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KRONECKER INDEX IN ABSTRACT DYNAMICAL SYSTEMS, I

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In this paper, the index of a simple loop and of a point with respect to an abstract dynamical system in open plane R^2 is defined, and the topological invariance theorem for the index proved. The main results are theorems on the index of the frontier of a + invariant Jordan domain (with theorems on the index of a cycle, of a closed transversal and on existence of critical points in the inner domain of a cycle as corollaries), on the index of a simple loop with inner domain not containing critical points, and on the relation between the index of a simple loop and the indexes of critical points in its inner domain. In the subsequent papers, some generalisations of these results for dynamical systems on R^p and on p-manifolds will be given.

In the qualitative theory of differential equations, in the investigation of isolated critical points of a vector field on R^2 defined by an autonomous system of differential equations, the notion of the index of a point or of a loop with respect to this vector field is rather useful. It is known that the set of all solutions of an autonomous system of differential equations in R^2 (which satisfies several further requirements) defines an dynamical system in R^2. Naturally there arises the question whether it is possible to define and apply the notion of index also in the investigation of critical points of abstract dynamical systems in R^2 which are not necessarily defined by a system of differential equations. It will be shown that the answer is affirmative, and that some main theorems regarding the index, valid in the case of differential dynamical systems, also apply in the case of abstract dynamical systems.

In the definition of the index in an abstract dynamical system, the following idea is used: trajectories define approximative "chordal" vector fields (in the obvious way: given $\theta > 0$, to the point x there is assigned the direction form x to $x \mathrm{T} \theta$). In the case of a non-differential dynamical system these fields need not converge to a "tangential" field of directions; but – under certain weak assumptions – they are mutually homotopic and thus enable us to define a homotopic invariant, the so-called index. The methods of proof differ, of course, from those in differential dynamical systems. As soon as it is shown that the index is topologically invariant, the proofs of all further propositions regarding index are almost trivial.

The main results of this chapter are formulated in theorems 10 – 17.

The following notation will be used.

Let P, Q be topological spaces; the notation $f : P \to Q$ states that f is a continuous mapping of the space P into Q, and $f : P \approx Q$ states that f is a homeomorphism P onto Q. Given $f : P \to Q$, $R \subset P$ (with the induced topology), denote by $f|R$ the partial mapping of the space R into Q. If $f_1 : P_1 \to Q$, $f_2 : P_2 \to Q$ are given with $f_1(x) = f_2(x)$ for each $x \in P_1 \cap P_2$, then $f = f_1 \cup f_2$ denotes the unique mapping $f : P_1 \cup P_2 \to Q$ such that $f \mid P_j = f_j$, $j = 1, 2$. The symbol R^p denotes p-dimensional euclidian space, I denotes the interval $\langle 0, 1 \rangle$.

1.1. First let us introduce some definitions and notations.

Let P, Q be topological spaces. A family of mappings

$$h_\lambda : P \to Q\,, \quad \lambda \in I\,,$$

is called a *homotopy* if the mapping $H : P \times I \to Q$ defined by

$$H(x, \lambda) = h_\lambda(x)\,, \quad x \in P\,, \quad \lambda \in I$$

is continuous. Two mappings $f, g : P \to Q$ are said to be *homotopic* (in Q; notation: $f \simeq g$), if there exists a homotopy, $h_\lambda : P \to Q$, such that $h_0 = f$, $h_1 = g$. In this case, h_λ is called a *homotopy connecting f and g*, and this is denoted by $h_\lambda : f \simeq g$. A homotopy, $h_\lambda : P \to Q$ is called an *isotopy*, if $h_\lambda : P \approx Q$ for each $\lambda \in I$. The relation $\simeq$ is an equivalence relation.

A mapping f is said to be *null-homotopic*, and this is denoted by $f \simeq 0$, if f is homotopic to a constant.

Observe that if $f, g : P \to Q$, $W : Q \to R$ and $h_\lambda : f \simeq g$, then $Wh_\lambda : Wf \simeq Wg$.

Let P be a topological space. A *path* in P is any continuous mapping $f : I \to P$. The points $f(0)$ and $f(1)$ are termed the *initial* and the *terminal* point of f respectively. A *loop* is a path whose initial and terminal points coincide. The set

$$|f| = \{z \in P : z = f(t), t \in I\}$$

is the *track* of the path f. If $|f|$ is a single point, f is a *point-path*. For every path f, the path $-f$ is defined by $-f(t) = f(1 - t)$. Let f_1, f_2 be paths such that the initial point of the path f_2 is the terminal point of the path f_1. Then $f_0 + f_1$ denotes the path f_0, where $f_0(t) = f_1(2t)$ for $0 \leqq t \leqq \frac{1}{2}$, $f_0(t) = f_2(2t - 1)$ for $\frac{1}{2} \leqq t \leqq 1$.

Two paths, f_1 and f_2, are *related by a change of parameter* if there exists a sense-preserving topological mapping $\varphi : I \approx I$ such that $f_2 = f_1\varphi$. Clearly then $|f_1| = |f_2|$ and $f_1 \simeq f_2$ (in their common track).

The just introduced operation of addition of paths is obviously not associative. Since we shall mainly be interested in classes of homotopical paths rather than in single paths, the following property of this operation is important. If f_1, f_2, f_3 are the paths such that $(f_1 + f_2) + f_3$ is defined, then $f_1 + (f_2 + f_3)$ is also defined, and both these paths are related by a change of parameter. Hence, $(f_1 + f_2) + f_3 \simeq \simeq f_1 + (f_2 + f_3)$ (in their common track).

f is a *simple path* if $f(t) \neq f(t')$ whenever $t \neq t'$; it is a *simple loop* if $f(0) = f(1)$ but $f(t) \neq f(t')$ whenever $0 < t < t'$.

According to Jordan's Theorem every simple path, f, in the open plane R^2 has two complementary domains, of each of which it is the complete frontier, i.e.

$$\mathsf{R}^2 - |f| = \text{Int}\,|f| \cup \text{Ext}\,|f|\,,$$

where Int $|f|$ (*inner domain* of $|f|$) and Ext $|f|$ (*outer domain* of $|f|$) are non-void disjoint sets with common frontier $|f|$. Thus the domain Int $|f|$ is bounded, Ext $|f|$ unbounded.

Let us prove the following lemma.

Lemma 1.1. *Let l_1, l_2, l be simple loops in* R^2, $|l_1| \cup |l_2| \subset \text{Int}\,|l|$, $a \in \text{Int}\,|l_1| \cap \cap \text{Int}\,|l_2|$. *Then* $l_1 \simeq l_2$ *in* $\overline{\text{Int}\,|l|} - \{a\}$.

Proof. First we shall prove the lemma supposing that $|l_1| \cap |l_2| = \emptyset$. There exists ([3]; p. 72) $g : \mathsf{R}^2 \approx \mathsf{R}^2$, mapping the closed annulus bounded by paths l_1, l_2, onto a ring. Clearly, $g(a)$ does not belong to this ring, and the boundary circles are homotopic in the ring, via a homotopy say h_λ. Then $g^{-1}h_\lambda$ is a homotopy (in the annulus bounded by l_1, l_2) of the paths l_1, l_2, hence $g^{-1}h_\lambda : l_1 \simeq l_2$ in $\overline{\text{Int}\,|l|} - \{a\}$.

If $|l_1| \cap |l_2| \neq \emptyset$, we use the auxiliary simple loop l containing $|l_1|$ and $|l_2|$ in its inner domain. From the first part of the proof it follows that $l_1 \simeq l$ in $\overline{\text{Int}\,|l|} - \{a\}$, and $l \simeq l_2$ in $\overline{\text{Int}\,|l|} - \{a\}$; thus $l_1 \simeq l_2$ in $\overline{\text{Int}\,|l|} - \{a\}$.

1.2. In this paragraph we shall introduce the concept of the order of a point with respect to a path in R^2.

If $a, b \in \mathsf{R}^2$, $a \neq b$, then the directed segment ab is the path defined by the relation $f(t) = (1 - t)\,a + t\,.\,b$. Let $b_0, b_1 \ldots, b_k$ be points in R^2, $b_0b_1, b_1b_2, \ldots, b_{k-1}b_k$ directed segments, then the path $\sigma = b_0b_1 + \ldots + b_{k-1}b_k$ is called a segmental path. For every integer m let c_m denote the unit circle described $|m|$ times in the direction given by sgn m, more precisely, the path $f : I \to \mathsf{R}^2$ defined by $f(t) = \exp(2m\pi it)$. For any two paths f_1, f_2, let $\eta(f_1, f_2)$ be the least distance of the end-points of one path from the track of the other. A segmental path $\sigma = b_0b_1 + \ldots + b_{k-1}b_k$ is an ε-approximation to a path f if $b_0 = f(0)$, $b_k = f(1)$ and there is a decomposition $0 = t_0 < t_1 \ldots < t_{k+1} = 1$ of the interval I such that, for each $r = 0, 1, \ldots, k$, the set $f(\langle t_r, t_{r+1}\rangle)$ is part of the open circle with center at b_r and radius ε.

Let $a \neq b$ be points in R^2. The left and the right side of the directed segment ab is that complementary domain of the set

$$\mathsf{R}^2 - \{z : z = a + t(b - a), t \in \mathsf{R}^1\}\,,$$

which contains the point $a + i(b - a)$ or $a - i(b - a)$ respectively. The directed segment ab crosses a directed segment cd positively if the interiors of these segments, ab, cd, have precisely one common point, and if a is on the right and b on the left side of cd; and then ba crosses cd negatively.

Let us define the intersection-number, $\nu(ab, cd)$, of the directed segments ab, cd (a, b, c, d distinct points in R^2, ab, cd have at most a common interior point) as follows: $\nu(ab, cd) = 1$ or -1 or 0 if ab crosses cd positively or negatively or ab and cd do not meet, respectively. If the directed segment ab and the segmental path $\sigma = b_0b_1 + b_1b_2 + \ldots + b_{k-1}b_k$ are such that $\nu(ab, b_jb_{j+1})$ is defined for $j = 0, 1, \ldots, k - 1$, then $\nu(ab, \sigma)$ is defined as

$$\nu(ab, \sigma) = \sum_{j=0}^{k-1} \nu(ab, b_jb_{j+1}) .$$

Given a path f, and a directed segment ab, assume that $\varepsilon = \frac{1}{3}\eta(ab, f) > 0$. Let σ_1, σ_2 be two ε-approximations of f, $\eta(ab, \sigma_1) > 0$, $\eta(ab, \sigma_2) > 0$. It can be shown [5, VII, § 2] that $\nu(ab, \sigma_1) = \nu(ab, \sigma_2)$. The intersection-number, $\nu(ab, f)$, of the directed segment ab and the path f is defined as the common value of $\nu(ab, \sigma)$ for all such ε-approximations σ, of f.

Let l be a loop, and a and b points not on $|l|$. Then $\nu(ab, l)$ remains constant if a or b is varied within a residual domain of $|l|$.

Let l be a loop, b a point not on $|l|$. The *order*, $\omega(b, l)$, *of* b *with respect to* l is defined as $\nu(bz, l)$, where z is any point of the unbounded residual domain of $|l|$.

In [5, VII, § 3] there are proved properties of the order $\omega(b, l)$; these are presented in the following lemma.

Lemma 1.2.

(i) $\omega(b, l) = -\omega(b, -l)$

(ii) $\omega(0, c_m) = m$

(iii) *If* $l_0 \simeq l_1 + l_2 + \ldots + l_n$ *in* $\mathsf{R}^2 - \{b\}$, *then* $\omega(b, l_0) = \sum_{j=1}^{n} \omega(b, l_j)$.

(iv) *If* l *is a loop not passing through* 0, *then* $l \simeq c_k$ *in* $\mathsf{R}^2 - \{0\}$ *if* $k = \omega(0, l)$.

(v) *For every homeomorphism* $f : \mathsf{R}^2 \approx \mathsf{R}^2$ *there exists a fixed number* $e_f = \pm 1$ *such that* $\omega(b, l) = e_f\omega(f(b), fl)$ *for all loops* l *and points* b *of the complement of* $|l|$.

If x *is a point of the inner domain of a simple loop* l, *set* $\varepsilon_l = \omega(x, l)$; *then*

(vi) $|\varepsilon_l| = 1$.

1.3. The propositions (v) and (vi) of lemma 1.2 will be used to classify simple loops in R^2. A simple loop is *positively* or *negatively* oriented if in (vi) $\varepsilon_l = 1$ or -1 respectively. Similarly, proposition (v) (with (vi)) allows one to classify homeomorphisms of R^2 as follows: a homeomorphism f is said to *preserve* or *reverse* orientation according as e_f in (v) is 1 or -1. Clearly, the composition of two orientation preserving or two orientation reversing homeomorphisms is orientation preserving, the composition of an orientation preserving homeomorphism with an orientation reversing homeomorphism is orientation reversing. An example of an orientation reversing homeomorphism is the mapping f defined by $f(z) = \bar{z}$, with $\bar{z}$ denoting the conjugate to z. Clearly, every orientation reversing homeomorphism can ab obtained as the composition of this homeomorphism f and of a suitable orientation preserving homeomorphism.

One immediate corollary of (v) will be mentioned. Let l be an arbitrary loop in R^2, f an orientation reversing homeomorphism, $f(0) = 0$. Then $\omega(0, l) = -\omega(0, fl)$.

Let l be simple loop in R^2, $f : \mathsf{R}^2 \approx \mathsf{R}^2$. Evidently, Int $|l|$ is compact and Ext $|fl|$ non-compact. Hence there easily follow the relations:

$$f(\text{Int } |l|) = \text{Int } |fl|, f(\text{Ext } |l|) = \text{Ext } |fl| .$$

The following lemma will be used in the proof of the very important theorem 1.10.

Lemma 1.3. *Every orientation preserving homeomorphism of* R^2 *is isotopic to the identity mapping of* R^2.

The proof of this lemma is an easy modification of the proof of Tietze's Deformation Theorem [3, pp. 186–190].

1.4. Now we shall introduce the notions of a dynamical system and of the vector fields associated with the dynamical system.

Definition 1.4. A *dynamical system on* R^2 is a mapping $\mathsf{T} : \mathsf{R}^2 \times \mathsf{R}^1 \to \mathsf{R}^2$ with the properties (i), (ii) described below:

(i) T is continuous onto; if the value of T at (x, θ) is denoted by $x\mathsf{T}\theta$, then

(ii) $(x\mathsf{T}\theta_1)\,\mathsf{T}\theta_2 = x\mathsf{T}(\theta_1 + \theta_2)$ for all $\theta_1, \theta_2 \in \mathsf{R}^1$, $x \in \mathsf{R}^2$.

We shall prove that $x\mathsf{T}0 = x$ for every $x \in \mathsf{R}^2$.

T maps onto R^2, so that for every $x \in \mathsf{R}^2$ there is a point $(y, \theta) \in \mathsf{R}^2 \times \mathsf{R}^1$ such that $y\mathsf{T}\theta = x$. Now, from (ii) there follows $(y\mathsf{T}\theta)\,\mathsf{T}0 = y\mathsf{T}(\theta + 0) = y\mathsf{T}\theta$ i.e. $x\mathsf{T}0 = x$.

For subsets $A \subset \mathsf{R}^1$, $X \subset \mathsf{R}^2$, let $X\mathsf{T}A$ denote the set $(z = x\mathsf{T}\theta : x \in X, \theta \in A\}$. If $x\mathsf{T}\mathsf{R}^1$ coincides with the singleton $\{x\}$, then x is said to be a *critical point* of the dynamical system T. Obviously, the set of all critical points of the dynamical system is closed [4, V, § 1]. The set $x\mathsf{T}\,\mathsf{R}^1$ is said to be a *trajectory* of the dynamical system T if there exists a $\theta \in \mathsf{R}^1 - \{0\}$ such that $x\mathsf{T}\theta \neq x$. The trajectory $x\mathsf{T}\mathsf{R}^1$ is called a *periodic trajectory* of the dynamical system T, if there exists a $\theta \in \mathsf{R}^1 - \{0\}$ such that $x\mathsf{T}\theta = x$. In this case the number $\theta_0 = \inf\{\theta : \theta > 0, x\mathsf{T}\theta = x\}$ will be called the *primitive period* of the periodic trajectory. Every periodic trajectory has a positive primitive period [4, V, § 1]. Let θ_0 be the primitive period of $x\mathsf{T}\mathsf{R}^1$; then for every $y \in x\mathsf{T}\mathsf{R}^1$ and integer k there holds $y = y\mathsf{T}(k\theta_0)$; conversely, if $y = y\mathsf{T}\theta$ holds for some $y \in x\mathrm{T}\mathrm{R}^1$, then $\theta = k\theta_0$ for some integer k. Every such number $k\theta_0$, $k > 0$, will be called a *period* of the periodic trajectory $x\mathrm{T}\mathrm{R}^1$.

If $x\mathrm{T}\mathrm{R}^1$ is a periodic trajectory of the dynamical system T, and θ_0 the primitive period of $x\mathrm{T}\mathrm{R}^1$, then every path f, $f(\theta) = x\mathrm{T}(\theta\theta_0)$, $\theta \in I$, will be called a *cycle* of the dynamical system T. The number θ_0 will then be called a *period* of the cycle.

A subset $A \subset \mathrm{R}^2$ is termed *+invariant* (or *−invariant*, *invariant*) if $A = A\mathrm{T}\langle 0, +\infty\rangle$, ($A = A\mathrm{T}(-\infty, 0\rangle$, $A = A\mathrm{T}\mathrm{R}^1$ respectively).

Let D be a subset of R^2. Consider a dynamical system T and any mapping $\vartheta : D \to \mathrm{R}^1$. A *vector field* of the dynamical system T on D is the mapping $W: D \to \mathrm{R}^2$ defined by

$$W(x) = x\mathrm{T}\vartheta(x) - x\,. \tag{1}$$

The mapping W is continuous. Clearly, W vanishes at a point x_0 if either x_0 is a critical point of the dynamical system T, or x_0 is on the track of a cycle with period $\vartheta(x_0)$, or $\vartheta(x_0) = 0$. In the sequel, vector fields which are, in certain sense, "small" will play very important rôle.

Lemma 1.5. *Given a dynamical system* T *and a compact subset* $F \subset \mathrm{R}^2$ *containing no critical point of* T. *Then there exists an* $A > 0$ *with the following property: The mapping* $W: F \to \mathrm{R}^2$ *defined by* (1) *is, for every* $\vartheta : F \to (0, A)$, *a vector field of the dynamical system* T, *continuous and vanishing nowhere on* F.

Proof. First prove the following proposition: the g. l. b. of the set of periods of cycles whose tracks intersect F is positive.

Suppose that the proposition does not hold. Then there are $x_n \in F$, $0 < \sigma_n \in \mathrm{R}^1$ such that $x_n \to x \in F$, $\sigma_n \to 0$, $x_n\mathrm{T}\sigma_n = x_n$. Given $\vartheta \in \mathrm{R}^1$, one has $\theta = k_n\sigma_n + \theta_n$ for some integer k_n.

Now, $x_n\mathrm{T}\theta = x_n\mathrm{T}(k_n\sigma_n + \theta_n) = x_n\mathrm{T}\theta_n \to x$ and hence $x\mathrm{T}\theta = x$. Thus x is a critical point of T on F; this is a contradiction.

Now it suffices to take for A in the assertion the g. l. b. just constructed.

It may be noted that in the situation described, for any pair of maps $\vartheta_0, \vartheta_1 : F \to (0, A)$ one has a linear homotopy, $\vartheta_\lambda : \vartheta_0 \simeq \vartheta_1$ defined by $\vartheta_\lambda(x) = (1 - \lambda) . \vartheta_0(x) + \lambda . \vartheta_1(x)$; then each ϑ_λ again maps into $(0, A)$, so that

$$W_\lambda(x) = x\mathrm{T}\vartheta_\lambda(x) - x$$

defines a vector field on F vanishing nowhere on F.

Still preserving the notation of lemma 1.5, every vector field W on F given by the relation (1) with $\vartheta : F \to (0, A)$ will be termed a *small vector field* of the dynamical system T on F.

From lemma 1.5 it follows that for every loop l whose track contains no critical point of T, there exist small vector fields on $|l|$. If W_1, W_2 are two small vector fields on $|l|$, then the tracks of the loops W_1f, W_2f do not contain the origin 0, and therefore $\omega(0, W_1 f) = \omega(0, W_2 f)$ according to lemma 1.2. Now we can set up the following definition.

Definition 1.6. Let T be a dynamical system, l a simple loop whose track contains no critical point of T, W a small vector field of T on $|l|$. The *Kronecker index* $\operatorname{ind}_{\mathrm{T}} l$ of l relative to T is defined as $\varepsilon_l \,.\, \omega(0, Wl)$.

Immediately from the definition 1.6 there follow these properties of the index:

(i) $\operatorname{ind}_{\mathrm{T}} l$ is an integer;

(ii) $\operatorname{ind}_{\mathrm{T}}(-l) = \operatorname{ind}_{\mathrm{T}} l$;

(iii) if simple loops l_j, $j = 0, 1, 2, \ldots, k$ are all positively or all negatively oriented and $l_0 \simeq l_1 + l_2 + \ldots + l_k$ in $\mathrm{R}^2 - \{x : x \text{ is a critical point of } \mathrm{T}\}$, then $\operatorname{ind}_{\mathrm{T}} l_0 = \sum_{j=1}^{k} \operatorname{ind}_{\mathrm{T}} l_j$; in particular, if the loops l_1, l_2 are related by a change of parameter, then $\operatorname{ind}_{\mathrm{T}} l_1 = \operatorname{ind}_{\mathrm{T}} l_2$.

Lemma 1.1 and property (iii) of the index allows one to set up the following definition.

Definition 1.7. Let T be a dynamical system, l simple loop, u the inner domain of $|l|$, $x \in u$ such that $\bar{u} - \{x\}$ does not contain critical points of T. The *Kronecker index* $\operatorname{ind}_{\mathrm{T}} x$ of the point x relative to T is defined as the number $\operatorname{ind}_{\mathrm{T}} l$.

In the following paragraphs we shall formulate the main results of this paper.

1.5. From the definition 1.4 of dynamical systems it follows immediately that the homeomorphic image of a dynamical system is again a dynamical system; more precisely, if T is a dynamical system in R^2 and $f : \mathrm{R}^2 \approx \mathrm{R}^2$, then the relation $f(x\mathrm{T}\theta) = f(x)\, \mathrm{T}_f \theta$ defines a dynamical system T_f in R^2. Clearly, every critical point of T is mapped onto a critical point of T_f, every periodic trajectory of T with primitive period θ_0 is mapped onto a periodic trajectory of T_f with primitive period θ_0 again and analogously for $+$ or $-$invariant sets. Naturally, there arises the question, how is the index changed by a homeomorphism. The solution of this problem will be preceded by several lemmas.

A dynamical system T' is said to be *isotopic* to a dynamical system T if there exists a homeomorphism, $f : \mathrm{R}^2 \approx \mathrm{R}^2$, isotopic with the identity mapping of R^2 and such that $\mathrm{T}' = \mathrm{T}_f$.

Lemma 1.8. *If* T, T' *are isotopic dynamical systems,* $f : \mathrm{R}^2 \approx \mathrm{R}^2$, $\mathrm{T}' = \mathrm{T}_f$, l *simple loop whose track contains no critical points of* T, *then* $\operatorname{ind}_{\mathrm{T}} l = \operatorname{ind}_{\mathrm{T}_f} fl$.

Proof. Let h_λ be an isotopy between the identity mapping id of R^2 and the homeomorphism f, i.e. $h_\lambda : id \simeq f$, $h_0 = id$, $h_1 = f$. Take a small vector field W on $|l|$,

$$W(x) = x\mathrm{T}\theta - x\,, \quad \theta \in (0, A)\,, \quad (A \text{ as in lemma 1.5.})$$

Then there hold the following relations:

$$Wf(x) = f(x)\, \mathrm{T}_f\theta - f(x)\,,$$

$$\operatorname{ind}_{\mathrm{T}} l = \varepsilon_l \,.\, \omega(0, Wl)\,, \quad \operatorname{ind}_{\mathrm{T}_f} fl = \varepsilon_{fl} \,.\, \omega(0, Wfl)\,.$$

Since $\varepsilon_{h_\lambda l}$ is an integer-valued continuous function of λ, it is constant on I, hence $\varepsilon_l = \varepsilon_{fl}$. Define the homotopy $Wh_\lambda : W \simeq Wf$ in $\mathrm{R}^2 - \{0\}$; this is described by

$$Wh_\lambda(x) = h_\lambda(x)\, \mathrm{T}_{h_\lambda}\theta - h_\lambda(x)\,.$$

These results $\omega(0, Wl) = (0, Wfl)$, so that $\operatorname{ind}_{\mathrm{T}} l = \operatorname{ind}_{\mathrm{T}_f} fl$. Lemma 1.8 is proved.

Lemma 1.9. *Let $f : \mathrm{R}^2 \approx \mathrm{R}^2$ be defined by $f(z) = \bar{z}$, $\bar{z}$ the complex conjugate to z. Then* $\operatorname{ind}_{\mathrm{T}} l = \operatorname{ind}_{\mathrm{T}_f} fl$.

Proof. Clearly, $\varepsilon_l = -\varepsilon_{fl}$. Thus it suffices to prove $\omega(0, W_f fl) = -\,\omega(Wl)$.

There is

$$W_f fl(t) = \overline{l(t)}_{\mathrm{T}_f}\, \theta - \overline{l(t)} = \overline{l(t)\,\mathrm{T}\theta} - \overline{l(t)} = \overline{l(t)\,\mathrm{T}\theta - l(t)} = \overline{Wl(t)} = fWl(t)\,,$$

hence

$$\omega(0, W_f fl) = \omega(0, fWl) = -\omega(0, Wl)\,,$$

since f is orientation reversing and $f(0) = 0$ (see proposition (v) of lemma 1.2). Therefore $\operatorname{ind}_{\mathrm{T}} l = \operatorname{ind}_{\mathrm{T}_f} fl$.

Now we shall formulate and prove the main results of this paper.

Theorem 1.10. *Let* T *be a dynamical system, l a simple loop whose track contains no critical points of* T, *$f : \mathrm{R}^2 \approx \mathrm{R}^2$. Then* $\operatorname{ind}_{\mathrm{T}} l = \operatorname{ind}_{\mathrm{T}_f} fl$.

Proof. Since every orientation reversing homeomorphism of R^2 can be composed of the homeomorphism of lemma 1.9 and some orientation preserving homeomorphism, we may suppose f to be orientation preserving.

Since the mapping f is isotopic with the identity mapping (see lemma 1.3), the dynamical systems T and T_f are also isotopic. Thus (lemma 1.8) $\operatorname{ind}_{\mathrm{T}} l = \operatorname{ind}_{\mathrm{T}_f} fl$. Theorem 1.10 is proved.

Theorem 1.11. *Let a simple loop l, not containing any critical points of* T *in its track, be the frontier of a +invariant set. Then* $\operatorname{ind}_{\mathrm{T}} l = 1$.

Proof. Having theorem 1.10 and the definition 3.6 we may assume that l is the positively oriented unit circle (since any simple loop is the image of the unit circle under some homeomorphism $\mathrm{R}^2 \approx \mathrm{R}^2$). Let W be a small vector field of T on $|l|$, $W(z) = z\mathrm{T}\theta - z$.

First let us suppose that Int $|l|$ is +invariat. Let $W_1 : |l| \to \mathrm{R}^2$ be defined by the relation $W_1(z) = -\,z$. Then $W \simeq W_1$ in $\mathrm{R}^2 - \{0\}$ (for the homotopy take the system

W_λ of the maps defined by $W_\lambda(z) = (1 - \lambda)(z\mathrm{T}\theta) - z$ as $(1 - \lambda)|z\mathrm{T}\theta| < |z|$ for $(\lambda, z) \in \langle 0, 1 \rangle \times |l|$). Now, $\varepsilon_l = 1$ and

$$\operatorname{ind}_\mathrm{T} l = \varepsilon_l \omega(0, Wl) = \omega(0, W_1 l) = \omega(0, C_1) = 1 ,$$

where $C_1(t) = -\exp(2\pi i t)$, $t \in I$.

Now, let us suppose that Ext $|l|$ is +invariant. Define $W_2 : |l| \to \mathrm{R}_2$ by the relation $W_2(z) = z$. Then $W \simeq W_2$ in $\mathrm{R}^2 - \{0\}$ (it suffices to take for the homotopy the system W_λ of maps defined by the relation $W_\lambda(z) - z\mathrm{T}[(1 - \lambda)\theta] - (1 - \lambda)z$, $z \in |l|$), hence by the same way as in preceding part of the proof, it follows $\operatorname{ind}_\mathrm{T} l = 1$. The theorem 1.11 is proved.

The following two theorems are immediate corollaries of the preceding.

Theorem 1.12. *If l is a cycle of dynamical system* T, *then* $\operatorname{ind}_\mathrm{T} l = 1$.

We recall the definition of transversals:

A *closed transversal* of a dynamical system T is a simple loop l with the following property: there exists an $\varepsilon > 0$ such that $|l| \cap (|l|\, \mathrm{T}\theta) = \emptyset$ whenever $0 < |\theta| < \varepsilon$.

Theorem 1.13. *If l is a closed transversal of dynamical system* T, *then* $\operatorname{ind}_\mathrm{T} l = 1$.

Let T be a dynamical system. A point $a \in \mathrm{R}^2$ is *orbitally stable* if, for every neighborhood U of a, there exists a neighborhood V of the point a such that $V\mathrm{T}\langle 0, +\infty) \subset U$.

Theorem 1.14. *If x is an orbitally stable isolated critical point of dynamical system* T, *then* $\operatorname{ind}_\mathrm{T} x = 1$.

Proof. Every isolated orbitally stable critical point of T is either a point of the inner domain of arbitrarily small cycles, or it has arbitrarily small neighborhoods bounded by an closed transversal [2].

Theorem 1.15. *If l is a simple loop such that the closure of its inner domain does not contain any critical point of* T, *then* $\operatorname{ind}_\mathrm{T} l = 0$.

Proof. Evidently we may suppose that l is the unit circle. Let W be a small vector field on $\overline{\mathrm{Int}\,|l|}$. Since $l \simeq 0$ in $\overline{\mathrm{Int}\,|l|}$ (e.g. via the projection into the center), there follows $Wl \simeq 0$ in $\mathrm{R}^2 - \{0\}$, and hence

$$\operatorname{ind}_\mathrm{T} l = \varepsilon_l \,.\, \omega(0, Wl) = 0 .$$

The theorem is proved.

Corollary 1.16. *At least one point of the interior of the +invariant set in* 1.11 *is a critical point of* T. *In particular, the inner domain of a cycle of a closed transversal of a dynamical system* T *contains at least one critical point.*

Theorem 1.17. *If a simple loop l contains no critical points of* T *in its track, and if the inner domain of $|l|$ contains only a finite number of critical points $x_1, x_2, \ldots$*

..., x_n *of* T, *then*

$$\mathrm{ind}_T\, l = \sum_{j=1}^{n} \mathrm{ind}_T\, x_j \,.$$

Proof. The theorem will be proved by induction on n. The validity of the theorem for $n = 1$ follows directly from definition 1.7. Suppose the assertion holds for $n - 1$ critical points; and let there be exactly n critical points $x_1, x_2, \ldots, x_n$ in the inner domain of $|l|$ (see fig. 1).

We may suppose that l is the unit circle, $\mathrm{Im}\, x_j < 0$ for $j = 1, 2, \ldots, n - 1$, $\mathrm{Im}\, x_n > 0$. Denote by $l_1, l_2, s_1, s_2, L_1, L_2$ the mappings given by the relations

$$l_1 = l \mid \langle 0, \tfrac{1}{2} \rangle \,, \quad l_2 = l \mid \langle \tfrac{1}{2}, 1 \rangle \,,$$

$$s_1(t) = 1 - 4(1 - t) \quad \text{for} \quad t \in \langle \tfrac{1}{2}, 1 \rangle \,,$$

$$s_2(t) = 1 - 4t \qquad \text{for} \quad t \in \langle 0, \tfrac{1}{2} \rangle \,,$$

$$L_1 = l_1 \cup s_1 \,, \qquad L_2 = s_2 \cup l_2 \,.$$

Clearly, L_1, L_2 are simple loops, $s_1 \cup s_2 \simeq 0$, $L_1 + L_2 \simeq l$, $\mathrm{ind}_T\, L_1 = \mathrm{ind}_T\, x_n$, $\mathrm{ind}_T\, L_2 = \sum_{j=1}^{n-1} \mathrm{ind}_T\, x_j$. Hence there follows

$$\mathrm{ind}_T\, l = \mathrm{ind}_T\, L_1 + \mathrm{ind}_T\, L_2 = \sum_{j=1}^{n} \mathrm{ind}_T\, x_j \,.$$

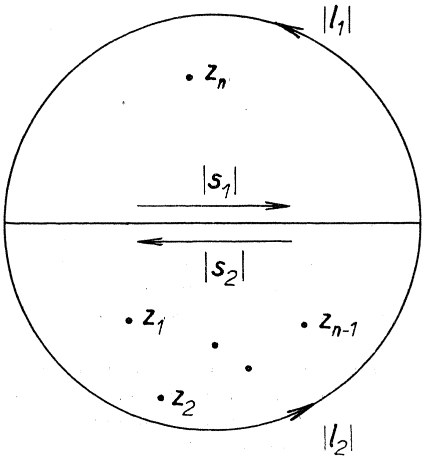

Fig. 1.

1.7. Finally, we shall prove that $\mathrm{ind}_T\, l$ is really a generalisation of the Kronecker index used in the qualitative theory of differential equations.

Consider a mapping $P : \mathrm{R}^2 \to \mathrm{R}^2$ of class C^1 on R^2 such that every solution of the equation

$$(2) \qquad \frac{\mathrm{d}x}{\mathrm{d}\theta} = P(x)$$

is prolongable on R^1. It is known [1, I, § 7] that every solution of this equation is of class C^2 on R^1, and that through every point $x \in \mathrm{R}^2$ there passes exactly one solution of (2). Define a mapping $T : \mathrm{R}^2 \times \mathrm{R}^1 \to \mathrm{R}^2$ thus: $T(z_0, \theta_0)$ is the value at $\theta = \theta_0$ of that solution of (2) which assumes the value x_0 for $\theta = 0$. Denoting $T(z, \theta) = zT\theta$, one obtains a dynamical system T. Such dynamical systems will be called *differential dynamical systems*.

Let l be a loop, $P(x) \neq 0$ for every $x \in |l|$. The number $\mathrm{Ind}_T\, l = \varepsilon_l \,.\, \omega(0, Pl)$ is called the Kronecker index of the loop l with respect to the differential dynamical system T [1, XVI, § 4]. We shall prove that $\mathrm{Ind}_T\, l = \mathrm{ind}_T\, l$. This implies that $\mathrm{ind}_T\, l$ is indeed a generalisation of the concept of the Kronecker index $\mathrm{Ind}_T\, l$.

The mapping $W_\theta : \mathrm{R}^2 \to \mathrm{R}^2$, defined for every $0 \neq \theta \in \mathrm{R}^1$ by

$$W_\theta(x) = \frac{1}{\theta}(x\mathsf{T}\theta - x)\,,$$

is a small vector field of T on $|l|$. Evidently, $\lim_{\theta \to 0+} W_\theta(x) = P(x)$. Set $W_0(x) = P(x)$; then there is a $A > 0$ such that, for every $\theta \in \langle 0, A\rangle$, W_θ is continuous and vanishes nowhere on $|l|$. Since $\omega(0, W_\theta l)$ is a continuous function of θ on $\langle 0, A\rangle$ assuming only integer values there, $\omega(0, W_\theta l)$ is constant on $\langle 0, A\rangle$. Hence

$$\mathrm{Ind}_\mathsf{T}\, l = \varepsilon_l\,.\,\omega(0, Pl) = \varepsilon_l\,.\,\omega(0, W_0 l) = \varepsilon_l\,.\,\omega(0, W_\theta l) = \mathrm{ind}_\mathsf{T}\, l\,.$$

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Резюме

ИНДЕКС КРОНЕКЕРА В АБСТРАКТНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ, I

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В работе сначала определяется (глобальная) *динамическах система* T в открытой плоскости R^2, обобщающая понятие дифференциальной динамической системы, известной из качественной теории дифференциальных уравнений (см., напр., [4]). Потом определяется (определение 1.6 и 1.7) *индекс* $\mathrm{ind}_\mathsf{T}\, l$ простого замкнутого пути l, не проходящего через критические точки динамической системы T, и *индекс* $\mathrm{ind}_\mathsf{T}\, x$ точки $x \in \mathrm{R}^2$ относительно динамической системы T. В теореме 1.10 доказывается, что эти понятия топологически инвариантны.

Дальнейшие важные результаты этой работы содержатся в следующих предложениях.

Теорема 1.11. *Пусть простая петля l, непроходящая через критические точки динамической системы* T, *является границей + инвариантной области. Тогда* $\operatorname{ind}_{\mathsf{T}} l = 1$.

Из теоремы 1.11 в качестве следствия вытекает следующее предложение (теоремы 1.12 и 1.13). *Если l — цикл или замкнутая трансверсаль динамической системы* T, *то* $\operatorname{ind}_{\mathsf{T}} l = 1$.

Теорема 1.14. *Если x — изолированная орбитально устойчивая критическая точка динамической системы* T, *то* $\operatorname{ind}_{\mathsf{T}} x = 1$.

Теорема 1.15. *Если l — простая петля, и замыкание ее внутренней области не содержит критических точек динамической системы* T, *то* $\operatorname{ind}_{\mathsf{T}} l = 0$.

Теорема 1.17. *Пусть простая петля l, непроходящая через критические точки динамической системы* T, *содержит в своей внутренней области только конечное число критических точек* $x_1, x_2, \ldots, x_n$ *системы* T. *Тогда имеет место соотношение*

$$\operatorname{ind}_{\mathsf{T}} l = \sum_{j=1}^{n} \operatorname{ind}_{\mathsf{T}} x_j .$$