Czechoslovak Mathematical Journal

Josef Král

The Fredholm radius of an operator in potential theory

Czechoslovak Mathematical Journal, Vol. 15 (1965), No. 3, 454–473

Persistent URL: http://dml.cz/dmlcz/100686

THE FREDHOLM RADIUS OF AN OPERATOR IN POTENTIAL THEORY

JOSEF KRÁL, Praha

(Received December 12, 1964)

Let D be a domain in the plane bounded by a finite number of non-intersecting rectifiable Jordan curves and let B be the oriented boundary of D. In [10] a simple necessary and sufficient condition was established for the logarithmic potential

$$W(z, F) = \operatorname{Im} \int_B \frac{F(\zeta)}{\zeta - z} \, d\zeta$$

of the double distribution with an arbitrary continuous density F to admit a continuous extension from D to $D \cup B$. If this condition holds then the potential $W(\zeta, F)$ can be defined for $\zeta \in B$ also and fulfils the usual equation

$$W(\zeta, F) = \lim_{\substack{z \to \zeta \\ z \in D}} W(z, F) \pm \pi F(\zeta), \quad \zeta \in B.$$

The operator

$$W : F(\zeta) \to W(\zeta, F)$$

acting on the Banach space of all continuous functions F on B with the supremum norm plays an important rôle in connection with some boundary value problems. In the present paper an expression for the Fredholm radius of W is derived exhibiting its dependence on the shape of B. This result is applied to obtain a solution of the modified Dirichlet problem for a sufficiently wide class of domains.

INTRODUCTION

Let $K_1, \ldots, K_q$ be clockwise oriented rectifiable Jordan curves in the plane and let D_j be the bounded complementary domain of K_j $(j = 1, \ldots, q)$. We suppose that the corresponding closed regions $\bar{D}_j = D_j \cup K_j$ $(1 \leqq j \leqq q)$ are mutually disjoint. Let E be either the whole Euclidean plane or a bounded complementary domain of a counterclockwise oriented rectifiable Jordan curve K_0 such that $\bigcup_{j=1}^{q} \bar{D}_j \subset E$ and put

$$D = E - \bigcup_{j=1}^{q} \bar{D}_j .$$

Let $B = \bigcup_{j=0}^{q} K_j$ be the oriented boundary of D. (We put $K_0 = \emptyset$ if D is unbounded;

in case $K_0 \neq \emptyset$ we allow $q = 0$ so that D may coincide with the bounded complementary domain of K_0.) Denoting by $C(B)$ the Banach space of all continuous real-valued functions F on B with the norm $\|F\| = \max_{\zeta \in B} |F(\zeta)|$ we consider, for every $F \in C(B)$, the corresponding logarithmic potential of the double distribution

$$(1) \qquad W(z, F) = \operatorname{Im} \int_B \frac{F(\zeta)}{\zeta - z} \, d\zeta \,, \quad z \in D \,.$$

It follows from [10] that a necessary and sufficient condition securing the uniform continuity of (1) (or, which is the same, its continuous extendability from D to $\bar{D}$) for every $F \in C(B)$ can be expressed in the following manner. Given $\zeta \in B$, $R > 0$ and $\alpha \in \langle 0, 2\pi \rangle$ denote by $\mu_R(\zeta, \alpha)$ the number ($0 \leqq \mu_R(\zeta, \alpha) \leqq +\infty$) of points in $B \cap \{\zeta + r \exp i\alpha;\ 0 < r < R\}$; $\mu_R(\zeta, \alpha)$ being Lebesgue measurable with respect to α (cf. [11]) we may put

$$(2) \qquad \mathscr{F}_R B = \sup_{\zeta \in B} \int_0^{2\pi} \mu_R(\zeta, \alpha) \, d\alpha \,.$$

Now the above mentioned condition reads as follows:

$$(3) \qquad \mathscr{F}_\infty B < \infty \,.$$

Imposing (3) on B we form the operator W on $C(B)$ by

$$(4) \qquad W F(\zeta) = \lim_{\substack{z \to \zeta \\ z \in D}} W(z, F) - \pi F(\zeta) \,, \quad F \in C(B) \,, \quad \zeta \in B \,;$$

in fact, $W F(\zeta)$ is merely the direct value of the logarithmic potential of the double distribution with density F at $\zeta \in B$. It is well known that some important boundary value problems reduce to solution of an equaiton of the form

$$(5) \qquad (I + \pi^{-1} W + T) F = G$$

(with a prescribed $G \in C(B)$ and unknown F), where I stands for the identity operator and T is a compact operator acting on $C(B)$. In order to be able to apply the Riesz-Schauder theory to the equation (5) it is useful to know the Fredholm radius of W which is the reciprocal of $\omega W = \inf_T \|W - T\|$, T ranging over all compact operators acting on $C(B)$. We show that

$$\omega W = \lim_{R \to 0+} \mathscr{F}_R B \quad (= \mathscr{F} B \text{ ex definitione}).$$

As an application we treat the modified Dirichlet problem consisting in determining – to a prescribed $G \in C(B)$ – a single-valued analytic function Φ in D such that $\operatorname{Im} \Phi$ extends continuously to a function Φ_2 on $D \cup B$ in such a way that $\Phi_2 = G$ on K_0 ($\Phi_2(\infty) = 0$ if $K_0 = \emptyset$) and $\Phi_2 - G$ reduces to a constant on every K_j, $j = 1, \ldots, q$.

We require Φ to be expressible in the form

$$\Phi(z) = \pi^{-1} \int_B \frac{F(\zeta)}{\zeta - z} \,\mathrm{d}\zeta\,, \quad z \in D \tag{6}$$

with an $F \in C(B)$. Following an idea of N.I. Muskhelishvili we introduce an operator T mapping $C(B)$ onto the subspace of all the functions vanishing K_0 and remaining constant on every K_j ($j = 1, \ldots, q$) and reduce the problem to the equation (5). In view of the Riesz-Schauder theory it is now natural to impose

$$\mathscr{F} B < \pi \tag{7}$$

on B. Then, by the Fredholm theorem, it is sufficient to show that the corresponding homogeneous equation

$$(I + \pi^{-1} W + T)\, F_0 = 0$$

has $F_0 = 0$ for its unique solution in order to obtain that, for every $G \in C(B)$, there is a unique F satisfying (5). This is done by means of the following theorem concerning the modified logarithmic potential of the single distribution

$$\mathrm{Re} \int_B \frac{F(\zeta)}{\zeta - z} \,\mathrm{d}\zeta = M(z, F)$$

established in § 2:

Assume (7). *Then, for* $F \in C(B)$, *the following conditions* (I) *and* (II) *are equivalent to each other:*

(I) $M(z, F)$ *is uniformly continuous in* D.

(II) *The integral*

$$M(\eta, F) = \text{V.p. } \mathrm{Re} \int_B \frac{F(\zeta)}{\zeta - \eta} \,\mathrm{d}\zeta = \lim_{r \to 0+} \mathrm{Re} \int_{B_r(\eta)} \frac{F(\zeta)}{\zeta - \eta} \,\mathrm{d}\zeta\,,$$

where $B_r(\eta) = \{\zeta;\, \zeta \in B,\, |\zeta - \eta| > r\}$, *converges uniformly in* $\eta \in B$.

If (II) *holds then* $M(z, F)$ *is uniformly continuous in the whole plane.*

As a final result we obtain that, for B submitted to (7) and every $G \in C(B)$, there is an $F \in C(B)$ such that (6) provides a solution of the corresponding modified Dirichlet problem, $F|_{K_j}$ being uniquely determined up to an additive constant a_j, where $a_0 = 0$ (provided $K_0 \neq \emptyset$) and $a_1, \ldots, a_q$ are arbitrary (compare [4], chapter III).

Let us remark that every B consisting of Lyapunov contours fulfils (3) and (7). If B consists of curves with bounded rotation (Kurven beschränkter Drehung) then (3) holds and, by the Radon theorem, $\mathscr{F} B < \pi$ if and only if there are no pin-points in B (cf. [6], $n°$ 91). It is interesting to observe that the Radon theorem is no longer valid for more general B submitted to (3) only. In § 1 an example is given showing that $\mathscr{F} B > \pi$ is possible for a B without angular points fulfilling (3).

§ 1

In the present paragraph we shall derive the above indicated results concerning the Fredholm radius of W for the case of a simply connected Jordan domain.

1.1. Notation. We shall assume throughout that ψ is a continuous complex-valued function of period $2k > 0$ on the real line E_1 satisfying the following condition:

$$0 < |u - v| < 2k \Rightarrow \psi(u) \neq \psi(v) .$$

We put $K = \psi(\langle 0, 2k \rangle)$. The same symbol K will be used to denote the simple closed oriented curve determined in an obvious way by ψ. Given $z \notin K$ we denote by $\vartheta_z(t)$ a single-valued continuous argument of $\psi(t) - z$ on E_1; ϑ_z is uniquely determined up to an additive constant. Noting that $2k$ is a period of ψ we see that

$$\vartheta_z(t + 2k) - \vartheta_z(t) = \Delta_u \arg [\psi(u) - z; \langle t, t + 2k \rangle] \tag{8}$$

must be constant on E_1. Since (8) is independent of t and of the choice of ϑ_z we are justified to define

$$\operatorname{ind}(z, K) = \frac{1}{2\pi} \Delta_u \arg [\psi(u) - z; \langle t, t + 2k \rangle] .$$

We have then $\operatorname{ind}(z, K) = 0$ for z in the unbounded complementary domain of K while $\operatorname{ind}(z, K) = \sigma$ for every z in the bounded complementary domain of K; the constant σ $(= \pm 1)$ characterizing the orientation of K will always have the meaning we have just described.

The variation of a (complex- or real-valued) function f on a set U open in an interval $J \subset E_1$ will be denoted by var $[f; U]$; it is defined as the least upper bound of all the sums $\sum_{j=1}^{n} |f(b_j) - f(a_j)|$, $\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle$ ranging over all finite systems of non-overlapping compact intervals contained in U. We suppose that var $[\psi; \langle 0, 2k \rangle] < +\infty$ (which amounts the same as the rectifiability of K); clearly, also var $[\psi; J] < +\infty$ for every bounded interval J. It follows from 1.12 in [10] that var $[\vartheta_z; J] < +\infty$ for any bounded interval J provided $z \notin K$.

If $M \neq \emptyset$ is a subset in the plane then $C(M)$ stands for the Banach space of all bounded continuous real-valued functions F on M with the norm $\|F\| = \sup \{|F(z)|; z \in M\}$.

Given $F \in C(K)$ and $z \notin K$ we define

$$W_K(z, F) = \int_0^{2k} F(\psi(t)) \, d\vartheta_z(t) \quad \left(= \operatorname{Im} \int_K \frac{F(\zeta)}{\zeta - z} \, d\zeta \right).$$

Noting that (8) is constant on E_1 we see that

$$W_K(z, F) = \int_I F(\psi(t))\, d\vartheta_z(t)$$

for any interval I of length $2k$.

The points (= vectors) in E_2, the Euclidean plane, are identified with the corresponding complex numbers. Given $\zeta \in E_2$, $R \in (0, +\infty\rangle$ and $\alpha \in \langle 0, 2\pi\rangle$ we denote by $\mu_R(\alpha, \zeta)$ the number $(0 \leqq \mu_R(\alpha, \zeta) \leqq +\infty)$ of points in $K \cap \{\zeta + r \exp i\alpha;\ 0 < r < R\}$. Since $\mu_R(\alpha, \zeta)$ is Lebesgue measurable with respect to α (cf. [11]) we may put

$$v_R^K(\zeta) = \int_0^{2\pi} \mu_R(\alpha, \zeta)\, d\alpha\,.$$

We write $v^K(\zeta)$ instead of $v_\infty^K(\zeta)$.

D will be a fixed component of $E_2 - K$. We know from [10] that

(9) $$\sup_{\zeta \in K} v^K(\zeta) < +\infty$$

is a necessary and sufficient condition to secure the uniform continuity of $W_K(z, F)$ on D for every $F \in C(K)$. Throughout § 1 we suppose (9) to be imposed on K. This implies that, for every $F \in C(K)$ and $\zeta \in K$, the limit

(10) $$\lim_{\substack{z \to \zeta \\ z \in D}} W_K(z, F)$$

exists. To obtain an expression for (10) we denote, for $\zeta \in K$, by ϑ_ζ a function on E_1 defined in the following manner. Fix a $t_0 \in E_1$ with $\psi(t_0) = \zeta$. Then $\psi(t) - \zeta$ is continuous and different from zero on $(t_0, t_0 + 2k)$. Let $\vartheta_\zeta(t)$ be a single-valued continuous argument of $\psi(t) - \zeta$ on $(t_0, t_0 + 2k)$. In view of

(11) $$\operatorname{var}\,[\vartheta_\zeta; (t_0, t_0 + 2k)] = v^K(\zeta) < +\infty$$

(cf. [11]), the limits

(12) $$\lim_{t \to t_0+} \vartheta_\zeta(t) = \vartheta_\zeta(t_0+)\,, \quad \vartheta_\zeta((t_0 + 2k)-) = \lim_{t \to (t_0 + 2k)-} \vartheta_\zeta(t)$$

are available. We define $\vartheta_\zeta(t_0) = \vartheta_\zeta(t_0+)$ and extend ϑ_ζ from $\langle t_0, t_0 + 2k)$ to E_1 by the requirement

(13) $$\vartheta_\zeta(t + 2k) = \vartheta_\zeta(t) + \sigma\pi\,, \quad t \in E_1\,.$$

It is easily seen that ϑ_ζ is uniquely determined up to an additive constant of the form $m\pi$, where m is an integer. On account of (11) and (13) we are justified to define for $F \in C(K)$ and $\zeta = \psi(t_0) \in K$

$$W_K(\zeta, F) = \int_I F(\psi(t))\, d\vartheta_\zeta(t)\,,$$

where I denotes an arbitrary compact interval of length $2k$.

Now (10) can be calculated as follows.

1.2. Proposition. *Given $F \in C(K)$ and $\zeta \in K$ we have*

$$\lim_{\substack{z \to \zeta \\ z \in D}} W_K(z, F) = W_K(\zeta, F) \pm \sigma\pi\, F(\zeta)\,, \tag{14}$$

where the sign "+" or "−" is taken according as D is bounded or not.

Proof. For the sake of brevity, let us consider here the case of a bounded domain only; the reader himself will easily complete the proof for an unbounded D. Let $\zeta = \psi(t_0)$. If F reduces to a constant γ on K then $z \in D \Rightarrow W_K(z, F) = 2\pi\sigma\gamma$; on the other hand, $W_K(\zeta, F) = \gamma . (\vartheta_\zeta(t_0 + 2k) - \vartheta_\zeta(t_0)) = \pi\sigma\gamma$, whence (14) follows at once. To complete the proof it is clearly sufficient to verify (14) for $F \in C(K)$ satisfying

$$F(\zeta) = 0\,. \tag{15}$$

Assuming (15) we shall show that

$$\lim_{\substack{z \to \zeta \\ z \in E_2 - K}} W(z, F) = W(\zeta, F)\,. \tag{16}$$

By theorem 1.11 in [10] it follows from (9) that

$$\sup_{z \in E_2 - K} \operatorname{var}\left[\vartheta_z; \langle t_0, t_0 + 2k\rangle\right] = \sup_{z \in E_2 - K} v^K(z) = c < +\infty\,.$$

Given $\varepsilon > 0$ we have a $\delta > 0$, $\delta < k$, such that

$$t \in \langle t_0, t_0 + \delta\rangle \cup \langle t_0 + 2k - \delta, t_0 + 2k\rangle \Rightarrow |F(\psi(t))| \leqq \varepsilon\,.$$

Hence we conclude that, for every $z \in E_2 - K_2$,

$$\left|\int_{t_0}^{t_0+\delta} F(\psi(t))\,\mathrm{d}\vartheta_z(t)\right| \leqq \varepsilon c\,, \quad \left|\int_{t_0+2k-\delta}^{t_0+2k} F(\psi(t))\,\mathrm{d}\vartheta_z(t)\right| \leqq \varepsilon c\,.$$

Since $\zeta \notin \psi(\langle t_0 + \delta, t_0 + 2k - \delta\rangle)$ we have by 1.12 in [10]

$$\lim_{z \to \zeta} \operatorname{var}\left[\vartheta_z - \vartheta_\zeta; \langle t_0 + \delta, t_0 + 2k - \delta\rangle\right] = 0$$

so that

$$\lim_{z \to \zeta} \int_{t_0+\delta}^{t_0+2k-\delta} F(\psi(t))\,\mathrm{d}(\vartheta_z(t) - \vartheta_\zeta(t)) = 0\,.$$

Summing up we obtain

$$\limsup_{\substack{z \to \zeta \\ z \in E_2 - K}} |W_K(z, F) - W_K(\zeta, F)| \leqq$$

$$\leqq \limsup \left\{\left|\int_{t_0}^{t_0+\delta} F(\psi(t))\,\mathrm{d}\vartheta_z(t)\right| + \left|\int_{t_0}^{t_0+\delta} F(\psi(t))\,\mathrm{d}\vartheta_\zeta(t)\right| + \right.$$

$$+ \left| \int_{t_0+2k-\delta}^{t_0+2k} F(\psi(t))\, \mathrm{d}\vartheta_z(t) \right| + \left| \int_{t_0+2k-\delta}^{t_0+2k} F(\psi(t))\, \mathrm{d}\vartheta_\zeta(t) \right| +$$

$$+ \left| \int_{t_0+\delta}^{t_0+2k-\delta} F(\psi(t))\, \mathrm{d}(\vartheta_z(t) - \vartheta_\zeta(t)) \right| \Bigg\} \leqq 4\varepsilon c \, .$$

Since $\varepsilon > 0$ was arbitrary we see that (16) is true.

1.3. Notation. Given $\zeta = \psi(t_0)$ then, as noted above, the limits (12) exist. Hence it follows that the following limits

(17) $$\lim_{t \to t_0+} \frac{\psi(t) - \zeta}{|\psi(t) - \zeta|} = \tau_K^+(\zeta) \, , \quad \lim_{t \to t_0-} \frac{\psi(t) - \zeta}{|\psi(t) - \zeta|} = -\tau_K^-(\zeta)$$

exist as well; it is easily seen that (17) do not depend upon the choice of $t_0 \in \psi^{-1}(\zeta)$. We shall denote by $\alpha_K(\zeta)$ ($\in \langle 0, \pi \rangle$) the radian measure of the non-oriented angle enclosed by the vectors $\tau_K^+(\zeta)$, $\tau_K^-(\zeta)$.

We are going to prove that $\alpha_K(\zeta) = |\vartheta_\zeta(t_0) - \vartheta_\zeta(t_0-)|$; first we prove two lemmas.

1.4. Lemma. *Let $z^1, z^2 \in E_2 - K$ and denote by S the segment with end-points z^1, z^2. Suppose that $S \cap K = \{\psi(t_0)\}$. Then there is a $\delta_0 > 0$ such that*

$$\operatorname{Im} \frac{\psi(t) - \psi(t_0)}{z^2 - z^1} = h(t)$$

has a constant sign on both $(t_0 - \delta, t_0)$ and $(t_0, t_0 + \delta)$; writing $S_+ = \operatorname{sign} h(t_0 + \frac{1}{2}\delta_0)$, $S_- = \operatorname{sign} h(t_0 - \frac{1}{2}\delta_0)$ we have

(18) $$\operatorname{ind}(z^2, K) - \operatorname{ind}(z^1, K) = \tfrac{1}{2}(S_- - S_+) \, .$$

Proof. Noting that $\operatorname{ind}(z, K)$ does not change if both z and K are submitted to a translation or rotation (cf. [8], chap. IV., § 6) we may clearly suppose that

$$z^1 = \operatorname{Re} z^1 < 0 = \psi(t_0) < \operatorname{Re} z^2 = z^2 \, .$$

Put $\psi_1 = \operatorname{Re} \psi$, $\psi_2 = \operatorname{Im} \psi$ and fix a $\delta \in (0, k)$ with $\psi_1(\langle t_0 - \delta, t_0 + \delta \rangle) \subset (z^1, z^2)$. Then $\operatorname{sign} \psi_2 = \operatorname{sign} h$ is constant on both $\langle t_0 - \delta, t_0)$ and $(t_0, t_0 + \delta\rangle$. Let ε be an arbitrary positive constant. There are points $\zeta^1 \in (z^1, 0)$, $\zeta^2 \in (0, z^2)$ such that

(19) $$|\Delta \arg [\psi(t) - \zeta^1; \langle t_0 - k, t_0 - \delta \rangle] - \Delta \arg [\psi(t) - \zeta^2; \langle t_0 - k, t_0 - \delta \rangle]| < \varepsilon \, ,$$

(20) $$|\Delta \arg [\psi(t) - \zeta^1; \langle t_0 + \delta, t_0 + k \rangle] - \Delta \arg [\psi(t) - \zeta^2; \langle t_0 + \delta, t_0 + k \rangle]| < \varepsilon \, .$$

We have

(21) $$\operatorname{ind}(z^j, K) = \operatorname{ind}(\zeta^j, K) \, , \quad j = 1, 2 \, ,$$

because the segment with end-points z^j, ζ^j does not meet K. We may clearly assume ζ^1, ζ^2 to be so close to each other that

$$\left| \arccos \frac{\psi_1(t_0 - \delta) - \zeta^1}{|\psi(t_0 - \delta) - \zeta^1|} - \arccos \frac{\psi_1(t_0 - \delta) - \zeta^2}{|\psi(t_0 - \delta) - \zeta^2|} \right| < \varepsilon , \tag{22}$$

$$\left| \arccos \frac{\psi_1(t_0 + \delta) - \zeta^1}{|\psi(t_0 + \delta) - \zeta^1|} - \arccos \frac{\psi_1(t_0 + \delta) - \zeta^2}{|\psi(t_0 + \delta) - \zeta^2|} \right| < \varepsilon . \tag{23}$$

Let $t_1 \in (t_0 - \delta, t_0)$, $t_2 \in (t_0, t_0 + \delta)$. We have by (21)

$$\begin{aligned} 2\pi \operatorname{ind}(z^j, K) = \Delta \arg[\psi(t) - \zeta^j; \langle t_0 - k, t_0 - \delta\rangle] + \\ + \Delta \arg[\psi(t) - \zeta^j; \langle t_0 - \delta, t_1\rangle] + \Delta \arg[\psi(t) - \zeta^j; \langle t_1, t_2\rangle] + \\ + \Delta \arg[\psi(t) - \zeta^j; \langle t_2, t_0 + \delta\rangle] + \Delta \arg[\psi(t) - \zeta^j; \langle t_0 + \delta, t_0 + k\rangle] . \end{aligned} \tag{24}$$

Further we have

$$t \in \langle t_0 - \delta, t_0) \Rightarrow \operatorname{sign} \psi_2(t) = \operatorname{sign} \operatorname{Im}(\psi(t) - \zeta^j) = S_- ,$$
$$t \in (t_0, t_0 + \delta\rangle \Rightarrow \operatorname{sign} \psi_2(t) = \operatorname{sign} \operatorname{Im}(\psi(t) - \zeta^j) = S_+ .$$

Noting that $(\operatorname{sign} y) . \arccos \dfrac{x}{|x + iy|}$ is a continuous argument of $x + iy$ on $\{x + iy; x, y \in E_1, y \neq 0\}$ we conclude that

$$\Delta \arg[\psi(t) - \zeta^j; \langle t_0 - \delta, t_1\rangle] =$$
$$= S_- . \left(\arccos \frac{\psi_1(t_1) - \zeta^j}{|\psi(t_1) - \zeta^j|} - \arccos \frac{\psi_1(t_0 - \delta) - \zeta^j}{|\psi(t_0 - \delta) - \zeta^j|} \right),$$
$$\Delta \arg[\psi(t) - \zeta^j; \langle t_2, t_0 + \delta\rangle] =$$
$$= S_+ . \left(\arccos \frac{\psi_1(t_0 + \delta) - \zeta^j}{|\psi(t_0 + \delta) - \zeta^j|} - \arccos \frac{\psi_1(t_2) - \zeta^j}{|\psi(t_2) - \zeta^j|} \right).$$

Hence it follows by (24)

$$\begin{aligned} 2\pi(\operatorname{ind}(z^2, K) - \operatorname{ind}(z^1, K)) = \Delta \arg[\psi(t) - \zeta^2; \langle t_1, t_2\rangle] - \\ - \Delta \arg[\psi(t) - \zeta^1; \langle t_1, t_2\rangle] + \\ + S_- . \left(\arccos \frac{\psi_1(t_1) - \zeta^2}{|\psi(t_1) - \zeta^2|} - \arccos \frac{\psi_1(t_1) - \zeta^1}{|\psi(t_1) - \zeta^1|} \right) - \\ - S_+ . \left(\arccos \frac{\psi_1(t_2) - \zeta^2}{|\psi(t_2) - \zeta^2|} - \arccos \frac{\psi_1(t_2) - \zeta^1}{|\psi(t_2) - \zeta^1|} \right) + c , \end{aligned} \tag{25}$$

where we put

$$c = \Delta \arg [\psi(t) - \zeta^2; \langle t_0 - k, t_0 - \delta\rangle] -$$
$$- \Delta \arg [\psi(t) - \zeta^1; \langle t_0 - k, t_0 - \vartheta\rangle] -$$
$$- S_- . \left(\arccos \frac{\psi_1(t_0 - \delta) - \zeta^2}{|\psi(t_0 - \delta) - \zeta^2|} - \arccos \frac{\psi_1(t_0 - \delta) - \zeta^1}{|\psi(t_0 - \delta) - \zeta^1|}\right) +$$
$$+ S_+ . \left(\arccos \frac{\psi_1(t_0 + \delta) - \zeta^2}{|\psi(t_0 + \delta) - \zeta^2|} - \arccos \frac{\psi_1(t_0 + \delta) - \zeta^1}{|\psi(t_0 + \delta) - \zeta^1|}\right) +$$
$$+ \Delta \arg [\psi(t) - \zeta^2; \langle t_0 + \delta, t_0 + k\rangle] - \Delta \arg [\psi(t) - \zeta^1; \langle t_0 + \delta, t_0 + k\rangle] .$$

Clearly,

$$\lim_{t_1 \to t_0 -} \left(\arccos \frac{\psi_1(t_1) - \zeta^2}{|\psi(t_1) - \zeta^2|} - \arccos \frac{\psi_1(t_1) - \zeta^1}{|\psi(t_1) - \zeta^1|}\right) = \arccos(-1) - \arccos 1 = \pi ,$$

$$\lim_{t_2 \to t_0 +} \left(\arccos \frac{\psi_1(t_2) - \zeta^2}{|\psi(t_2) - \zeta^2|} - \arccos \frac{\psi_1(t_2) - \zeta^1}{|\psi(t_2) - \zeta^1|}\right) = \arccos(-1) - \arccos 1 = \pi ,$$

while $\Delta \arg [\psi(t) - \zeta^j; \langle t_1, t_2\rangle] \to 0$ as $t_1 \to t_0 -, t_2 \to t_0 +$ $(j = 1, 2)$.
Noting that c does not depend on t_1, t_2 and making $t_1 \to t_0-$, $t_2 \to t_0+$ in (25) we obtain $2\pi(\operatorname{ind}(z^2, K) - \operatorname{ind}(z^1, K)) = c + \pi(S_- - S_+)$.

Now (19), (20), (22) and (23) imply $|c| < 4\varepsilon$; since $\varepsilon > 0$ was arbitrary, (18) is proved.

Remark. In the above proof, neither (9) nor the rectifiability of K were exploited. For another proof of a similar lemma concerning rectifiable curves cf. section 7 in [2].

1.5. Lemma. *Let* $\zeta = \psi(t_0)$. *Then* $|\vartheta_\zeta(t_0) - \vartheta_\zeta(t_0 -)| \leqq \pi$.

Proof. Consider r fulfilling

(26) $$0 < r < |\zeta - \psi(t_0 - k)| = |\zeta - \psi(t_0 + k)|$$

and put

$$c_r = \inf \{t; t \in (t_0 - k, t_0), |\psi(t) - \zeta| \leqq r\} ,$$
$$d_r = \sup \{t; t \in (t_0, t_0 + k), |\psi(t) - \zeta| \leqq r\} .$$

It is easily seen that $c_r \in (t_0 - k, t_0)$, $d_r \in (t_0, t_0 + k)$ (so that $\psi(c_r) \neq \psi(d_r)$), $|\psi(c_r) - \zeta| = r = |\psi(d_r) - \zeta|$ and $\lim_{r \to 0+} c_r = t_0 = \lim_{r \to 0+} d_r$.

We shall denote by K_r the simple closed oriented curve which is obtained by replacing the arc $\psi(\langle c_r, d_r\rangle)$ in K by the arc of the circle $\{z; |z - \zeta| = r\}$ with origin at $\psi(c_r)$ and end at $\psi(d_r)$ whose orientation coincides with that of K; to be more precise we proceed as follows. Put $\varphi_r(\tau) = \zeta + r \exp i(\alpha_r \tau + \beta_r)$, $c_r \leqq \tau \leqq d_r$, where α_r, β_r are

real numbers which are so chosen that φ_r be simple on $\langle c_r, d_r \rangle$ and the following conditions be satisfied: sign $\alpha_r = \sigma$, $\varphi(c_r) = \psi(c_r)$, $\varphi(d_r) = \psi(d_r)$.
We have then

$$(27) \qquad |\pi\sigma - \Delta \arg [\varphi_r(t) - \zeta; \langle c_r, d_r \rangle]| < \pi .$$

Next denote by ψ^r the continuous complex-valued function of period $2k$ on E_1 which coincides with ψ and φ_r on $\langle t_0 - k, c_r \rangle \cup \langle d_r, t_0 + k \rangle$ and $\langle c_r, d_r \rangle$ respectively. It is easily seen that ψ^r determines a simple closed oriented curve which, as well a the set $\psi^r(\langle t_0 - k, t_0 + k \rangle) = \psi^r(E_1)$, will be denoted by K_r. If $z \notin K$ and $\operatorname{ind}(z, K) = \sigma$ then $\lim_{r \to 0+} \operatorname{ind}(z, K_r) = \sigma$ because $\psi^r \to \psi$ uniformly as $r \to 0+$ (cf. [8], chap. IV, (6.3)). In particular, ind $(\ldots, K_r)$ assumes the value σ for sufficiently small r. Let us fix such an r with (26). We have

$$\psi^r(\langle t_0 - k, c_r \rangle \cup \langle d_r, t_0 + k \rangle) \subset \{z; |z - \zeta| \geqq r\} - \varphi_r((c_r, d_r)) .$$

Let $\zeta_r = \varphi_r(\frac{1}{2}(c_r + d_r))$. Then there is a $\delta_r > 0$ such that

$$\{\zeta + t(\zeta_r - \zeta); 0 \leqq t \leqq 1 + \delta_r\} \cap \psi^r(\langle t_0 - k, c_r \rangle \cup \langle d_r, t_0 + k \rangle) = \emptyset .$$

Put $z = \zeta + (1 + \delta_r)(\zeta_r - \zeta)$. Applying 1.4, where z^1, z^2, K and t_0 are changed for ζ, z, K_r and $\frac{1}{2}(c_r + d_r)$ respectively, we obtain $\operatorname{ind}(z, K_r) - \operatorname{ind}(\zeta, K_r) = -\sigma$. Noting that ind $(\ldots, K_r)$ cannot assume a value different from 0 and σ we conclude that $\operatorname{ind}(\zeta, K_r) = \sigma$ (and $\operatorname{ind}(z, K_r) = 0$). We have thus

$$\Delta \arg [\psi(t) - \zeta; \langle t_0 - k, c_r \rangle] + \Delta \arg [\varphi_r(t) - \zeta; \langle c_r, d_r \rangle] + \\ + \Delta \arg [\psi(t) - \zeta; \langle d_r, t_0 + k \rangle] = 2\pi\sigma ,$$

$$\vartheta_\zeta(c_r) - \vartheta_\zeta(t_0 - k) + \Delta \arg [\varphi_r(t) - \zeta; \langle c_r, d_r \rangle] + \\ + \vartheta_\zeta(t_0 + k) - \vartheta_\zeta(d_r) = 2\pi\sigma ,$$

whence $\vartheta_\zeta(c_r) - \vartheta_\zeta(d_r) = \pi\sigma - \Delta \arg [\varphi_r(t) - \zeta; \langle c_r, d_r \rangle]$. It follows from (27) that $|\vartheta_\zeta(c_r) - \vartheta_\zeta(d_r)| < \pi$. Making $r \to 0+$ we obtain $|\vartheta_\zeta(t_0-) - \vartheta_\zeta(t_0+)| = |\vartheta_\zeta(t_0-) - \vartheta_\zeta(t_0)| \leqq \pi$ which concludes the proof.

Now we are able to prove the promised

1.6. Proposition. *Given* $\zeta = \psi(t_0)$ *then* $\alpha_K(\zeta) = |\vartheta_\zeta(t_0) - \vartheta_\zeta(t_0-)|$.

Proof. We may suppose that $\vartheta_\zeta(t)$ is an argument of $\psi(t) - \zeta$ on $(t_0, t_0 + 2k)$. Then $\psi(t) - \zeta = |\psi(t) - \zeta| \exp i\, \vartheta_\zeta(t)$, $t \in (t_0, t_0 + 2k)$, whence

$$\tau_K^+(\zeta) = \lim_{t \to t_0+} \frac{\psi(t) - \zeta}{|\psi(t) - \zeta|} = \exp i\, \vartheta_\zeta(t_0) ,$$

$$\tau_K^-(\zeta) = \lim_{t \to t_0-} \frac{\zeta - \psi(t)}{|\zeta - \psi(t)|} = \lim_{t \to (t_0 + 2k)-} \frac{\zeta - \psi(t)}{|\zeta - \psi(t)|} = \\ = -\exp i\, \vartheta_\zeta((t_0 + 2k)-) = \exp i\, \vartheta_\zeta(t_0-) .$$

Using 1.5 we conclude that the non-oriented angle enclosed by $\tau_K^+(\zeta)$ and $\tau_K^-(\zeta)$ equals $|\vartheta_\zeta(t_0) - \vartheta_\zeta(t_0-)|$.

Remark. Every $\zeta \in K$ with $\alpha_K(\zeta) > 0$ will be called an angular point. Let us include here the proof of the following

1.7. Proposition. *The set of all angular points in K is at most countable.*

Proof. With every angular point $\zeta \in K$ we can associate rational numbers $\varrho(\zeta)$, $a(\zeta)$, $b(\zeta)$ with $\varrho(\zeta) > 0$, $0 < b(\zeta) - a(\zeta) < \pi$ such that, for $A(\zeta, \varrho, a, b) = \{z; |z-\zeta| < \varrho\} - \{\zeta + r \exp i\gamma; 0 \leqq r < \varrho, a(\zeta) \leqq \gamma \leqq b(\zeta)\}$ one has $K \cap A(\zeta, \varrho, a, b) = \emptyset$. Let us admit that the set $\mathscr{A}$ of all angular points in K is non-denumerable. Then there must be a triple of rational numbers ϱ, a, b such that

$$\varrho(\zeta) = \varrho\,, \quad a(\zeta) = a\,, \quad b(\zeta) = b \tag{28}$$

for an infinity of $\zeta \in \mathscr{A}$. In view of the compactness of K in the set of all $\zeta \in \mathscr{A}$ fulfilling (28) there must be two points, to be denoted by ζ_1 and ζ_2, such that

$$0 < |\zeta_1 - \zeta_2| < \varrho\,. \tag{29}$$

It is easily seen that (29) implies that either $\zeta_2 \in A(\zeta_1, \varrho, a, b)$ or $\zeta_1 \in A(\zeta_2, \varrho, a, b)$ which contradicts $K \cap \{A(\zeta_1, \varrho, a, b) \cup A(\zeta_2, \varrho, a, b)\} = \emptyset$.

As a consequence of the preceding proposition we obtain the following corollary which will be needed below.

1.8. Corollary. *The set of all $\zeta \in K$ with $\alpha_K(\zeta) = 0$ is dense in K.*

Remark. The above corollary could also be derived from the known fact that a rectifiable curve K possesses a unique tangent almost everywhere with respect to the linear measure (= length) on K.

1.9. Notation. We shall denote by C_p the Banach space of all real-valued continuous functions on E_1 with period p; the norm of an $f \in C_p$ is defined by $\|f\| = \max_t |f(t)|$. With every $F \in C(K)$ we can associate an $f \in C_{2k}$ defined by

$$f(t) = F(\psi(t))\,, \quad t \in E_1\,. \tag{30}$$

It is easily seen that, conversely, to any $f \in C_{2k}$ there is a unique $F \in C(K)$ fulfilling (30) and the correspondence

$$F \leftrightarrow f \tag{31}$$

determined by (30) is an isometric isomorphism between $C(K)$ and C_{2k}.

Given $s \in E_1$ we put $\vartheta^s = \vartheta_{\psi(s)}$ and define for every $f \in C_{2k}$

$$w f(s) = \int_{s-k}^{s+k} f(t) \, \mathrm{d}\vartheta^s(t) \; ;$$

we have thus

$$w f(s) = W_K(\psi(s), F)$$

for $F \in C(K)$ corresponding to f in (31). It follows from 1.2 that $W_K(\zeta, F)$ is a continuous function of the variable ζ in K; consequently, $wf \in C_{2k}$.

If $r > 0$ we put

$$M_{rs} = \{t;\, t \in \langle s - k,\, s + k\rangle,\, |\psi(t) - \psi(s)| \geqq r\} \tag{32}$$

and define

$$w_r f(s) = \int_{M_{rs}} f(t) \, \mathrm{d}\vartheta^s(t) \, , \quad f \in C_{2k} \, , \quad s \in E_1 \, .$$

We shall show later that $w_r f \in C_{2k}$ whenever r does not belong to an at most countable set $\mathscr{R}$ defined below; moreover, the operator

$$w_r : f \to w_r f \tag{33}$$

acting on C_{2k} will be shown to be compact provided $r \notin \mathscr{R}$.

The (outer) Hausdorff linear measure (= length) of a set $M \subset E_2$ (as defined in [7], chap. II, § 8) will be denoted by λM. It follows from known properties of λ that var $[\psi; I] = \lambda\psi(I)$ for any interval I with length $\leqq 2k$ (cf. [12] for references on the subject). Extending var $[\psi; \ldots]$ by the standard procedure to a Carathéodory outer measure (complare [11], section 1) we conclude easily that var $[\psi; M] = \lambda\psi(M)$ for every $M \subset E_1$ with diameter not exceeding $2k$. In particular, var $[\psi; \langle 0, 2k\rangle] = \lambda K$ and, for $M \subset E_1$, var $[\psi; M] = 0 \Leftrightarrow \lambda\psi(M) = 0$ (compare also 3.4 in [13]). We shall denote by var ψ the measure determined in a usual way by the outer measure var $[\psi; \ldots]$ (cf. [7], chap. II).

Let $\mathscr{R}$ be the set of all $r > 0$ for which there is a circumference S^r with radius r such that $\lambda(K \cap S^r) > 0$.

1.10. Lemma. *$\mathscr{R}$ is at most countable.*

Proof. Let us denote by $\mathscr{R}_n$ the set of all $r > 0$ for which there is a circumference S^r with radius r such that $\lambda(K \cap S^r) \geqq 1/n$. If $r_1, \ldots, r_m$ are different elements of $\mathscr{R}_n$ then there are circumferences $S_1, \ldots, S_m$ with radii $r_1, \ldots, r_m$ respectively such that $\lambda(K \cap S_j) \geqq 1/n$, $1 \leqq j \leqq m$. Noting that $S_i \cap S_j$ contains at most two points (and, consequently, $\lambda(S_i \cap S_j) = 0$) whenever $i \neq j$ we conclude that $m/n \leqq \sum_j \lambda(K \cap S_j) = \lambda(\bigcup_j K \cap S_j) \leqq \lambda K$. We see that the number of elements in $\mathscr{R}_n$ does not exceed $n\lambda K < +\infty$ so that $\mathscr{R} = \bigcup_n \mathscr{R}_n$ is at most countable.

1.11. Lemma. *Given* $f \in C_{2k}$ *and* $r > 0$ *define*

$$\tilde{w}_r f(s) = \int_{M_{rs}} \frac{f(t)}{\psi(t) - \psi(s)} \, d\psi(t) \,, \quad s \in E_1$$

(*cf.* (32); *the integral is taken in the sense of Lebesgue-Stieltjes*). *Let*

(34) $$\mathscr{B} = \{f; f \in C_{2k}, \|f\| \leqq 1\} .$$

If $r \in (0, +\infty) - \mathscr{R}$ *then all the functions* $\tilde{w}_r f$ *with* $f \in \mathscr{B}$ *are equicontinuous and uniformly bounded.*

Proof. It is easily seen that, for every $f \in C_{2k}$, $r > 0$ and $s \in E_1$,

$$|\tilde{w}_r f(s)| \leqq \|f\| \,.\, r^{-1} \,.\, \text{var}\,[\psi; M_{rs}] \leqq \|f\| \, r^{-1} \lambda K \,.$$

Fix now $r \in (0, +\infty) - \mathscr{R}$. In order to make the proof or our lemma complete it is sufficient to verify

(35) $$\lim_{x \to s} \sup_{f \in \mathscr{B}} |\tilde{w}_r f(x) - \tilde{w}_r f(s)| = 0 \,, \quad s \in E_1 \,.$$

Making use of the uniform continuity of ψ we fix a $\delta \in (0, k)$ such that

$$|a - b| < \delta \Rightarrow |\psi(a) - \psi(b)| < \tfrac{1}{2} r \,.$$

Fix $s \in E_1$ and consider $x \in (s - \delta, s + \delta)$. Let $\chi_r^x(t)$ stand for the characteristic function of $\{t; t \in E_1, |\psi(t) - \psi(x)| \geqq r\}$. If $|t - s| < \delta$ then $|\psi(t) - \psi(x)| \leqq \leqq |\psi(t) - \psi(s)| + |\psi(s) - \psi(x)| < r$ and, consequently, $\chi_r^x(t) = 0$. We see that

$$\tilde{w}_r f(x) = \int_{x-k}^{s-\delta} \frac{f(t)\, \chi_r^x(t)}{\psi(t) - \psi(x)} \, d\psi(t) + \int_{s+\delta}^{x+k} \ldots = \int_{s+\delta}^{s+2k-\delta} \frac{f(t)\, \chi_r^x(t)}{\psi(t) - \psi(r)} \, d\psi(t) \,,$$

(36) $$\tilde{w}_r f(s) - \tilde{w}_r f(x) = \int_{s+\delta}^{s+2k-\delta} f(t) \left(\frac{\chi_r^s(t)}{\psi(t) - \psi(s)} - \frac{\chi_r^x(t)}{\psi(t) - \psi(x)} \right) d\psi(t) = = J_1(x, f) + J_2(x, f) \,,$$

where we put

(37) $$J_1(x, f) = \int_{s+\delta}^{s+2k-\delta} f(t) \frac{\chi_r^s(t) - \chi_r^x(t)}{\psi(t) - \psi(s)} \, d\psi(t) \,,$$

(38) $$J_2(x, f) = \int_{s+\delta}^{s+2k-\delta} f(t)\, \chi_r^x(t) \left((\psi(t) - \psi(s))^{-1} - (\psi(t) - \psi(x))^{-1} \right) d\psi(t) \,.$$

Taking $c > 0$ small enough we have

$$s + \delta \leqq t \leqq s + 2k - \delta \Rightarrow |\psi(t) - \psi(s)| \geqq c$$

whence we conclude on account of (37)

$$(39)\qquad |J_1(x,f)| \leqq \|f\|\, c^{-1} \int_{s+\delta}^{s+2k-\delta} |\chi_r^s - \chi_r^x| \,\mathrm{d}\,\mathrm{var}\,\psi, \quad |x - s| < \delta .$$

It is easily seen that $\{t;\ s < t < s + 2k, \limsup_{x\to s} |\chi_r^s(t) - \chi_r^x(t)| > 0\} \subset \{t;\ s < t < s + 2k, |\psi(t) - \psi(s)| = r\}$; let us denote the last set by M. Since $r \notin \mathscr{R}$ we have

$$0 = \lambda\{\zeta;\ \zeta \in K, |\zeta - \psi(s)| = r\} = \lambda\psi(M) = \mathrm{var}\,[\psi; M]$$

so that, by (39),

$$(40)\qquad \lim_{x\to s} \sup_{f\in\mathscr{B}} |J_1(x,f)| = 0 .$$

Employing (38) and defining

$$h_x(t) = |(\psi(t) - \psi(s))^{-1} - (\psi(t) - \psi(x))^{-1}| ,$$
$$s + \delta \leqq t \leqq s + 2k - \delta,\ |x - s| < \delta ,$$

we arrive at

$$|J_2(x,f)| \leqq \|f\| \int_{s+\delta}^{s+2k-\delta} h_x \,\mathrm{d}\,\mathrm{var}\,\psi , \quad |x - s| < \delta .$$

Since $h_x(t) \to 0$ uniformly in $t \in \langle s + \delta, s + 2k - \delta\rangle$ as $x \to s$ we obtain

$$(41)\qquad \lim_{x\to s} \sup_{f\in\mathscr{B}} |J_2(x,f)| = 0 .$$

Finally, (40) and (41) together with (36) imply (35) which concludes the proof.

The following remark will be used later:

Remark. Put $N_r(s) = \{t;\ t \in \langle 0, 2k\rangle, |\psi(t) - \psi(s)| > 2r\}$. If $2r \in (0, +\infty) - \mathscr{R}$ then

$$\int_{N_r(s)} \frac{f(t)}{\psi(t) - \psi(s)} \,\mathrm{d}\psi(t) = \tilde{w}_{2r} f(s) , \quad s \in E_1 , \quad f \in C_{2k} .$$

Indeed, we have with the notation from the proof of 1.11

$$\tilde{w}_{2r} f(s) = \int_s^{s+2k} \frac{f(t)\,\chi_{2r}^s(t)}{\psi(t) - \psi(s)} \,\mathrm{d}\psi(t) = \int_0^{2k} \frac{f(t)\,\chi_{2r}^s(t)}{\psi(t) - \psi(s)} \,\mathrm{d}\psi(t) .$$

Noting that the variation of ψ on $\{t;\ t \in \langle 0, 2k\rangle, \chi_{2r}^s(t) \neq 0\} - N_r(s) = \{t;\ t \in \langle 0, 2k\rangle, |\psi(t) - \psi(s)| = 2r\}$ vanishes (cf. 1.9) we obtain that the last integral equals

$$\int_{N_r(s)} \frac{f(t)}{\psi(t) - \psi(s)} \,\mathrm{d}\psi(t) .$$

1.12. Lemma. *If* $r \in (0, +\infty) - \mathscr{R}$ *then*

$$f \in C_{2k} \Rightarrow w_r f \in C_{2k} \tag{42}$$

and the operator (33) *acting on* C_{2k} *is compact.*

Proof. Let $r \in (0, +\infty) - \mathscr{R}$ and let χ_r^s have the meaning described in the proof of 1.11. We have then

$$w_r f(s) = \int_s^{s+2k} f(t)\,\chi_r^s(t)\,\mathrm{d}\vartheta^s(t)\,, \quad \tilde{w}_r f(s) = \int_s^{s+2k} \frac{f(t)\,\chi_r^s(t)}{\psi(t) - \psi(s)}\,\mathrm{d}\psi(t), \quad s \in E_1\,, \quad f \in C_{2k}\,.$$

Noting that, for any pair of points $a < b$ in $(s, s + 2k)$,

$$\vartheta^s(b) - \vartheta^s(a) = \Delta_t \arg\left[\psi(t) - \psi(s); \langle a, b\rangle\right] = \operatorname{Im} \int_a^b \frac{\mathrm{d}\psi(t)}{\psi(t) - \psi(s)}\,,$$

we see that

$$w_r f(s) = \operatorname{Im} \tilde{w}_r f(s)\,, \quad s \in E_1\,, \quad f \in C_{2k}\,,$$

whence it follows (42) by 1.11. On account of 1.11 we conclude that all the functions $w_r f$ with $f \in \mathscr{B}$ are equicontinuous and uniformly bounded which, by the Arzelà theorem, implies the compactness of w_r.

1.13. Lemma. *Let* $\psi(s) = \zeta$, $r > 0$ *and put*

$$U_r^s = \{t;\, t \in \langle s - k, s + k\rangle,\, |\psi(t) - \zeta| < r\}\,. \tag{43}$$

Defining $\mathscr{B}$ *by* (34) *we have*

$$v_r^K(\zeta) + \alpha_K(\zeta) = \operatorname{var}\left[\vartheta^s; U_r^s\right] = \sup_{f \in \mathscr{B}} \left(w f(s) - w_r f(s)\right)\,; \tag{44}$$

if $r \notin \mathscr{R}$ *then* $v_r^K(\zeta) + \alpha_K(\zeta)$ *is a lower-semicontinuous function of the variable* ζ *in* K.

Proof. We have

$$w f(s) - w_r f(s) = \int_{U_r^s} f(t)\,\mathrm{d}\vartheta^s(t) \tag{45}$$

whence

$$\operatorname{var}\left[\vartheta^s; U_r^s\right] = \sup_{f \in \mathscr{B}} \left(w f(s) - w_r f(s)\right)\,. \tag{46}$$

It follows from 1.6 that $\operatorname{var}\left[\vartheta^s; U_r^s\right] = \alpha_K(\zeta) + \operatorname{var}\left[\vartheta_r; U_r^s - \{s\}\right]$ which together with the equality $\operatorname{var}\left[\vartheta^s; U_r^s - \{s\}\right] = v_r^K(\zeta)$ (cf. section 2 in [11]) and (46) provides (44).

Suppose now that $r \notin \mathscr{R}$. Then, by 1.12, (45) is continuous in s whenever $f \in C_{2k}$. Consequently, $\sup_{f \in \mathscr{B}} \left(w f(s) - w_r f(s)\right) = v_r^K(\psi(s)) + \alpha_K(\psi(s))$ is lower semicontinuous in s whence our assertion easily follows.

1.14. Notation. If A is a linear operator defined on a Banach space E with norm $\|\ldots\|$ we let T range over all compact linear operators acting on E and put

$$\omega A = \inf \|A - T\| .$$

1.15. Proposition. *Put* $\mathscr{F}_r K = \sup_{\zeta \in K} v_r^K(\zeta)$, $\mathscr{F} K = \lim_{r \to 0+} \mathscr{F}_r K$. *Then* $\omega w \leqq \mathscr{F} K$ *and*

$$r \in (0, +\infty) - \mathscr{R} \Rightarrow \mathscr{F}_r K = \sup_{\zeta \in K} (v_r^K(\zeta) + \alpha_K(\zeta)) . \tag{47}$$

Proof. Fix $r \in (0, +\infty) - \mathscr{R}$ and denote by H the set of all $\zeta \in K$ with $\alpha_K(\zeta) = 0$. We know from 1.8 that H is dense in K. In view of the lower-semicontinuity of (44) we have $\mathscr{F}_r K \geqq \sup_{\zeta \in H} v_r^K(\zeta) = \sup_{\zeta \in H} (v_r^K(\zeta) + \alpha_K(\zeta)) = \sup_{\zeta \in K} (v_r^K(\zeta) + \alpha_K(\zeta)) = \|w - w_r\|$ whence (47) easily follows; w_r being compact (cf. 1.12) we obtain

$$\mathscr{F}_r K \geqq \omega w , \quad r \in (0, +\infty) - \mathscr{R} .$$

Noting that $\mathscr{R}$ is at most countable (cf. 1.10) and $\mathscr{F}_r K$ is non-decreasing in r we conclude that also $\mathscr{F} K \geqq \omega w$.

Now we are going to prove the opposite inequality. Its proof will be based on approximation of compact operators acting on C_{2k} by operators of finite rank which is enabled by known results of J. Radon.

1.16. Notation. Let us denote by $\mathfrak{P}$ the class of all the operators $P: f \to Pf$ having the form

$$P f(s) = \sum_{j=1}^{n} f_j(s) \int_{s-k}^{s+k} f(t) \, \mathrm{d}g_j(t) , \quad s \in E_1 , \tag{48}$$

where $f_1, \ldots, f_n \in C_{2k}$ and g_j are (real-valued) functions with locally finite variation on E_1 fulfilling the following conditions (I), (II):

$$g_j(t+) = g_j(t) , \quad t \in E_1 , \tag{I}$$

$$g_j(t + 2k) - g_j(t) \quad \text{is constant on } E_1 \tag{II}$$

$(j = 1, \ldots, n)$. Thus $\mathfrak{P}$ is the class of all operators of finite rank acting on C_{2k}.

It follows from results established by J. Radon (cf. [6], chap. V, $n°$ 90) that the following assertion is true:

1.17. Proposition. *Let R be a linear operator acting on C_{2k}. Then (cf.* 1.14 *for notation)* $\omega R = \inf \{\|R - P\|;\ P \in \mathfrak{P}\}$.

Using this proposition we shall derive the following lemma which will be useful below:

1.18. Lemma. *Let $\mathfrak{G} \subset \mathfrak{P}$ be the class of all the operators P of the form* (48) *where $f_1, \ldots, f_n \in C_{2k}$ and $g_1, \ldots, g_n$ are continuous functions with locally finite variation fulfilling* (II). *Then*

$$\omega w = \inf \{\|w - Q\|; \; Q \in \mathfrak{G}\} . \tag{49}$$

Proof. Let P be an arbitrary operator in $\mathfrak{P}$ and suppose that P has the form (48) with $f_1, \ldots, f_n \in C_{2k}$ and $g_1, \ldots, g_n$ fulfilling (I) and (II). Put

$$h(s) = \operatorname{var}_t \left[\vartheta^s(t) - \sum_{j=1}^{n} f_j(s)\, g_j(t); \; \langle s - k, s + k \rangle\right], \quad s \in E_1 .$$

Defining $\mathscr{B}$ by (34) we have $h(s) = \sup_{f \in \mathscr{B}} (w - P) f(s) \leqq \liminf_{x \to s} \sup_{f \in \mathscr{B}} (w - P) f(x) = \liminf_{x \to s} h(x)$ so that h is lower semicontinuous on E_1. Clearly, $\|w - P\| = \sup \{h(s); s \in E_1\}$. Let $\mathscr{S}$ be the set of all $s \in E_1$ with $\alpha_K(\psi(s)) = 0$. It follows from 1.8 that $\mathscr{S}$ is dense in E_1 whence

$$\|w - P\| = \sup \{h(s); \; s \in \mathscr{S}\} . \tag{50}$$

Put for $t \in (0, 2k)$

$$s_j(t) = \sum_u [g_j(u) - g_j(u-)] , \quad u \in (0, t\rangle$$

(the sum being extended over $u \in (0, t\rangle$ with $g_j(u) - g_j(u-) \neq 0$) and extend s_j to E_1 by the requirement

$$s_j(t + 2k) - s_j(t) = s_j(2k) , \quad t \in E_1 .$$

Thus s_j is the saltus-function of g_j and we obtain the decomposition $g_j = q_j + s_j$, where q_j is continuous on E_1 and $q_j(t + 2k) - q_j(t)$ is constant on E_1 (compare (I), (II)), $j = 1, \ldots, n$. Further define the operator $Q \in \mathfrak{G}$ by

$$Q f(x) = \sum_{j=1}^{n} f_j(x) \int_{x-k}^{x+k} f(t)\, \mathrm{d}q_j(t)$$

and put

$$p(x) = \operatorname{var}_t \left[\vartheta^x(t) - \sum_{j=1}^{n} f_j(x)\, q_j(t); \; \langle x - k, x + k \rangle\right]$$

$$\left(= \sup_{f \in \mathscr{B}} (w - Q) f(x)\right) .$$

Then p is lower-semicontinuous on E_1 (compare the argument used for the proof of the lower-semicontinuity of h) so that

$$\|w - Q\| = \sup \{p(s); \; s \in \mathscr{S}\} . \tag{51}$$

Fix now $s \in \mathscr{S}$. Noting that ϑ^s is continuous on E_1 (cf. 1.6) we have the following decomposition

$$\vartheta^s - \sum_{j=1}^{n} f_j(s)\, g_j = (\vartheta^s - \sum_{j=1}^{n} f_j(s)\, q_j) - (\sum_{j=1}^{n} f_j(s)\, s_j)\,,$$

where $\vartheta^s - \sum_{j=1}^{n} f_j(s)\, q_j$ is continuous and $\sum_{j=1}^{n} f_j(s)\, s_j$ is a saltus-function. Consequently,

$$h(s) = \operatorname{var}_t [\vartheta^s(t) - \sum_{j=1}^{n} f_j(s)\, q_j(t);\ \langle s - k,\ s + k\rangle] +$$
$$+ \operatorname{var}_t [\sum_{j=1}^{n} f_j(s)\, s_j(t);\ \langle s - k,\ s + k\rangle] \geqq p(s)\,, \quad s \in \mathscr{S}.$$

Combining this with (51) and (50) we arrive at

$$\|w - P\| \geqq \|w - Q\|\,. \tag{52}$$

We have thus seen that with any $P \in \mathfrak{P}$ there can be associated a $Q \in \mathfrak{G}$ fulfilling (52). Hence it follows by 1.17 the equality (49).

1.19. Theorem. $\omega w = \mathscr{F}K = \lim_{r \to 0+} \sup_{\zeta \in K} (v_r^K(\zeta) + \alpha_K(\zeta))\,.$

Proof. Let Q be an arbitrary operator in $\mathfrak{G}$,

$$Q f(s) = \sum_{j=1}^{n} f_j(s) \int_{s-k}^{s+k} f(t)\, \mathrm{d}g_j(t)\,,$$

where $f_1, \ldots, f_n \in C_{2k}$ and $g_1, \ldots, g_n$ are continuous functions with locally finite variation fulfilling (II). Define U_r^s by (43). In view of 1.13, 1.15 and 1.18 it is sufficient to show that

$$\|w - Q\| \geqq \lim_{r \to 0+} \sup_{s \in E_1} \operatorname{var} [\vartheta^s;\ U_r^s]\,. \tag{53}$$

Clearly,

$$\|w - Q\| = \sup_s \operatorname{var} [\vartheta^s - \sum_{j=1}^{n} f_j(s)\, g_j;\ \langle s - k,\ s + k\rangle]\,,$$

$$\operatorname{var} [\vartheta^s - \sum_{j=1}^{n} f_j(s)\, g_j;\ \langle s - k,\ s + k\rangle] \geqq \operatorname{var} [\vartheta^s - \sum_{j=1}^{n} f_j(s)\, g_j;\ U_r^s] \geqq$$
$$\geqq \operatorname{var} [\vartheta^s;\ U_r^s] - \operatorname{var} [\sum_{j=1}^{n} f_j(s)\, g_j;\ U_r^s]\,.$$

Writing $c = \max_{1 \leqq j \leqq n} \sup_s |f_j(s)|$ we have $\operatorname{var} [\sum_{j=1}^{n} f_j(s)\, g_j;\ U_r^s] \leqq c \sum_{j=1}^{n} \operatorname{var} [g_j;\ U_r^s]$ so that

$$\|w - Q\| \geqq \sup_s \operatorname{var} [\vartheta^s;\ U_r^s] - c \sup_s \sum_{j=1}^{n} \operatorname{var} [g_j;\ U_r^s]\,. \tag{54}$$

Put $\delta_r = \sup_s \operatorname{diam} U_r^s$ (= diameter of U_r^s). It is easily seen that $\lim_{r \to 0+} \delta_r = 0$. Fix now a $j \in \langle 1, n \rangle$ and define $h_r(s) = \operatorname{var}[g_j; \langle s - \delta_r, s + \delta_r \rangle]$; $h_r(s)$ is continuous in s (for fixed r) and non-decreasing in r (for fixed s). Since $\lim_{r \to 0+} h_r(s) = 0$ we conclude by the Dini theorem that $\lim_{r \to 0+} \sup_s h_r(s) = 0$. Taking into account that $U_r^s \subset \subset \langle s - \delta_r, s + \delta_r \rangle$ we see that $\lim_{r \to 0+} \sup_s \sum_{j=1}^r \operatorname{var}[g_j; U_r^s] = 0$. Making $r \to 0+$ in (54) we arrive at (53) which concludes the proof.

Remark. As explained in 1.9, the operator $W_K : F(\zeta) \to W_K(\zeta, F)$ acting on $C(K)$ corresponds to w (acting on C_{2k}) in the isometric isomorphism (31) between $C(K)$ and C_{2k}. Hence we obtain easily that $\omega W_K = \omega w$ (cf. 1.14). In particular, we have the following corollary of 1.19):

1.20. Theorem. $\omega W_K = \mathscr{F} K = \lim_{r \to 0+} \sup_{\zeta \in K} (v_r^K(\zeta) + \alpha_K(\zeta))$.

1.21. Remark. Noting that W_K is compact if and only if $\omega W_K = 0$ we see that there must be no angular points in K in order that W_K be compact; on the other hand, we shall show by an example that $\mathscr{F} K > 0$ is possible for a K without angular points fulfilling (9). Let us first prove a simple lemma.

1.22. Lemma. *Let f be a (real-valued) continuous function of bounded variation on $\langle a, b \rangle$, $f(a) = 0$. For every $\varepsilon > 0$ denote by f^ε the non-parametric curve which is defined by the equation*

$$y = \varepsilon f(a + \varepsilon^{-1}(x - a)), \quad a \leqq x \leqq a + \varepsilon(b - a).$$

Then, for every $\zeta \in E_2$ and $\varepsilon > 0$,

$$v^{f^1}(a + \zeta) = v^{f^\varepsilon}(a + \varepsilon \zeta) \tag{55}$$

and, for every $c > 0$,

$$\lim_{\varepsilon \to 0+} v^{f^\varepsilon}(z) = 0 \quad \text{uniformly in} \quad \{z; z \in E_2, |\operatorname{Re} z - a| \geqq c\}. \tag{56}$$

Proof. Let us observe that the number of points at which a half-ray issuing at $a + \zeta$ meets f^1 coincides with the number of points at which the parallel half-ray issuing at $a + \varepsilon\zeta$ meets f^ε; hence (55) follows at once. It follows from 1.12 in [10] that for any curve K of length λK and every $z \in E_2$ with

$$\operatorname{dist}(z, K) = \inf \{|z - \zeta|; \zeta \in K\} > 0$$

the following estimate

$$v^K(z) \leqq \frac{\lambda K}{\operatorname{dist}(z, K)}$$

is valid. Since $v^{f^\varepsilon}(z) = v^{f^1}(a + \varepsilon^{-1}(z - a))$ and $1/\text{dist}\,(a + \varepsilon^{-1}(z - a), f^1) \to 0$ uniformly in $\{z;\ z \in E_2,\ |\text{Re}\, z - a| \geqq c\}$ $(c > 0)$ as $\varepsilon \to 0+$ we obtain (56).

1.23. Example. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of positive real numbers tending to 0 as $n \to \infty$ and let f be a continuously differentiable (real-valued) function on $\langle 0, 1\rangle$ such that $f(0) = f'(0) = f(1) = f'(1) = 0$ and, with the notation described in 1.22, $v^{f^1}(0) = \delta > 0$, $\sup\,\{v^{f^1}(z);\ z \in E_2\} = \gamma < +\infty$.

Put $a_0 = a_1 + 1$ and, for every $n \geqq 1$, fix an $\varepsilon^n > 0$ such that

$$a_n + \varepsilon^n < \tfrac{1}{2}(a_n + a_{n-1}), \quad a_n^{-1} \,.\, \varepsilon^n < 2^{-n}. \tag{57}$$

Defining f_n on $\langle a_n, a_n + 1\rangle$ by

$$f_n(x) = f(x - a_n), \quad a_n \leqq x \leqq a_n + 1,$$

we write $K^n = f_n^{\varepsilon^n}$ for the non-parametric curve corresponding to f_n and ε^n in the way described in 1.22. It follows easily from 1.22 that we may assume ε^n to be small enough to secure that

$$\text{Re}\, z \notin (\tfrac{1}{2}(a_{n+1} + a_n),\ \tfrac{1}{2}(a_n + a_{n-1})) \Rightarrow v^{K^n}(z) < 2^{-n}. \tag{58}$$

Now we denote by L the curve obtained by joining together all K^n and the segments $\langle -1, 0\rangle$, $\langle a_n + \varepsilon^n, a_{n-1}\rangle$ $(n = 1, 2, \ldots)$. The reader will easily verify that L is a rectifiable curve without angular points (cf. (57)). If $\text{Re}\, z \in \langle \frac{1}{2}(a_{m+1} + a_m), \frac{1}{2}(a_m + a_{m-1})\rangle$ then $v^{K^m}(z) \leqq \gamma$ and, by (58), $v^{K^n}(z) < 2^{-n}$ for $n \neq m$; hence we conclude easily that $v^L(z) \leqq \pi + \gamma + \sum_{n \neq m} 2^{-n} < \pi + \gamma + 1$. If $\text{Re}\, z \notin \bigcup_m \langle \frac{1}{2}(a_{m+1} + a_m), \frac{1}{2}(a_m + a_{m-1})\rangle$ then, for every m, $v^{K^m}(z) < 2^{-m}$ and, consequently, $v^L(z) \leqq \pi + \sum_m 2^{-m} < \pi + 1$. We see that $\sup\,\{v^L(z);\ z \in E_2\} < \pi + \gamma + 1 < +\infty$.

Fix now an $r > 0$. Then there is an n such that the diameter of K^n is less then r. Employing 1.22 we obtain $\delta = v^{K^n}(a_n) \leqq v_r^L(a_n)$ whence $\sup\,\{v_r^L(\zeta);\ \zeta \in L\} \geqq \delta$.

The reader will easily observe that L can be completed by a suitable arc so as to obtain a simple closed rectifiable curve K without angular points satisfying

$$\sup\,\{v^K(z);\ z \in E_2\} < +\infty, \quad \lim_{r \to 0+} \sup_{\zeta \in K} v_r^K(\zeta) \geqq \delta.$$

(To be continued)