \tfrac{1}{2}\delta \geqq b_2 - \pi + \tfrac{1}{2}\delta \geqq u - \pi + \tfrac{1}{2}\delta\,,$$
$$\gamma \leqq \tilde{a} - \tfrac{1}{2}\delta \leqq a_2 - \tfrac{1}{2}\delta \leqq u - \tfrac{1}{2}\delta\,,$$

so that

(75) $$u \in \langle a_2, b_2 \rangle \Rightarrow \tfrac{1}{2}\delta \leqq \gamma + \pi - u \leqq \pi - \tfrac{1}{2}\delta\,.$$

Finally, (67) and (75) imply (63) while (74), (68) imply (64). Let now $x \in E_1$. Given $t \in (\alpha, t_0)$ we have $\psi(t) = \zeta + \xi \exp iu$ with $\xi = |\psi(t) - \zeta|$ and $u = \vartheta_\zeta(t) - \pi \in \langle a_1 - \pi, b_1 - \pi \rangle$ (note that, with the notation from 1.1, $\exp i(\vartheta_\zeta(t) - \sigma\pi) = \exp i(\vartheta_\zeta(t) - \pi)$) whence we obtain by (74)

$$|\psi(t) - (\zeta + x \exp i\gamma)| = |x \exp i\gamma - \xi \exp iu| =$$
$$= |x \exp i(\gamma - u) - \xi| \geqq |\mathrm{Im}\,(x \exp i(\gamma - u) - \xi)| \geqq |x| \sin \tfrac{1}{2}\delta\,.$$

If $t \in (t_0, \beta)$ then $\psi(t) = \zeta + \xi \exp iu$ with $u = \vartheta_\zeta(t) \in \langle a_2, b_2 \rangle$ and we conclude from (75) $|\zeta + x \exp i\gamma - \psi(t)| = |x \exp i(\gamma + \pi - u) + \xi| \geqq |\mathrm{Im}\,(x \exp i(\gamma + \pi - u) + \xi)| \geqq |\sin \frac{1}{2}\delta|$. Thus (65) is checked and the the proof is complete.

2.7. Lemma. *Suppose that $\mathscr{F}K < \pi$ (cf. 1.15). Then there is a $\delta \in (0, \pi)$ and a (finite) real-valued function $\gamma(\zeta)$ such that*

(76) $$(\zeta \in K,\ x \in E_1,\ |x| \leqq \delta) \Rightarrow \mathrm{dist}\,(\zeta + x \exp i\gamma(\zeta),\ K) \geqq |x| \sin \delta$$

and, for every $s \in E_1$ (cf. 1.9 for notation),

(77) $$0 < \vartheta^s(s) - \gamma(\psi(s)) < \pi\,,$$

(78) $$0 < \vartheta^s(s-) - \gamma(\psi(s)) < \pi\,.$$

Proof. Let $r \in (0, +\infty) - \mathscr{R}$ be so chosen that $\mathscr{F}_r K < \pi$ (cf. 1.15) and fix a $\Delta > 0$ with $\mathscr{F}_r K + \Delta < \pi$. Suppose that r is less than the diameter of K and, for every $s \in E_1$, denote by $(\alpha(s), \beta(s))$ the component of $\{t;\ t \in E_1,\ |\psi(t) - \psi(s)| < r\}$ containing s. Defining U_r^s by (43) we have clearly $(\alpha(s), \beta(s)) \subset U_r^s$ and on account of 1.13 and 1.15 we conclude that $\mathrm{var}\,[\vartheta^s; (\alpha(s), \beta(s))] < \pi - \Delta,\quad s \in E_1$. Next we put $A_s = \langle s - k, \alpha(s) \rangle \cup \langle \beta(s), s + k \rangle$ and show that

(79) $$\inf\,\{\mathrm{dist}\,(\psi(s), \psi(A_s));\ s \in E_1\} > 0\,.$$

Indeed, suppose that $\inf\limits_s \mathrm{dist}\,(\psi(s), \psi(A_s)) = 0$. Then there are $s_n \in \langle 0, 2k \rangle$ and

(80) $$q_n \in A_{s_n}$$

such that $\lim_{n\to\infty} |\psi(s_n) - \psi(q_n)| = 0$. Passing to subsequences, if necessary, we can achieve that $\{s_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ be convergent. Put $s_0 = \lim_{n\to\infty} s_n$, $q_0 = \lim_{n\to\infty} q_n$. Clearly, $\psi(s_0) = \psi(q_0)$ and $q_0 \in \langle s_0 - k, s_0 + k \rangle$, so that $s_0 = q_0$. Let c, d be so chosen that

$$\alpha(s_0) < c < s_0 < d < \beta(s_0)\,.$$

Then $\psi(\langle c, d\rangle) \subset \{\zeta; \zeta \in K, |\zeta - \psi(s_0)| < r\}$ and we have an n_0 such that

$$n > n_0 \Rightarrow (\psi(\langle c, d\rangle) \subset \{\zeta; \zeta \in K, |\zeta - \psi(s_n)| < r\},\ s_n \in \langle c, d\rangle)\,.$$

Consequently,

$$n > n_0 \Rightarrow \langle c, d\rangle \subset (\alpha(s_n), \beta(s_n))\,. \tag{81}$$

Noting that $q_n \in \langle c, d\rangle$ for sufficiently large n we observe that (81) violates (80). Thus (79) is proved. Let $2\varepsilon = \inf_s \operatorname{dist}(\psi(s), \psi(A_s))$. Applying 2.6 we associate with every $s \in \langle 0, 2k)$ a $\gamma^s \in E_1$ such that

$$\tfrac{1}{2}\Delta \leqq \vartheta^s(s) - \gamma^s \leqq \pi - \tfrac{1}{2}\Delta\,, \quad \tfrac{1}{2}\Delta \leqq \vartheta^s(s-) - \gamma^s \leqq \pi - \tfrac{1}{2}\Delta\,,$$

$$x \in E_1 \Rightarrow \operatorname{dist}(\psi(s) + x \exp i\gamma^s, \psi((\alpha(s), \beta(s)))) \geqq |x| \sin \tfrac{1}{2}\Delta\,. \tag{82}$$

If $x \in E_1$, $|x| \leqq \varepsilon$, then $\operatorname{dist}(\psi(s) + x \exp i\gamma^s, \psi(A^s)) \geqq \varepsilon \geqq |x| \sin \tfrac{1}{2}\Delta$ for every $s \in \langle 0, 2k)$ which combined with (82) yields

$$(x \in E_1,\ |x| \leqq \varepsilon,\ s \in \langle 0, 2k)) \Rightarrow \operatorname{dist}(\psi(s) + x \exp i\gamma^s, K) \geqq |x| \sin \tfrac{1}{2}\Delta\,.$$

Defining for $\zeta \in K$

$$\gamma(\zeta) = \gamma^s\,, \quad \text{where} \quad s \in \langle 0, 2k)\,, \quad \psi(s) = \zeta\,,$$

we see that it is sufficient to put $\delta = \min(\varepsilon, \tfrac{1}{2}\Delta)$ in order to make the proof complete.

Now we are able to present the following

2.8. Proof of proposition 2.4. Let us keep the notation and the assumptions introduced in 2.4. With every $\zeta \in K$ we associate a $\gamma(\zeta)$ possessing all the properties described in 2.7. We shall show that, for every $\zeta \in K$, the points $\zeta + \delta \exp i\gamma(\zeta)$, $\zeta - \delta \exp i\gamma(\zeta)$ (where $\delta > 0$ is taken from 2.7) belong to different complementary domains of K. Let $s \in E_1$, $\psi(s) = \zeta$. We shall apply 1.4 where we put $z^1 = \zeta - \delta \exp i\,\gamma(\zeta)$, $z^2 = \zeta + \delta \exp i\,\gamma(\zeta)$. We may assume that $\vartheta_\zeta(t)$ is an argument of $\psi(t) - \zeta$ for $t \in (s, s + 2k)$ (cf. 1.1), so that

$$t \in (s, s + 2k) \Rightarrow \frac{\psi(t) - \zeta}{z^2 - z^1} = \left|\frac{\psi(t) - \zeta}{2\delta}\right| . \exp i(\vartheta_\zeta(t) - \gamma(\zeta))\,,$$

$$t \in (s - 2k, s) \Rightarrow \frac{\psi(t) - \zeta}{z^2 - z^1} = \left|\frac{\psi(t) - \zeta}{2\delta}\right| . \exp i(\vartheta_\zeta(t) + \sigma\pi - \gamma(\zeta))\,.$$

Hence we obtain on account of (77), (78)

$$\lim_{t\to s+} \operatorname{sign} \operatorname{Im} \frac{\psi(t) - \zeta}{z^2 - z^1} = \quad \operatorname{sign} \sin (\vartheta_\zeta(s) \quad - \gamma(\zeta)) > 0 ,$$

$$\lim_{t\to s-} \operatorname{sign} \operatorname{Im} \frac{\psi(t) - \zeta}{z^2 - z^1} = - \operatorname{sign} \sin (\vartheta_\zeta(s-) - \gamma(\zeta)) < 0 ,$$

so that, by 1.4, $\operatorname{ind}(z^1, K) \neq \operatorname{ind}(z^2, K)$. For every $a \in K$ we denote by $b(a)$ that of the points $a + \delta \exp i\gamma(a)$, $a - \delta \exp i\gamma(a)$ which belongs to D. If $b = a + \xi(b(a) - a)$, $0 < \xi \leqq 1$, then, in view of (76),

$$\frac{\operatorname{dist}(b, K)}{|b - a|} \geqq \sin \delta$$

and the proof is complete.

2.9. Notation. Given $z \in E_2$ and $\varrho > 0$ we denote by $\nu(\varrho, z)$ the number (possibly zero or infinite) of points in $\{\zeta;\ \zeta \in K,\ |\zeta - z| = \varrho\}$. Since $\nu(\varrho, z)$ is Lebesgue measurable with respect to ϱ (cf. section 2 in [11]) we are justified to define

$$u_r^K(z) = \int_0^r \nu(\varrho, z)\, \mathrm{d}\varrho , \quad r > 0 .$$

The following proposition will be needed below:

2.10. Proposition. *Let* $\mathscr{F}K < \pi$. *Then* $\sup\{r^{-1} u_r^K(\zeta);\ r > 0,\ \zeta \in K\} < +\infty$.

Proof. On account of 1.11 in [10] we conclude from (9) that

(83) $$\sup\{v^K(z);\ z \in E_2\} = V < +\infty .$$

Let now δ and $\gamma(\zeta)$ have the same meaning as in 2.7. It follows from (76) that, for every $\zeta \in K$,

$$(\gamma(\zeta) - \delta < \beta < \gamma(\zeta) + \delta,\ 0 < r < \delta) \Rightarrow \zeta \pm r \exp i\gamma(\zeta) \notin K .$$

Hence it follows by [11] that $\sup\{r^{-1} u_r^K(\zeta);\ 0 < r < \frac{1}{2}\delta,\ \zeta \in K\} = c(\delta) < +\infty$ with a $c(\delta)$ depending on δ and V only.

On the other hand, we have for $r \geqq \frac{1}{2}\delta$ and every $\zeta \in K$ $r^{-1} u_r^K(\zeta) \leqq 2\delta^{-1} u_\infty^K(\zeta) =$ = (cf. [11]) $= 2\delta^{-1} \operatorname{var}[|\psi(t) - \zeta|;\ \langle 0, 2k\rangle] \leqq 2\delta^{-1} \operatorname{var}[\psi;\ \langle 0, 2k\rangle] = 2\delta^{-1}\lambda K$ whence we conclude that $\sup\{r^{-1} u_r^K(\zeta);\ r > 0,\ \zeta \in K\} \leqq \max(c(\delta),\ 2\delta^{-1} . \lambda K)$.

Now we shall concentrate on the proof of proposition 2.5. First we prove three lemmas.

2.11. Lemma. *Let* $f_1, \ldots, f_n$ *be continuous functions on a set* U *open in an interval* I *and denote by* $f = [f_1, \ldots, f_n]$ *the corresponding map of* U *into* E_n, *the Euclidean* *n-space. Suppose that F is a function of n real variables possessing continuous and bounded first order derivatives* F'_j $(j = 1, \ldots, n)$ *in a neighbourhood of* $f(U)$. *Let every* f_j $(j = 1, \ldots, n)$ *have finite variation on U and define* $h(t) = F(f(t))$, $t \in U$. *Then h has finite variation on U and, for every bounded Borel-measurable function H on U, the equality*

$$\int_U H \, \mathrm{d}h = \sum_{j=1}^{n} \int_U H(t) \, F'_j(f(t)) \, \mathrm{d}f_j(t) \tag{84}$$

is true (the integrals are taken in the sense of Lebesgue-Stieltjes). Further we have for any non-negative Borel-measurable function H on U

$$\int_U H \, \mathrm{d} \operatorname{var} h \leqq \sum_{j=1}^{n} \int_U H(t) \left| F'_j(f(t)) \right| \mathrm{d} \operatorname{var} f_j(t) \,. \tag{85}$$

Proof. Fix a compact interval $\langle \alpha, \beta \rangle \subset U$ and consider an arbitrary subdivision $D = \{\alpha = t_0 < \ldots < t_p = \beta\}$ of $\langle \alpha, \beta \rangle$. We have for sufficiently small $|D| = \max \{t_m - t_{m-1};\ 1 \leqq m \leqq p\}$

$$h(t_m) - h(t_{m-1}) = \sum_{j=1}^{n} \left[F'_j(f(t_m)) + z_j^m \right] . \left[f_j(t_m) - f_j(t_{m-1}) \right],$$

where $z_j^m = F'_j(x^m) - F'_j(f(t_m))$ and x^m is a suitable point of the segment with end-points $f(t_{m-1}), f(t_m)$. It is easily seen that

$$z(D) = \max \{ |z_j^m|;\ 1 \leqq m \leqq p,\ 1 \leqq j \leqq n \} \to 0 \quad \text{as} \quad |D| \to 0 \,. \tag{86}$$

Noting that

$$h(\beta) - h(\alpha) = \sum_{j=1}^{n} \sum_{m=1}^{p} F'_j(f(t_m)) . \left[f_j(t_m) - f_j(t_{m-1}) \right] + \\ + \Theta(D) \, z(D) \sum_{j,m} \left| f_j(t_m) - f_j(t_{m-1}) \right|$$

with $|\Theta(D)| \leqq 1$ and making $|D| \to 0$ we obtain

$$h(\beta) - h(\alpha) = \sum_{j=1}^{n} \int_\alpha^\beta F'_j(f(t)) \, \mathrm{d}f_j(t) \,.$$

In a similar way

$$\sum_m \left| h(t_m) - h(t_{m-1}) \right| \leqq \sum_{j,m} \left| F'_j(f_j(t_m)) \right| . \left| f_j(t_m) - f_j(t_{m-1}) \right| + \\ + z(D) \sum_{j,m} \left| f_j(t_m) - f_j(t_{m-1}) \right| . \tag{87}$$

By lemma 1.3 in [11] we have

$$\sum_m \left| F'_j(f(t_m)) \right| . \left| f_j(t_m) - f_j(t_{m-1}) \right| \to \int_\alpha^\beta \left| F'_j(f(t)) \right| \mathrm{d} \operatorname{var} f_j(t) \quad \text{as} \quad |D| \to 0 \,.$$

Making $|D| \to 0$ in (87) we obtain by (86)

$$\operatorname{var}[h;\ \langle\alpha, \beta\rangle] \leqq \sum_j \int_\alpha^\beta |F_j'(f(t))| \, \mathrm{d} \operatorname{var} f_j(t) .$$

We see that (84) and (85) are true if H is the characteristic function of a compact interval contained in U. Making linear combinations and passing to limits one easily shows that (85) is valied for any non-negative function of Baire (compare [9], chap. V, § 10). In particular, (85) with $H \equiv 1$ shows that h has finite variation on U. Similar reasonings show that also (84) is valid for every bounded function H of Baire.

As a direct corollary we obtain the following

2.12. Lemma. *Let f_1, f_2 be bounded continuous functions of finite variation on a set U open in an interval I and put $h(t) = f_1(t) \cdot f_2(t)$, $t \in U$. Then h has finite variation on U and for every boundedr Boel-measurable function H on U the equality*

$$\int_U H \, \mathrm{d}h = \int_U H f_1 \, \mathrm{d}f_2 + \int_U H f_2 \, \mathrm{d}f_1$$

is true. Further we have for any non-negative Borel-measurable function H on U

$$\int_U H \, \mathrm{d} \operatorname{var} h \leqq \int_U H |f_1| \, \mathrm{d} \operatorname{var} f_2 + \int_U H |f_2| \, \mathrm{d} \operatorname{var} f_1 .$$

2.13. Lemma. *Let U be open in $\langle 0, 2k\rangle$, $a, b \in E_2 - \psi(U)$ and suppose that $d = \operatorname{dist}(a, \psi(U)) > 0$. Then*

(88) $$\operatorname{var}_t [|\psi(t) - a| - |\psi(t) - b|;\ U] \leqq 4d^{-1}|a - b| \operatorname{var}[\psi;\ U] .$$

Proof. Write $\psi_1 = \operatorname{Re} \psi$, $\psi_2 = \operatorname{Im} \psi$ and let a_j, b_j $(j = 1, 2)$ have a similar meaning with respect to a, b. Put $F(z) = |z - a| - |z - b|$, $z \in E_2$. We have by 2.11

$$\operatorname{var}_t[|\psi(t) - a| - |\psi(t) - b|;\ U] = \operatorname{var}_t [F(\psi(t));\ U] \leqq$$
$$\leqq \sum_{j=1}^{2} \int_U \left| \frac{\psi_j(t) - a_j}{|\psi(t) - a|} - \frac{\psi_j(t) - b_j}{|\psi(t) - b|} \right| \mathrm{d} \operatorname{var} \psi_j(t) .$$

Since

$$\left| \frac{\psi_j(t) - a_j}{|\psi(t) - a|} - \frac{\psi_j(t) - b_j}{|\psi(t) - b|} \right| =$$
$$= \frac{|(b_j - a_j)|\psi(t) - b| + (\psi_j(t) - b_j)(|\psi(t) - b| - |\psi(t) - a|)|}{|\psi(t) - a| \cdot |\psi(t) - b|} \leqq$$
$$\leqq \frac{|b_j - a_j|}{|\psi(t) - a|} + \frac{|b - a|}{|\psi(t) - a|} \leqq 2d^{-1}|b - a|$$

we conclude easily that (88) is true.

Now we can pass to

2.14. Proof of proposition 2.5. Let $[a, b] \in H$. We shall show that

$$(89) \qquad \int_K \frac{1}{|\zeta - b|} \, d|\zeta - b| = 0 = \int_{K_{ab}} \frac{1}{|\zeta - a|} \, d|\zeta - a| \, .$$

Indeed, using 2.11 with $n = 1$, $F(x) = \log x$ we obtain

$$(90) \qquad \int_K \frac{1}{|\zeta - b|} \, d|\zeta - b| = \int_0^{2k} d_t \log |\psi(t) - b| =$$
$$= \log |\psi(2k) - b| - \log |\psi(0) - b| = 0 \, .$$

In a similar way we derive for $U = \{t;\ t \in \langle 0, 2k \rangle,\ |\psi(t) - a| > 2|b - a|\}$

$$(91) \qquad \int_{K_{ab}} \frac{1}{|\zeta - a|} \, d|\zeta - a| = \int_U d_t \log |\psi(t) - a| \, .$$

Let $\mathscr{S}$ be the system of all components of U. If $|\psi(0) - a| \leqq 2|a - b|$ then every $I \in \mathscr{S}$ has the form $I = (\alpha, \beta)$ with

$$(92) \qquad |\psi(\alpha) - a| = 2|a - b| = |\psi(\beta) - a| \, ,$$

so that

$$(93) \qquad \int_I d_t \log |\psi(t) - a| = \log |\psi(\beta) - a| - \log |\psi(\alpha) - a| = 0$$

and

$$\int_U d_t \log |\psi(t) - a| = \sum_{I \in \mathscr{S}} \int_I d_t \log |\psi(t) - a| = 0 \, .$$

Consider now the case $|\psi(0) - a| > 2|a - b|$. Noting that $a \in K$ we see that $K - K_{ab} \neq \emptyset$. There are exactly two components $\langle 0, \alpha_1)$, $(\beta_1, 2k\rangle$ in $\mathscr{S}$ containing the end-points of $\langle 0, 2k\rangle$ while every other $I \in \mathscr{S}$ has the form $I = (\alpha, \beta)$ with (92) (so that (93) is true). Consequently,

$$\int_U d_t \log |\psi(t) - a| = \int_0^{\alpha_1} d_t \log |\psi(t) - a| + \int_{\beta_1}^{2k} d_t \log |\psi(t) - a| =$$
$$= \log 2|a - b| - \log |\psi(0) - a| + \log |\psi(2k) - a| - \log 2|a - b| = 0$$

which together with (91), (90) yields (89) again. We have seen that, in any case, the equality (89) is true. Let us fix an $F \in C(K)$. On account of (89) we have for $[a, b] \in H$

$$(94) \qquad \Phi(a, b, F) = \int_K \frac{F(\zeta) - F(a)}{|\zeta - b|} \, d|\zeta - b| - \int_{K_{ab}} \frac{F(\zeta) - F(a)}{|\zeta - a|} \, d|\zeta - a| \, .$$

The rest of the proof will be divided into three lemmas 2.15–2.17.

2.15. Lemma. *Given* $[a, b] \in H$ *and* $\Delta > 0$ *put*

$$K_\Delta = K_\Delta(a) = \{\zeta;\ \zeta \in K,\ |\zeta - a| > 2\Delta\}$$

and define

$$\Phi_\Delta(a, b) = \int_{K_\Delta} \frac{F(\zeta) - F(a)}{|\zeta - b|} \, \mathrm{d}|\zeta - b| - \int_{K_\Delta} \frac{F(\zeta) - F(a)}{|\zeta - a|} \, \mathrm{d}|\zeta - a| .$$

Then, for any fixed $\Delta > 0$,

$$\Phi_\Delta(a, b) \to 0 \quad \text{as} \quad |a - b| \to 0 , \quad [a, b] \in H .$$

Proof. Let us define for $[a, b] \in H$

$$J_1(a, b) = \int_{K_\Delta(a)} \left(\frac{F(\zeta) - F(a)}{|\zeta - b|} - \frac{F(\zeta) - F(a)}{|\zeta - a|} \right) \mathrm{d}|\zeta - b| ,$$

$$J_2(a, b) = \int_{K_\Delta(a)} \frac{F(\zeta) - F(a)}{|\zeta - a|} \, \mathrm{d}(|\zeta - a| - |\zeta - b|) ,$$

so that

$$\Phi_\Delta(a, b) = J_1(a, b) - J_2(a, b) .$$

Let $\max \{|F(\zeta)|;\ \zeta \in K\} = m$. Given $[a, b] \in H$ with $|b - a| < \Delta$ we have for every $\zeta \in K_\Delta(a)$

$$\left| \frac{1}{|\zeta - b|} - \frac{1}{|\zeta - a|} \right| \leqq \frac{|a - b|}{|\zeta - b| \, . \, |\zeta - a|} \leqq \tfrac{1}{2} |a - b| \, . \, \Delta^{-2}$$

whence

$$|J_1(a, b)| \leqq m |a - b| \, \Delta^{-2} \, . \, \mathrm{var}_t \, [|\psi(t) - b|;\ \langle 0, 2k \rangle] \leqq$$
$$\leqq m |a - b| \, \Delta^{-2} \, . \, \mathrm{var} \, [\psi;\ \langle 0, 2k \rangle] \to 0 \quad \text{as} \quad |a - b| \to 0 .$$

Employing 2.13 with $U = \{t;\ t \in \langle 0, 2k \rangle,\ |\psi(t) - a| > 2\Delta\}$ we obtain

$$J_2(a, b) \leqq m \Delta^{-1} \, \mathrm{var}_t \, [|\psi(t) - a| - |\psi(t) - b|;\ U] \leqq$$
$$\leqq 4 |a - b| \, m \Delta^{-2} \, \mathrm{var} \, [\psi;\ \langle 0, 2k \rangle] \to 0 \quad \text{as} \quad |a - b| \to 0 ,$$

which completes the proof.

2.16. Lemma. *Given* $\Delta > 0$ *and* $[a, b] \in H$ *with* $|a - b| < \Delta$ *put*

$$L = L_\Delta(a, b) = \{\zeta;\ \zeta \in K,\ 2|a - b| < |\zeta - a| \leqq 2\Delta\}$$

and define

$$\Psi_\Delta(a, b) = \int_L \frac{F(\zeta) - F(a)}{|\zeta - b|} \, \mathrm{d}|\zeta - b| - \int_L \frac{F(\zeta) - F(a)}{|\zeta - a|} \, \mathrm{d}|\zeta - a| .$$

Then $\Psi_\Delta(a, b) \to 0$ as $\Delta \to 0+$, $[a, b] \in H$, $|a - b| < \Delta$.

Proof. Given $[a, b] \in H$ we denote by $\vartheta(t; a, b)$ a continuous single-valued argument of $(\psi(t) - a)/(b - a)$ on

$$T_a = \{t;\ t \in \langle 0, 2k \rangle,\ |\psi(t) - a| > 0\}.$$

We have thus for every $t \in T_a$

$$(95) \qquad |\psi(t) - b|^2 = |a - b|^2 \cdot \left| \left| \frac{\psi(t) - a}{b - a} \right| \exp i\,\vartheta(t; a, b) - 1 \right|^2 =$$

$$= |\psi(t) - a|^2 + |a - b|^2 - 2|a - b| \cdot |\psi(t) - a| \cos \vartheta(t; a, b)$$

(which, in fact, is the elementary cosine theorem). Let us fix a $\Delta > 0$ and a $[a, b] \in H$ with $|a - b| < \Delta$; further put $T = \{t;\ t \in \langle 0, 2k \rangle,\ 2|a - b| < |\psi(t) - a| \leqq 2\Delta\}$. It follows easily from 2.11, 2.12 and (95) that

$$\int_T \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|} \mathrm{d}_t |\psi(t) - b| =$$

$$= \frac{1}{2} \int_T \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|^2} \mathrm{d}_t [|\psi(t) - a|^2 - 2|a - b| \cdot |\psi(t) - a| \cos \vartheta(t; a, b)] =$$

$$= \int_T \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|^2} (|\psi(t) - a| - |a - b| \cos \vartheta\,(t; a, b))\, \mathrm{d}_t |\psi(t) - a| +$$

$$+ |a - b| \int_T \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|^2} |\psi(t) - a| \sin \vartheta(t; a, b)\, \mathrm{d}_t\, \vartheta(t; a, b).$$

Writing

$$K_1(a, b, \Delta) = \int_T (F(\psi(t)) -$$

$$- F(a)) \left[\frac{|\psi(t) - a| - |a - b| \cos \vartheta(t; a, b)}{|\psi(t) - b|^2} - \frac{1}{|\psi(t) - a|} \right] \mathrm{d}_t |\psi(t) - a|,$$

$$K_2(a, b, \Delta) = |a - b| \int_T \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|^2} |\psi(t) - a| \sin \vartheta(t, a, b)\, \mathrm{d}_t \vartheta(t; a, b)$$

we obtain

$$(96) \qquad \Psi_\Delta(a, b) = K_1(a, b, \Delta) + K_2(a, b, \Delta).$$

Put, for the sake of brevity, $r = |\psi(t) - a|$, $\vartheta = \vartheta(t; a, b)$. Then, by (95),

$$\frac{|\psi(t) - a| - |a - b| \cos \vartheta(t; a, b)}{|\psi(t) - b|^2} - \frac{1}{|\psi(t) - a|} = \frac{r|a - b| \cos \vartheta - |a - b|^2}{r(r^2 + |a - b|^2 - 2r|a - b| \cos \vartheta)}.$$

Let us define for every $\delta \geqq 0$

$$\Omega_F(\delta) = \sup \{|F(\zeta) - F(\eta)|;\ \zeta, \eta \in K,\ |\zeta - \eta| \leqq \delta\}.$$

Noting that $2|a - b| < r$ $(t \in T)$ and, consequently,

$$\left| \frac{r|a - b| \cos \vartheta - |a - b|^2}{r(r^2 + |a - b|^2 - 2r|a - b| \cos \vartheta)} \right| \leqq 2 \frac{|a - b|}{(r - |a - b|)^2},$$

we obtain the following estimate for $K_1(a, b, \Delta)$:

(97) $$|K_1(a, b, \Delta)| \leqq 2\Omega_F(2\Delta) \, . \, |a - b| \int_T \frac{\mathrm{d}_t \operatorname{var} |\psi(t) - a|}{(|\psi(t) - a| - |a - b|)^2} \, .$$

Denoting by $\chi(y)$ the characteristic function of the interval $\{y; 2|a - b| < y \leqq 2\Delta\}$ we have

(98) $$\int_T \frac{\mathrm{d} \operatorname{var} |\psi(t) - a|}{(|\psi(t) - a| - |a - b|)^2} = \int_0^{2k} \frac{\chi(|\psi(t) - a|)}{(|\psi(t) - a| - |a - b|)^2} \mathrm{d} \operatorname{var} |\psi(t) - a| \, .$$

Let us observe that $\nu(y, a)$ (cf. 2.9 for notation) equals the number of points in $\{t; t \in \langle 0, 2k); |\psi(t) - a| = y\}$. Employing 1.1 in [11] we arrive at

(99) $$\int_0^{2k} \frac{\chi(|\psi(t) - a|)}{(|\psi(t) - a| - |a - b|)^2} \mathrm{d} \operatorname{var} |\psi(t) - a| = \int_{2|a-b|}^{2\Delta} \frac{\nu(y, a)}{(y - |a - b|)^2} \mathrm{d}y \, .$$

Let us recall that $\int_0^q \nu(y, a) \, \mathrm{d}y = u_q^K(a)$ and, by 2.10,

(100) $$\sup \{q^{-1} u_q^K(\zeta); \, q > 0, \, \zeta \in K\} = Q < +\infty \, .$$

Integrating by parts in (99) we obtain

$$\int_{2|a-b|}^{2\Delta} \frac{\nu(y, a)}{(y - |a - b|)^2} \mathrm{d}y = \frac{u_{2\Delta}^K(a)}{(2\Delta - |a - b|)^2} - \frac{u_{2|a-b|}^K(a)}{|a - b|^2} +$$

$$+ 2 \int_{2|a-b|}^{2\Delta} (y - |a - b|)^{-3} u_y^K(a) \, \mathrm{d}y \leqq 2Q \left(\Delta^{-1} + \int_{2|a-b|}^{2\Delta} y(y - |a - b|)^{-3} \right) \mathrm{d}y \leqq$$

$$\leqq 2Q/\Delta + 3Q/|a - b| \, ,$$

which together with (99), (98) and (97) gives

(101) $$|K_1(a, b, \Delta)| \leqq 10\Omega_F(2\Delta) \, . \, Q \, , \quad [a, b] \in H, \; |a - b| < \Delta \, .$$

To derive an estimate for K_2 (a, b, Δ) we keep the notation introduced above and notice that, by (95),

$$\frac{|a - b| \, . \, |\psi(t) - a| \, . \, |\sin \vartheta(t; a, b)|}{|\psi(t) - b|^2} = \frac{|a - b| \, . \, r \, . \, |\sin \vartheta|}{r^2 + |a - b|^2 - 2r|a - b| \cos \vartheta} \leqq$$

$$\leqq \frac{|a - b| \, r}{(r - |a - b|)^2} \leqq \frac{r}{r - |a - b|} < 2$$

whence

(102) $$K_2(a, b, \Delta) \leqq 2\Omega_F(2\Delta) \, . \operatorname{var}_t [\vartheta(t; a, b); \, T_a] \, .$$

Let $\mathscr{V}$ be the system of all components of T_a and, for every $I \in \mathscr{V}$, denote by $\vartheta_I(t)$ a single valued continuous argument of $\psi(t) - a$ on I. It is easily seen that, for every $I \in \mathscr{V}$, $\vartheta_I(t) - \vartheta(t; a, b)$ reduces to a constant on I so that

$$\operatorname{var}_t [\vartheta(t; a, b); I) = \operatorname{var} [\vartheta_I; I], \quad I \in \mathscr{V}$$

and, by 2.2–2.4 in [11],

$$(103) \qquad \operatorname{var} [\vartheta(t; a, b); T_a] = \sum_{I \in \mathscr{V}} \operatorname{var} [\vartheta_I; I] = v^K(a).$$

Employing (83) and (102) we arrive at

$$(104) \qquad |K_2(a, b, \Delta)| \leqq 2V\Omega_F(2\Delta), \quad [a, b] \in H, \; |a - b| < \Delta.$$

In view of (104), (101), (96) and the uniform continuity of F our lemma is proved.

2.17. Lemma. *Given* $[a, b] \in H$ *put* $M = M_{ab} = \{\zeta; \zeta \in K, \; |\zeta - a| \leqq 2|a - b|\}$ *and define*

$$\Gamma(a, b) = \int_M \frac{F(\zeta) - F(a)}{|\zeta - b|} \, \mathrm{d}|\zeta - b|.$$

Then $\Gamma(a, b) \to 0$ *as* $|a - b| \to 0$, $[a, b] \in H$.

Proof. Let us recall that

$$(105) \qquad \inf \{\operatorname{dist}(b, K)/|a - b|; \; [a, b] \in H\} = l > 0.$$

Fix a $[a, b] \in H$ and put

$$A = \{t; \; t \in \langle 0, 2k \rangle, \; |\psi(t) - a| \leqq 2|a - b|\},$$
$$B = \{t; \; t \in A, \; |\psi(t) - a| > 0\} = A \cap T_a.$$

Noting that $\operatorname{var} [|\psi(t) - b|; A - B] = 0$ because $|\psi(t) - b| = |a - b|$ is constant on $A - B$ (cf. 1.9), we see that

$$\Gamma(a, b) = \int_A \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|} \, \mathrm{d}|\psi(t) - b| = \int_B \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|} \, \mathrm{d}|\psi(t) - b|.$$

Let $\vartheta(t; a, b)$ and Ω_F have the meaning described in the proof of 2.16. Applying 2.11 we obtain by the same reasoning as in the proof of 2.16

$$\int_B \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|} \, \mathrm{d}|\psi(t) - b| =$$
$$= \int_B \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|^2} (|\psi(t) - a| - |a - b| \cos \vartheta(t; a, b)) \, \mathrm{d}_t |\psi(t) - a| +$$
$$+ |a - b| \int_B \frac{F(\psi(t)) - F(a)}{|\psi(t) - b|^2} |\psi(t) - a| \sin \vartheta(t; a, b) \, \mathrm{d}_t \, \vartheta(t; a, b) =$$
$$= I_1(a, b) + I_2(a, b).$$

Noting that, by (105),

$$t \in B \Rightarrow |\psi(t) - b| \geqq l|a - b| \tag{106}$$

and (compare (100) and section 2 in [11])

$$\mathrm{var}_t \left[|\psi(t) - a|;\ B\right] = u^K_{2|a-b|}(a) \leqq 2Q|a - b|\,,$$

we conclude easily that

$$|I_1(a, b)| \leqq 6Ql^{-2}\Omega_F(2|a - b|)\,, \quad [a, b] \in H\,. \tag{107}$$

Further we have on account of (106), (103) and (83)

$$|I_2(a, b)| \leqq 2Vl^{-2}\Omega_F(2|a - b|)\,, \quad [a, b] \in H\,. \tag{108}$$

Since $\Gamma(a, b) = I_1(a, b) + I_2(a, b)$, (107) and (108) complete the proof of our lemma.

Now we are able to finish the proof of 2.5.

In view of (94) and 2.15–2.17 we have

$$\Phi(a, b, F) = \Phi_\Delta(a, b) + \Psi_\Delta(a, b) + \Gamma(a, b)$$

provided $[a, b] \in H$, $|a - b| < \Delta$. Let ε be an arbitrary positive number. By 2.16 we can fix a $\Delta > 0$ such that

$$([a, b] \in H,\ |a - b| < \Delta) \Rightarrow |\Psi_\Delta(a, b)| < \varepsilon/3\,.$$

Applying 2.15 we find a $\delta_1 > 0$, $\delta_1 < \Delta$, such that

$$([a, b] \in H,\ |a - b| < \delta_1) \Rightarrow |\Phi_\Delta(a, b)| < \varepsilon/3\,.$$

Employing 2.17 we obtain a $\delta_2 > 0$, $\delta_2 < \delta_1$, with

$$([a, b] \in H,\ |a - b| < \delta_2) \Rightarrow |\Gamma(a, b)| < \varepsilon/3\,.$$

Thus $([a, b] \in H,\ |a - b| < \delta_2) \Rightarrow |\Phi(a, b, F)| < \varepsilon$ and 2.5 is proved.

§ 3

In this paragraph we shall be engaged with the modified Dirichlet problem as described in the introduction. We shall show that results obtained in §§ 1, 2 permit us to express its solution as a logarithmic potential of the double distribution for a sufficiently wide class of domains.

We start with an auxiliary result:

3.1. Proposition. *Let G be an arbitrary domain in E_2 with a compact boundary B and let Ψ be bounded analytic function in G. Suppose that there exists a continuous function Ψ_1 on $G \cup B = \bar{G}$ coinciding with* Re Ψ on *G and put $m = \inf\{\Psi_1(z);\ z \in \bar{G}\}$, $M = \sup\{\Psi_1(z);\ z \in \bar{G}\}$. Then $\Psi_1(B) = \langle m, M\rangle$.*

Proof. Let us first remark that it follows from known properties of harmonic functions that $\inf \Psi_1(B) = m$, $\sup \Psi_1(B) = M$. Hence we, however, cannot conclude that

$$\Psi_1(B) = \langle m, M \rangle\,, \tag{109}$$

because B need not be connected. Since $\Psi_1(B)$ is closed it is sufficient to show that $\Psi_1(B)$ is dense in (m, M) in order to obtain (109). Let us admit that there is an interval (a, b) with $m < a < b < M$ containing no points of $\Psi_1(B)$. Let Z be the set of all $z \in G$ with $\Psi'(z) = 0$. Ψ being non-constant, Z is at most countable and, consequently, $\Psi_1(Z)$ is at most countable as well. Let us fix $\alpha < \beta$ in $(a, b) - \Psi_1(Z)$. If G is unbounded then there exists the limit $\lim_{|z|\to\infty} \Psi(z) = \Psi(\infty)$ (note that Ψ is bounded and B is compact) and we may assume α and β to be so chosen that $\Psi_1(\infty) \notin (\alpha, \beta)$. Put $G_1 = \{z;\ z \in G,\ \alpha < \Psi_1(z) < \beta\}$ and denote by B_1 the boundary of G_1. It is easily seen that G_1 is a bounded open set. G_1 is non-void. Indeed, there are points z_1, z_2 in G with $m < \Psi_1(z_1) < \alpha < \beta < \Psi_1(z_2) < M$; consequently, there must be some points in G_1, too, because G is connected and Ψ_1 is continuous in G. Since $B_1 \subset \{z;\ z \in \bar{G},\ \Psi_1(z) = \alpha\} \cup \{z;\ z \in \bar{G},\ \Psi_1(z) = \beta\} \subset \bar{G} - B = G$ (note that $(a, b) \cap \Psi_1(B) = \emptyset$), we have $\bar{G}_1 = G_1 \cup B_1 \subset G$. Consider now an arbitrary point $\zeta = x_0 + iy_0 \in B_1$ $(x_0,\ y_0 \in E_1)$. Then either $\Psi_1(\zeta) = \alpha$ or $\Psi_1(\zeta) = \beta$ so that $\Psi'(\zeta) = \frac{\partial \Psi_1}{\partial x}(\zeta) - i\frac{\partial \Psi_1}{\partial y}(\zeta) \neq 0$, because $\alpha, \beta \notin \Psi_1(Z)$. Hence we conclude by the classical theorem on implicit functions that, for some neighbourhood

$$U = (x_0 - \delta,\ x_0 + \delta) \times (y_0 - \Delta,\ y_0 + \Delta) \subset G\,(\delta, \Delta > 0)\,,$$

$U \cap B_1$ coincides with the set of all $x + iy$ fulfilling one of the equations

$$y = f(x)$$

(with an infinitely differentiable function $f(x)$ of the variable x on $(x_0 - \delta,\ x_0 + \delta)$) or

$$x = g(y)$$

(with an infinitely differentiable function $g(y)$ of the variable y on $(y_0 - \Delta,\ y_0 + \Delta)$) and that $U \cap G_1$ and $U - \bar{G}_1$ are exactly the two parts into which U is naturally divided by $B_1 \cap U$ (= the graph of the corresponding function). We may thus speak of an exterior normal n_ζ to G_1 for every $\zeta \in B_1$, and, B_1 being smooth, we are justified to employ the classical Green formula. Noting that $\Psi_2 = \operatorname{Im} \Psi$ is harmonic in $G \supset \bar{G}_1$ we obtain

$$\iint_{G_1} \left[\left(\frac{\partial \Psi_2}{\partial x}\right)^2 + \left(\frac{\partial \Psi_2}{\partial y}\right)^2\right] \mathrm{d}x\, \mathrm{d}y = \int_{B_1} \Psi_2(\zeta) \frac{\partial \Psi_2}{\partial n_\zeta}(\zeta)\, \mathrm{d}\lambda(\zeta)\,, \tag{110}$$

where λ stands for the linear measure (= length) on B_1. It follows from the Cauchy-Riemann equations that, for every $\zeta \in B_1$,

$$\frac{\partial \Psi_2}{\partial n_\zeta}(\zeta) = \frac{\partial \Psi_1}{\partial \tau_\zeta}(\zeta), \tag{111}$$

where τ_ζ is a tangent to B_1 at ζ. Ψ_1 being constant on every component of B_1 we have $\partial \Psi_1 / \partial \tau_\zeta = 0$ ($\zeta \in B_1$) and, by (111), the integral occuring in (110) must vanish. Consequently, $\Psi' = \dfrac{\partial \Psi_2}{\partial y} + i \dfrac{\partial \Psi_2}{\partial x} = 0$ in G_1, which is a contradiction concluding the proof.

Remark. Reasonings applied in the course of the above proof are known; compare N. I. Muskhelishvili's monograph [4], p. 248.

From now on we shall assume that the symbols $D, K_0, \ldots, K_q, B$ have the meaning described in the introduction. On account of 3.1 we obtain the following

3.2. Corollary. *Given $G \in C(B)$ there is at most one function Φ_2 which is continuous on $\bar{D}$ and fulfils the following conditions* 1) *and* 2):

1) *$\Phi_2 - G$ is constant on K_j ($j = 0, \ldots, q$) and*

$$\Phi_2 = G \quad \text{on} \quad K_0 \tag{112}$$

(*in case $K_0 = \emptyset$ we require $\Phi_2(\infty) = \lim_{|z| \to \infty} \Phi_2(z) = 0$ instead of* (112)).

2) *There is a single-valued analytic function Φ in D for which*

$$\operatorname{Im} \Phi = \Phi_2 \quad \text{on} \quad D .$$

Proof. Suppose that, besides Φ_2, Φ, we have another pair $\tilde{\Phi}_2, \tilde{\Phi}$ possessing all the properties described in 3.2 and put $\Psi = (1/i)(\Phi - \tilde{\Phi})$, $\Psi_1 = \Phi_2 - \tilde{\Phi}_2$. $\Psi_1(B)$ being finite we conclude on account of 3.1 that the interval $\Psi_1(\bar{D}) = \Psi_1(B)$ must reduce to a single point. Since $\Psi_1 = 0$ on K_0 (or $\Psi_1(\infty) = 0$ if $K_0 = \emptyset$) we see that $\Psi_1 = 0$ on $\bar{D}$.

3.3. Remark. Every Φ_2 enjoying the properties 1) and 2) from 3.2 will be termed a solution of the modified Dirichlet problem corresponding to G. It follows at once from 3.2 that, for every $G \in C(B)$, the corersponding solution of the modified Dirichlet problem is uniquely determined. Now we proceed to examine the existence of the solution and its representability in the form Im Φ, where Φ is defined by (6). We impose (3) on B and, for every $F \in C(B)$, we define

$$W(z, F) = \sum_j W_{K_j}(z, F), \quad z \in E_2$$

(j ranging over $0, \ldots, q$ if $K_0 \neq \emptyset$ and over $1, \ldots, q$ if $K_0 = \emptyset$); further define WF on B by (4). It follows from 1.2 that, for $\zeta \in K_j$,

$$\lim_{\substack{z \to \zeta \\ z \in D}} W_{K_j}(z, F) = W_{K_j}(\zeta, F) + \pi F(\zeta);$$

further we have

$$\lim_{\substack{z \to \zeta \\ z \in D}} W_{K_l}(z, F) = W_{K_l}(\zeta, F)$$

for $l \neq j$, because $\zeta \notin K_l$ (compare 2.1). Consequently,

$$\lim_{\substack{z \to \zeta \\ z \in D}} W(z, F) = W(\zeta, F) + \pi F(\zeta).$$

We see that $WF(\zeta) = W(\zeta, F)$, $\zeta \in B$, $F \in C(B)$. $WF \in C(B)$ provided $F \in C(B)$ and

$$W : F \to WF$$

is a linear operator acting on $C(B)$. Defining $\mathscr{F}B = \lim_{R \to 0+} \mathscr{F}_R B$ (compare (2)) and using the notation introduced in 1.14 we have

3.4. Theorem. $\omega W = \mathscr{F}B$.

Proof. For every j we denote by W_{K_j} the operator on $C(K_j)$ defined by

$$W_{K_j} F(\zeta) = W_{K_j}(\zeta, F), \quad \zeta \in K_j, \quad F \in C(K_j).$$

We first show that

$$\omega W = \max_j \omega W_{K_j}. \tag{113}$$

For $F \in C(K_j)$ we denote by $E_j F (\in C(B))$ the function on B coinciding with F on K_j and vanishing on $B - K_j$. Given $F \in C(B)$ let $R_j F = F|_{K_j} (\in C(K_j))$ be its restriction to K_j. If T is any compact operator acting on $C(B)$ then $R_j T E_j$ is a compact operator on $C(K_j)$ so that

$$\omega W_{K_j} \leqq \|W_{K_j} - R_j T E_j\| = \|R_j W E_j - R_j T E_j\| \leqq \|W - T\|$$

which shows that

$$\max \omega W_{K_j} \leqq \omega W. \tag{114}$$

Put $\Psi_j = R_j W - W_{K_j} R_j$. Then, for every $F \in C(B)$ and $z \in K_j$,

$$\Psi_j F(z) = \sum_{l \neq j} W_{K_l}(z, F) = \sum_{l \neq j} \operatorname{Im} \int_{K_l} \frac{F(\zeta)}{\zeta - z} \, d\zeta$$

whence we conclude on account of 2.1 and the Arzelà theorem that Ψ_j is a compact operator from $C(B)$ into $C(K_j)$. It follows from the definition of Ψ_j that

$$W = \sum_j E_j (W_{K_j} R_j + \Psi_j).$$

Let now T_j be arbitrary compact operators acting on $C(K_j)$ each and put

$$T = \sum_j E_j(\Psi_j + T_j R_j) .$$

T being a compact operator on $C(B)$ we have

$$\omega W \leqq \|W - T\| = \|\sum_j E_j(W_{K_j} - T_j) R_j\| = \max_j \|W_{K_j} - T_j\|$$

whence $\omega W \leqq \max_j \omega W_{K_j}$ which together with (114) implies (113).

It is easily seen that

$$0 < r < \min_{j \neq l} \operatorname{dist}(K_j, K_l) \Rightarrow \mathscr{F}_r B = \max_j \mathscr{F}_r K_j$$

whence

$$\mathscr{F} B = \max_j \mathscr{F} K_j . \tag{115}$$

We know from 1.20 that $\mathscr{F} K_j = \omega W_{K_j}$ so that (113) and (115) conclude the proof.

3.5. Notation. Given $G \in C(B)$ we denote $\mathbf{D} G$ the class of all $F \in C(B)$ for which the function Φ_2 defined on $\bar{D}$ by

$$\Phi_2(z) = \pi^{-1} W(z, F) , \quad z \in D , \quad \Phi_2(\zeta) = \pi^{-1} \lim_{\substack{z \to \zeta \\ z \in D}} W(z, F) , \quad \zeta \in B ,$$

possesses the property 1) from 3.2.

Let Q be the class of all $F \in C(B)$ which are constant on every K_j. Further, let Q_0 be the class of all $F \in Q$ vanishing on K_0 ($Q_0 = Q$ if $K_0 = \emptyset$). Clearly, Q_0 is a q-dimensional subspace in $C(B)$. We shall denote by I the identity operator on $C(B)$.

3.6. Lemma. *Let $F, G \in C(B)$. Then*

$$F \in \mathbf{D}\, G \Leftrightarrow (I + \pi^{-1} W) F - G \in Q_0 .$$

Proof follows at once from the relation

$$\lim_{\substack{z \to \zeta \\ z \in D}} W(z, F) = W F(\zeta) + \pi F(\zeta) , \quad \zeta \in B . \tag{116}$$

3.7. Notation. Let T be a linear operator on $C(B)$ enjoying the following properties (117), (118):

$$T C(B) \subset Q_0 , \tag{117}$$

$$T Q_0 = Q_0 . \tag{118}$$

To obtain such an operator T it is sufficient to put

$$TF(\zeta) = \sum_{j=1}^{q} e_j(\zeta) F(\zeta_j), \quad \zeta \in B,$$

where e_j denotes the characteristic function of K_j on B and ζ_j is a fixed point in K_j $(j = 1, \ldots, q)$.

The following lemma follows at once from 3.6 and (117)

3.8. Lemma. *Let* $G \in C(B)$. *Then* $\{F;\ F \in C(B),\ (I + \pi^{-1}W + T) F = G\} \subset \mathbf{D}G$.

3.9. Remark. In view of 3.8 it is sufficient to show that, for every $G \in C(B)$, the equation

(119) $$(I + \pi^{-1}W + T) F = G$$

has a solution $F \in C(B)$ in order to prove the existence theorem for the modified Dirichlet problem. To be able to apply the Riesz-Schauder theory to the equation (119) we now impose (7) on B (compare (3.4). Then the Fredholm alternative is valid (note that T is compact) and it is sufficient to verify that

(120) $$(F_0 \in C(B),\ (I + \pi^{-1}W + T) F_0 = 0) \Rightarrow F_0 = 0$$

in order to prove that, for any $G \in C(B)$, there si a unique solution $F \in C(B)$ of (119) (cf. [6], n° 89 and [1], chap. XIII). We first prove the following

3.10. Lemma. *Assume* (7). *Then*

(121) $$(F \in C(B),\ (I + \pi^{-1}W) F \in Q) \Rightarrow F \in Q.$$

Proof. Let $F \in C(B)$, $(I + \pi^{-1}W) F \in Q$. Consider the function

$$\Psi(z) = \frac{1}{\pi i} \sum_j \int_{K_j} \frac{F(\zeta)}{\zeta - z} d\zeta, \quad z \in E_2 - B.$$

We have by 1.2 for every $\zeta \in B$

$$\lim_{\substack{z \to \zeta \\ z \in D}} \operatorname{Re} \Psi(z) = (I + \pi^{-1}W) F(\zeta)$$

so that $\operatorname{Re} \Psi$ extends from D to a continuous function Ψ_1 on $D \cup B$. $\Psi_1(B)$ being finite we conclude on account of 3.1 that Ψ is constant on D. Fix now a_j and $r > 0$ with $r < \min_{l \neq j} \operatorname{dist}(K_j, K_l)$ and define $U_j = \{z;\ z \in E_2,\ \operatorname{dist}(z, K_j) < r\}$. Further denote by A_j, B_j the complementary domains of K_j; let the notation be so chosen that $D \subset A_j$, $D \cap B_j = \emptyset$. By 2.1 every

$$\int_{K_l} \frac{F(\zeta)}{\zeta - z} d\zeta$$

with $l \neq j$ is uniformly continuous in U_j. Since Ψ is uniformly continuous on $U_j \cap \cap A_j = U_j \cap D$ (for it is constant on D) so must be

$$\text{Re} \int_{K_j} \frac{F(\zeta)}{\zeta - z} \, d\zeta = \text{Re} \left(\pi i \, \Psi(z) - \sum_{l \neq j} \int_{K_l} \frac{F(\zeta)}{\zeta - z} \, d\zeta \right). \tag{122}$$

By theorem 2.3 (cf. also 2.1) we conclude that (122) is uniformly continuous on $U_j - K_j$ so that also

$$\text{Im} \, \Psi(z) = -\frac{1}{\pi} \text{Re} \int_{K_j} \frac{F(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{\pi} \sum_{l \neq j} \text{Re} \int_{K_l} \frac{F(\zeta)}{\zeta - z} \, d\zeta$$

is uniformly continuous on $U_j - K_j$. We have thus

$$\lim_{\substack{z \to \zeta \\ z \in B_j}} \text{Im} \, \Psi(z) = \lim_{\substack{z \to \zeta \\ z \in A_j}} \text{Im} \, \Psi(z) = a_j, \quad \zeta \in K_j,$$

where a_j is a constant (note that Ψ is constant on D). Hence Im Ψ is constant on B_j (in case $j = 0$ the domain $B_j = B_0$ is unbounded and we notice that $\Psi(\infty) = 0$). Consequently, also Re Ψ is constant on B_j and we obtain from 1.2 (cf. also 2.1) that

$$2F(\zeta) = \lim_{\substack{z \to \zeta \\ z \in A_j}} \text{Re} \, \Psi(z) - \lim_{\substack{z \to \zeta \\ z \in B_j}} \text{Re} \, \Psi(z)$$

is constant on K_j. Since j was arbitrary we see that $F \in Q$.

3.11. Lemma. *Let $\mathscr{F}B < \pi$. Then*

$$(F \in C(B), \, (I + \pi^{-1}W) \, F \in Q_0) \Rightarrow F \in Q_0, \tag{123}$$

$$F \in Q_0 \Rightarrow (I + \pi^{-1}W) \, F = 0. \tag{124}$$

Proof. Given $F \in Q$ we denote by $a_0(F)$ the value assumed by F on K_0 if $K_0 \neq \emptyset$ and put $a_0(F) = 0$ if $K_0 = \emptyset$.

We shall first show that

$$F \in Q \Rightarrow (I + \pi^{-1} W) \, F(\zeta) = 2a_0(F) \quad \text{for every} \quad \zeta \in B. \tag{125}$$

Indeed, we have for $F \in Q$ and $z \in D$

$$W_{K_j}(z, F) = 0, \quad j = 1, \ldots, q; \tag{126}$$

in case $K_0 \neq \emptyset$ we have besides (126) $W_{K_0}(z, F) = 2\pi a_0(F)$, whence we obtain $\sum_j W_{K_j}(z, F) = W(z, F) = 2\pi a_0(F)$, $z \in D$. This together with (116) implies (125). (124) is merely a special case of (125). Finally, (123) follows at once from 3.10 and (125).

3.12. Lemma. *Let* $\mathscr{F}B < \pi$. *Then* $Q_0 = \mathbf{D}0$.

Proof. (124) and (116) imply $Q_0 \subset \mathbf{D}0$ while (116) and (123) imply $\mathbf{D}0 \subset Q_0$.

3.13. Theorem. *Let* $\mathscr{F}B < \pi$. *Then* $\mathbf{D}G \neq \emptyset$ *for every* $G \in C(B)$. *Given* $F \in \mathbf{D}G$ $(G \in C(B))$ *we have* $\mathbf{D}G = F + Q_0 = \{F + H;\ H \in Q_0\}$.

Proof. Let $F_0 \in C(B)$ and suppose that $(I + \pi^{-1}W + T) F_0 = 0$. In view of (117) we have $(I + \pi^{-1}W) F_0 = -TF_0 \in Q_0$ whence it follows by (123) that $F_0 \in Q_0$. Using (124) we conclude that $TF_0 = -(I + \pi^{-1}W) F_0 = 0$ and, on account of (118) (note that Q_0 is finite-dimensional), we arrive at $F_0 = 0$. Thus (120) is verified and by 3.9 and 3.8 we have $\mathbf{D}G \neq \emptyset$ for every $G \in C(B)$. The rest is an immediate consequence of 3.12.

3.14. Remark. Main reasonings used above for the proof of 3.13 are classical for B consisting of sufficiently smooth curves (compare [3], § 32 and [4], chap. III, § 61). The idea of employing the equation (119) with an operator T enjoying (117), (118) for solution of the modified Dirichlet problem is due to N. I. Muskhelishvili (cf. [4], footnote on p. 253 for the bibliography).

An $F \in \mathbf{D}G$ $(G \in C(B))$ having been found the Dirichlet problem corresponding to G can be easily solved as described in [4], chap. III, § 63 (cf. also [3], §§ 31, 32).

References

[1] *Л. В. Канторович — Г. П. Акилов:* Функциональный анализ в нормированных пространствах, Москва 1959.

[2] *J. Král - J. Mařík:* Der Greensche Satz, Czech. mat. J. 7 (*82*), 1957, 235—247.

[3] *S. G. Michlin:* Integrální rovnice, Praha 1952 (*С. Г. Михлин:* Интегральные уравнения и их приложения к некоторым проблемам механики, математической физики и техники, Москва-Ленинград 1949).

[4] *Н. И. Мусхелишвили:* Сингулярные интегральные уравнения, Москва 1962.

[5] *I. I. Priwalow* (red. A. I. Markuschewitsch); Randeigenschaften analytischer Funktionen, Berlin 1956 (*И. И. Привалов:* Граничные свойства аналитических функций, Москва-Ленинград 1950).

[6] *F. Riesz - B. Sz. Nagy:* Leçons d'Analyse Fonctionelle, Budapest 1952.

[7] *S. Saks:* Theory of the integral, New York 1937.

[8] *S. Saks - A. Zygmund:* Funkcje analityczne, Stockholm 1948.

[9] *R. Sikorski:* Funkcje rzeczywiste I, Warszawa 1958.

[10] *J. Král:* On the logarithmic potential of the double distribution Czech. Mat. J. *14* (*89*), 1964, 306—321.

[11] *J. Král:* Some inequalities concerning the cyclic and radial variations of a plane path-curve, Czech. mat. J. *14* (*89*), 1964, 271—280.

[12] *J. Král:* Poznámka o lineární míře a délce cesty v metrickém prostoru, Acta Univ. Carolinae — Math. et Phys., 1963, No 1, 1—10.

[13] *J. Král:* Non-tangential limits of the logarithmic potential, Czech. mat. J. *14* (*89*), 1964, 455—482.

Резюме

РАДИУС ФРЕДГОЛМА ОДНОГО ОПЕРАТОРА ТЕОРИИ ПОТЕНЦИАЛА

ИОСЕФ КРАЛ (Josef Král), Прага

Пусть $D_1, \ldots, D_q$ — ограниченные дополнительные области простых замкнутых спрямляемых кривых $K_1, \ldots, K_q$. Предположим, что кривые K_j отрицательно ориентированы и множества $\bar{D}_j = D_j \cup K_j$ взаимно не пересекаются. Пусть, далее, E — евклидова плоскость или ограниченная дополнительная область положительно ориентированной спрямляемой кривой Жордана K_0 такая, что $\bigcup_{j=1}^{q} \bar{D}_j \subset E$. Положим $D = E - \bigcup_{j=1}^{q} \bar{D}_j$ и обозначим через $B = \bigcup_{j=0}^{q} K_j$ ориентированную границу области D. (Мы полагаем $K_0 = \emptyset$, если D неограничена. В случае $K_0 \neq \emptyset$ допускается $q = 0$; таким образом, D может совпадать с ограниченной дополнительной областью кривой K_0.)

Пространство Банаха всех непрерывных действительных функций F на B с нормой $\|F\| = \max \{|F(\zeta)|; \zeta \in B\}$ обозначим через $C(B)$. Для каждой функции $F \in C(B)$ рассмотрим соответствующий потенциал двойного слоя

$$W(z, F) = \operatorname{Im} \int_B \frac{F(\zeta)}{\zeta - z} \, d\zeta \,, \quad z \in D \,. \tag{1}$$

Если обозначить через $\mu_R(\zeta, \alpha)$ число точек, в которых отрезок $\{\zeta + r \exp i\alpha; 0 < r < R\}$ пересекает B, то $\mu_R(\zeta, \alpha)$ является измеримой (по Лебегу) функцией переменного α и мы можем полагать по определению

$$\mathscr{F}_R B = \sup_{\zeta \in B} \int_0^{2\pi} \mu_R(\zeta, \alpha) \, d\alpha \,.$$

Из результатов статьи [10] вытекает, что

$$\mathscr{F}_\infty B < \infty \tag{2}$$

является необходимым и достаточным условием для того, чтобы потенциал (1) с произвольной плотностью $F \in C(B)$ допускал непрерывное расширение с D на $\bar{D} = D \cup B$. В дальнейшем предполагаем, что условие (2) соблюдается. Определим на $C(B)$ оператор $W : F \to WF$ полагая

$$WF(\zeta) = \lim_{\substack{z \to \zeta \\ z \in D}} W(z, F) - \pi F(\zeta) \,, \quad \zeta \in B \,, \quad F \in C(B) \,.$$

Хорошо известно, что некоторые важные краевые задачи сводятся к решению уравнения вида

$$(I + \pi^{-1} W + T)\, F = G \tag{3}$$

(с заданной функцией $G \in C(B)$ и неизвестной функцией $F \in C(B)$), где I обозначает тождественный оператор и T — определенный вполне непрерывный оператор действующий на $C(B)$. В связи с применением теории Рисса-Шаудера к уравнению (3) полезно изучить величину

$$\omega W = \inf_Q \|W - Q\|,$$

где нижняя грань берется по всем вполне непрерывным операторам, действующим на $C(B)$. В статье доказывается равенство

$$\omega W = \lim_{R \to 0+} \mathscr{F}_R B\,.$$

В качестве применения доказывается единственность, существование и представимость в виде потенциала двойного слоя решения модифицированной проблемы Дирихле для областей, граница которых подчиняется условию

$$\lim_{R \to 0+} \mathscr{F}_R B < \pi\,.$$

В доказательстве используются некоторые установленные в статье свойства модифицированного потенциала простого слоя

$$M(z, F) = \operatorname{Re} \int_B \frac{F(\zeta)}{\zeta - z}\, \mathrm{d}\zeta\,.$$