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THE LATTICE OF ALL CONVEX L-SUBGROUPS OF A LATTICE-ORDERED GROUP[1])

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1. Introduction. In [5] it is shown that a commutative lattice-ordered group G ("l-group") can be embedded in a Hahn-type group of real valued functions. Moreover, whether or not there exists a minimal such embedding depends only on the lattice $\mathscr{L}$ of all l-ideals of G. In [6] it is shown that whether or not G is completely distributive depends only on $\mathscr{L}$. It is well known that $\mathscr{L}$ is a complete distributive lattice, and K. LORENZ [12] has shown that if we discard the commutative hypothesis, then the set Γ of all convex l-subgroups of G is also a complete distributive lattice. Most of the known structure and representation theorems for G follow from properties of Γ or from putting restrictions on Γ. For example, C. HOLLAND [10] has shown that each l-group G is l-isomorphic to a group of order preserving permutations of a totally ordered set. Here the ordered set is built up from ordered sets of right cosets of convex l-subgroups of G. These results indicate quite clearly the need for an investigation of the structure of Γ for an arbitrary l-group G.

In section 2 we investigate those lattices that are freely generated by their meet irreducible elements. In section 3 it is shown that the lattice Γ of all convex l-subgroups of an l-group G is generated by its set Γ_1 of meet irreducible elements, and that Γ_1 is a root system. Thus it follows (Theorem 3.4) that there is a natural l-isomorphism of Γ into the lattice that is freely generated by Γ_1. Theorem 3.9 asserts that Γ_1 freely generates Γ if and only if for each element g in G there exists at most a finite number of convex l-subgroups M of G that are maximal with respect to $g \notin M$. Also Γ_1 freely generates Γ if and only if

$$B \vee (\bigwedge A_\sigma) = \wedge (B \bigwedge A_\sigma) \quad \text{for all} \quad A_\sigma, B \in \Gamma_1 \, (\sigma \in \Sigma) .$$

The basic concept used in proving these results is that of a prime convex l-subgroup. A convex l-subgroup M of G is called *prime* if whenever a and b belong to G^+ but not to M, then $a \wedge b > 0$. Theorem 3.2 gives six equivalent definitions of a prime

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convex l-subgroup. In particular, the elements of Γ_1 are prime, and if M is an l-ideal of G, then G/M is an o-group if and only if M is prime. For each $g \in G$ let $C(g)$ be the convex l-subgroup of G that is generated by g. Then (Theorem 3.5) the mapping of M upon $M \cap C(g)$ is a one to one mapping of the set of all convex l-subgroups of G that are maximal without g onto the set of all maximal convex l-subgroups of $C(g)$. If $M_1, \ldots, M_n$ are the only convex l-subgroups of G that are maximal without g, then (Theorem 3.7)

$$C(g) = C(g_1) \oplus C(g_2) \oplus \ldots \oplus C(g_n)$$

where M_i is the only convex l-subgroup of G that is maximal without g_i, $C(g_i)$ is a lexicographical extension of $C(g_i) \cap M_i$ and $C(g_i)/(C(g_i) \cap M_i)$ is an archimedean o-group ($i = 1, \ldots, n$). Thus we have a local structure theorem for G.

In section 4 we show that if L is a lattice that is freely generated by its set Λ of meet irreducible elements and if Λ is a root system, then L is (isomorphic to) the lattice of all convex l-subgroups of an l-group (Theorem 4.2). In particular, a finite distributive lattice is (isomorphic to) the lattice of all convex l-subgroups of an l-group if and only if its set Λ of meet irreducible elements form a root system, and if this is the case, then the lattice is freely generated by Λ.

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Notation. We shall denote the null set by $\square$ and the fact that $a, b \in G$ are not comparable by $a \parallel b$ or that the subsets A and B of G are not comparable (with respect to inclusion) by $A \parallel B$. Also $A \setminus B$ will denote the elements that are in A but not in B. We shall denote the lattice operations by $\wedge$, $\vee$, $<$, $\leqq$ and the set theoretic operations by $\cap$, $\cup$, $\subset$, $\subseteq$. A subset D of a po-set P is called a *dual ideal* if whenever $d < p$ for $d \in D$ and $p \in P$, it follows that $p \in D$. R will always denote the naturally ordered additive group of real numbers, and $\oplus$ will always denote the cardinal sum. If S is a subset of a group G, then $[S]$ will always denote the subgroup of G that is generated by S.

2. Lattices that are generated by their meet irreducible elements. Throughout this section let Λ be a po-set and let Λ' be the set of all dual ideals of Λ including the null set $\square$. For α' and β' in Λ' we define $\alpha' \leqq \beta'$ if $\alpha' \supseteq \beta'$ as subsets of Λ. If follows easily that Λ' is a complete distributive sublattice of the Boolean algebra 2^{Λ} of all subsets of Λ, where $\alpha' \vee \beta' = \alpha' \cap \beta'$ and $\alpha' \wedge \beta' = \alpha' \cup \beta'$. Also Λ is the least element and $\square$ is the greatest element in Λ'.

Proposition 2.1. *Λ' satisfies the generalised distributive law*

$$\bigwedge_{\Delta}(\bigvee_{A_\delta} u_{\delta,a}) = \bigvee_{F}(\bigwedge_{\Delta} u_{\delta,\tau(\delta)})$$

and dually, where for each δ in the set Δ, A_δ is a set, and F is the set of all mappings τ of Δ into the join of the A_δ such that $\tau(\delta) \in A_\delta$ for each δ in Δ.

This is an immediate consequence of the validity of the generalized distributive law in 2^{Λ}. Clearly the mapping π of $\lambda \in \Lambda$ onto the principal dual ideal $\lambda' = \{\alpha \in \Lambda : : \alpha \geqq \lambda\}$ of Λ is one to one and $\alpha \leqq \beta$ in Λ if and only if $\alpha' \leqq \beta'$ in Λ'. Thus the lattice Λ' contains an isomorphic copy of the given po-set Λ.

Proposition 2.2. *Each element in Λ' is the greatest lower bound of a unique dual ideal in $\Lambda\pi$. If $\alpha' \in \Lambda'$, then $\alpha' = \bigwedge \beta\pi(\beta \in \alpha')$, and if α' is not principal, then each $\beta\pi > \alpha'$.*

An element a of a lattice L will be called *meet irreducible* if a is not the greatest element in L and if $a < \bigwedge b(b \in L$ and $b > a)$. This is more restrictive than the usual concept of finite meet irreducible $(b, c \in L,\ b > a$ and $c > a$ imply $b \wedge c > a)$.

Proposition 2.3. *$\Lambda\pi$ is the set of all meet irreducible elements in Λ'.*

Proof. If $\lambda' \in \Lambda\pi$, then $\lambda' = \{\alpha \in \Lambda : \alpha \geqq \lambda\}$. Let $\Delta = \{\sigma' \in \Lambda' : \sigma' > \lambda'\}$. Then $\mu' = \bigwedge \sigma'(\sigma' \in \Delta)$ is the dual ideal $\{\alpha \in \Lambda : \alpha > \lambda\}$ of Λ and hence $\mu' > \lambda'$. Thus λ' is meet irreducible, and by Proposition 2.2 the elements in $\Lambda' \setminus \Lambda\pi$ are meet reducible.

Let L be a lattice and let S be the set of all meet irreducible elements in L. If each element in L is the greatest lower bound of a dual ideal of S (including S and the null ideal) and if $\bigwedge a_\alpha$ exists for each dual ideal $\{a_\alpha\}$ of S, then we say the L is *generated by its meet irreducible elements.* In particular, L has a greatest and a least element, and in all that follows we shall only consider lattices that have greatest and least elements. If in addition, for each pair $\{a_\alpha\}$ and $\{b_\beta\}$ of dual ideals of S, $\bigwedge a_\alpha = \bigwedge b_\beta$ implies that $\{a_\alpha\} = \{b_\beta\}$, then we say that L is *freely generated* by S. Note that Λ' is freely generated by $\Lambda\pi$.

Theorem 2.1. *If L is a lattice that is generated by its set S of meet irreducible elements, then the following are equivalent.*

(a) *L is freely generated by S.*
(b) *L satisfies the generalized distributive law.*
(c) *$b \vee (\bigwedge a_\sigma) = \bigwedge(b \vee a_\sigma)$ for all $a_\sigma, b \in S(\sigma \in \Sigma)$.*

Proof. Let S' be the lattice of all dual ideals of S. If L is freely generated by S, then there is a natural l-isomorphism between L and S', and so by Proposition 2.1. L satisfies the generalized distributive law. Therefore (a) implies (b) and clearly (b) implies (c). Finally suppose that (c) is satisfied and that $\bigwedge a_\alpha = \bigwedge b_\beta$, where $\{a_\alpha\}$ and $\{b_\beta\}$ are dual ideals of S. For $b \in \{b_\beta\}$

$$\bigwedge(b \vee a_\alpha) = b \vee (\bigwedge a_\alpha) = b \vee (\bigwedge b_\beta) = b\,.$$

Thus since b is meet irreducible in L, there exists an element $a \in \{a_\alpha\}$ such that $b \vee a = b$ and hence $a \leqq b$. But $\{a_\alpha\}$ is a dual ideal of S and hence $b \in \{a_\alpha\}$. Therefore $\{b_\beta\} \subseteq \{a_\alpha\}$ and similarly $\{a_\alpha\} \subseteq \{b_\beta\}$. Thus S freely generates L.

Corollary. *If L is a lattice that satisfies both chain conditions, then L is generated by its set S of meet irreducible elements. Moreover, S freely generates L if and only if L is distributive, and if S freely generates L, then L is finite.*

Proof. It is known ([2] p. 38) that L is a complete lattice, and in fact, for each subset $\{a_\sigma\}$ of L, $\bigwedge a_\sigma = \bigwedge a_{\sigma_i}$ $(i = 1, \ldots, n)$ for some finite subset $\{a_{\sigma_i}\}$ of $\{a_\sigma\}$. If $a \in L \setminus S$, then $a \leqq \bigwedge a_\sigma (a < a_\sigma \in L)$ and we may assume that each a_σ is the meet of elements in S. Thus each $a \in L$ is the meet of elements in S, and hence S generates L. If S freely generates L, then by our theorem L is distributive. If L is distributive, then $b \vee (\bigwedge a_\sigma) = b \vee (\bigwedge a_{\sigma_i})$ and $\bigwedge(b \vee a_\sigma) = \bigwedge(b \vee a_{\sigma_i})$ for $i = 1, \ldots, n$. Thus it follows that L satisfies condition (c) of our theorem, and hence S freely generates L.

Suppose that S freely generates L and assume (by way of contradiction) that S is infinite. Then there exists an infinite trivially ordered subset $a_1, a_2, \ldots$ of S and hence $a_1 > a_1 \wedge a_2 > a_1 \wedge a_2 \wedge a_3 \ldots$, which is impossible. Therefore S is finite and hence L is finite.

Let L be a lattice that is generated by its set S of meet irreducible elements, and for each $a \in L$ let $D(a)$ be the dual ideal of S consisting of all elements that exceed a $D(a) = \{s \in S : s \geqq a\}$.

Proposition 2.4. *For each subset $\{a_\alpha : \alpha \in \Delta\}$ of L*

(1) $\bigwedge a_\alpha = \bigwedge u(u \in \bigcup D(a_\alpha))$ *and* $\bigvee a_\alpha = \bigwedge v(v \in \bigcap D(a_\alpha))$.

In particular, L is a complete lattice and

(2) $D(\bigvee a_\alpha) = \bigcap D(a_\alpha)$ *and* $D(\bigwedge a_\alpha) \supseteq \bigcup D(a_\alpha)$.

Proof. Since $U = \bigcup D(a_\alpha)$ is a dual ideal of S, $\bigwedge u$ exists and since $U \supseteq D(a_\alpha)$, $\bigwedge u \leqq \bigwedge a_{\alpha,\beta} = a_\alpha (a_{\alpha,\beta} \in D(a_\alpha))$ for all α. If $c \leqq a_\alpha$ for all α, then $c \leqq a_\alpha = \bigwedge a_{\alpha,\beta} \leqq$ $\leqq a_{\alpha,\beta}$ for all $a_{\alpha,\beta} \in U$ and hence $D(c) \supseteq U$. Thus $c = \bigwedge q \leqq \bigwedge u$ $(q \in D(c)$ and $u \in U)$. Therefore $\bigwedge u = \bigwedge a_\alpha$ and similarly $\bigwedge v = \bigvee a_\alpha$. In any complete lattice $x \geqq \bigvee a_\alpha$ if and only if for all α, $x \geqq a_\alpha$, and if $x \geqq a_\alpha$ for some α, then $x \geqq \bigwedge a_\alpha$. Therefore (2) is also satisfied.

Corollary. *The following are equivalent.*

(a) *S freely generates L.*

(b) *$D(\bigwedge a_\alpha) = \bigcup D(a_\alpha)$ for all subsets $\{a_\alpha\}$ of L.*

(c) *If $x \in S$ and $x \geqq \bigwedge a_\alpha (a_\alpha \in L)$, then x exceeds some a_α.*

Proof. Since every element in L is the intersection of a dual ideal of S, $\bigwedge a_\alpha =$ $= \bigwedge w$ $(w \in D(\bigwedge a_\alpha))$ and by (1), $\bigwedge a_\alpha = \bigwedge u$ $(u \in \bigcup D(a_\alpha))$. Thus $\bigwedge w = \bigwedge u$ and if S is freely generated by L, then clearly (b) is satisfied. If $x \in S$ and $x \geqq \bigwedge a_\alpha$, then by (b) $x \in D(\bigwedge a_\alpha) = \bigcup D(a_\alpha)$ and hence x exceeds some a_α. Finally if $\bigwedge x_\delta = \bigwedge y_\gamma$, where $\{x_\delta\}$ and $\{y_\gamma\}$ are dual ideals of S, then $x_\delta \geqq \bigwedge y_\gamma$ and hence by (c), $x_\delta \geqq y_\gamma$ for some y_γ. Thus x_δ belongs to the dual ideal $\{y_\gamma\}$, and it follows that $\{x_\delta\} = \{y_\gamma\}$. Therefore S freely generates L.

Proposition 2.5. *If $D(a \wedge b) = D(a) \cup D(b)$ for all a, b in L, then*

$$a \wedge (\bigvee b_\beta) = \bigvee(a \wedge b_\beta) \quad \textit{for all} \quad a, b_\beta \in L \quad (\beta \in \Delta)$$

and in particular, L is a distributive lattice.

Proof. $\bigvee b_\beta = \bigwedge t$ for all t in $\bigcap D(b_\beta) = D(\bigvee b_\beta)$. Thus $a \wedge (\bigvee b_\beta) = \bigwedge t$ for all t in $D(a) \cup (\bigcap D(b_\beta)) = \bigcap(D(a) \cup D(b_\beta)) = \bigcap D(a \wedge b_\beta)$. But $\bigvee(a \wedge b_\beta) = \bigwedge t$ for all t in $\bigcap D(a \wedge b_\beta)$, and so $a \wedge (\bigvee b_\beta) = \bigvee(a \wedge b_\beta)$.

The following is a summary of the preceding results.

Theorem 2.2. *If L is a lattice that is generated by its set S of meet irreducible elements, then L is a complete lattice and the following are equivalent.*

(1) *$D(a \wedge b) = D(a) \cup D(b)$ for all a, b in L.*

(2) *L is finitely freely generated by S in the sense that if two finitely generated dual ideals of S have the same greatest lower bound in L, then the dual ideals are equal.*

(3) *The mapping $\lambda : a \to D(a)$ is an l-isomorphism of L into the lattice S' of all dual ideals of S. $L\lambda$ is a sublattice of S' which is complete and such that arbitrary joins agree with those in S', but arbitrary meets agree with those in S' if and only if $L\lambda = S'$ (or equivalently if L is freely generated by S).*

Moreover if (1) holds, then L is distributive, and in fact

$$a \wedge (\bigvee b_\beta) = \bigvee(a \wedge b_\beta) \quad \textit{for all} \quad a, b_\beta \in L \quad (\beta \in \Delta),$$

but the dual of this law holds if and only if $L\lambda = S'$.

In section 3 we show that the lattice Γ of all convex l-subgroups of an l-group satisfy the hypotheses and property (1) of Theorem 2.2. Moreover, the meet irreducible elements of Γ form a *root system*; that is, a po-set for which each principal dual ideal is a chain or equivalently in which each pair of incomparable elements have no lower bound. A maximal chain in a root system will be called a *root*. The next theorem completely characterizes those Γ in which the generating root system contains only a finite number of roots (see Theorem 3.4).

Theorem 2.3. *Let L be a distributive lattice that is generated by its set S of meet irreducible elements. If S is a root system that contains only a finite number of roots, and if for each chain $\{c_\gamma\}$ of elements in S*

$$D(\bigwedge c_\gamma) = \bigcup D(c_\gamma),$$

then L is freely generated by S.

Proof. Suppose that $\bigwedge a_\alpha = \bigwedge b_\beta$, where $\{a_\alpha\}$ and $\{b_\beta\}$ are dual ideals of S, and let n be the number of roots in S. For each $j = 1, \ldots, n$ let $\{a(j)_\alpha\}$ be the set of all the a_α in the j-th root of S. Then for each j, $\{a(j)_\alpha\}$ is a dual ideal of S and hence $a(j) = \bigwedge a(j)_\alpha$ belongs to L and $\bigwedge a_\alpha = \bigwedge a(j)$ for all j. For $b \in \{b_\beta\}$

$$b = b \vee (\bigwedge b_\beta) = b \vee (\bigwedge a_\alpha) = b \vee (\bigwedge_{j=1}^{n} a(j)) = \bigwedge_{j=1}^{n} (b \vee a(j)) .$$

Since b is meet irreducible we may, without loss of generality, assume that $b = b \vee a(1) = b \vee (\bigwedge a(1)_\alpha)$ and hence $b \geqq \bigwedge a(1)_\alpha$. Thus $b \in D(\bigwedge a(1)_\alpha) = \bigcup D(a(1)_\alpha)$ and hence $b \geqq a(1)_\alpha$ for some α. But $\{a_\alpha\}$ is a dual ideal of S and hence $b \in \{a_\alpha\}$. Therefore $\{b_\beta\} \subseteq \{a_\alpha\}$ and similarly $\{a_\alpha\} \subseteq \{b_\beta\}$, and hence S freely generates L.

Corollary. *If, as in the theorem, L is generated by a root system S that contains only a finite number of roots, and if each root contains a least element, then S freely generates L.*

Proof. A chain $\{c_\gamma\}$ in S must belong to one of the roots Y of S and hence $\bigwedge c_\gamma \geqq a$, where a is the least element in Y. If $\bigwedge c_\gamma = d \in Y$, then since the elements of S are irreducible, $c_\gamma = d$ for some γ and hence $D(\bigwedge c_\gamma) = \bigcap D(c_\gamma)$. Suppose that $\bigwedge c_\gamma \notin Y$ and consider $x \in D(\bigwedge c_\gamma)$. Then $a \leqq \bigwedge c_\gamma \leqq x$ and $x \in Y$. If $x < c_\gamma$ for all γ, then $x = \bigwedge c_\gamma \in Y$, a contradiction. Thus $x \geqq c_\gamma$ for some γ and hence $x \in \bigcup D(c_\gamma)$. Therefore $D(\bigwedge c_\gamma) = \bigcup D(c_\gamma)$, and S freely generates L.

The following example shows that the hypothesis that $D(\bigwedge c_\gamma) = \bigcap D(c_\gamma)$ for chains $\{c_\gamma\}$ in S cannot be omitted from the last theorem.

Example 2.1. Let S consist of a point v and a desceding sequence of points $u_1, u_2, \ldots$ then we have the following picture of L. (Fig. 1.)

L is generated by S but it is not freely generated by S because

$$v \wedge (\bigwedge_{i=1}^{\infty} u_i) = \bigwedge_{i=1}^{\infty} u_i = \Theta$$

and $\{u_i\}$ and $\{u_i\} \cup \{v\}$ are distinct dual ideals in S. Also L is a distributive lattice and in fact satisfies

$$b \wedge (\bigvee a_\sigma) = \bigvee(b \wedge a_\sigma) \;\; b, a_\sigma \in L \, (\sigma \in \Sigma) .$$

The lattice that is freely generated by S is (Fig. 2)

3. The lattice of convex l-subgroups of an l-group. Throughout this section let $G \neq 0$ be an l-group. A subgroup C of G is an *l-subgroup* provided that C is a sublattice of G, and C is a *convex* subgroup if $0 < g < c \in C$ and $g \in G$ imply that $g \in C$. A normal convex l-subgroup is called an *l-ideal*. The following three propositions are well known for l-ideals (see [7] for proofs) and the generalization to convex l-subgroups is straightforward.

Proposition 3.1. *For a subgroup C of G the following are equivalent.*

(1) *C is a convex l-subgroup of G.*
(2) *C is a directed convex subgroup of G.*
(3) *C is convex and $c \vee 0 \in C$ for each c in C.*
(4) *If $c \in C$, $g \in G$ and $|g| \leqq |c|$, then $g \in G$.*

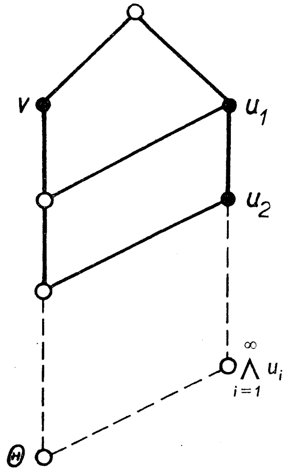

Fig. 1.

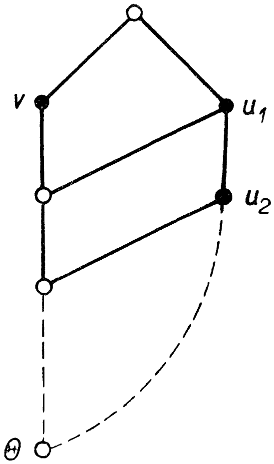

Fig. 2.

Proposition 3.2. *If $\{B_\lambda : \lambda \in \Lambda\}$ is a set of convex l-subgroups of G, then the subgroup of G that is generated by the B_λ is also a convex l-subgroup of G. Thus the convex l-subgroups form a complete sublattice of the lattice of all subgroups of G.*

Let C be a convex l-subgroup of G and let $R(C)$ be the set of all right cosets of C in G. For x and y in G define

$$C + x \leqq C + y \quad \text{if} \quad c + x \leqq y \quad \text{for some } c \text{ in } C\,.$$

Proposition 3.3. *$R(C)$ is a distributive lattice, and*

$$C + x \vee C + y = C + x \vee y \text{ and dually}\,.$$

Moreover, if A and B are convex l-subgroups of G and $A \subseteq B$, then the mapping $A + x \to B + x$ is an l-homomorphism of $R(A)$ onto $R(B)$, and for each $g \in G$, the mapping $A + x \to A + x + g$ is an l-automorphism of $R(A)$.

Let G^+ denote the positive cone of G. If S is a subsemigroup of G^+ that contains 0 and $a \in G^+$, then let $\langle S, a\rangle$ be the subsemigroup of G that is generated by S and a. Thus $\langle S, a\rangle$ consists of all elements of the form

$$u_1 + a + u_2 + a + \ldots + u_{n-1} + a + u_n \;(u_i \in S)\,.$$

Proposition 3.4. (Clifford) *If M is a convex l-subgroup of G and if $a \in G^+ \setminus M$, then*

$$C(M, a) = \{x \in G : |x| \leqq p \quad \textit{for some} \quad p \in \langle M^+, a\rangle\}$$

is the smallest convex l-subgroup of G that contains M and a. If $a, b \in G^+ \setminus M$, then $C(M, a) \cap C(M, b) = C(M, a \wedge b)$. In particular, when $M = 0$

$$C(a) = \{x \in G : |x| \leqq na \quad \textit{for some positive integer } n\} .$$

Proof. If $x, y \in C(M, a)$, then $|x| \leqq p$ and $|y| \leqq q$, where $p, q \in \langle M^+, a\rangle$. Thus $|x - y| \leqq |x| + |y| + |x| \leqq p + q + p \in \langle M^+, a\rangle$ and hence $x - y$ belongs to $C(M, a)$. Thus $C(M, a)$ is a group, and clearly if $|g| \leqq |c|$ for $g \in G$ and $c \in C(M, a)$, then $g \in C(M, a)$. Therefore by Proposition 3.1, $C(M, a)$ is a convex l-subgroup of G that contains M and a, and it is the smallest such sub- group.

Now consider $0 < x \in C(M, a) \cap C(M, b)$.

$$x \leqq m_1 + a + m_2 + \ldots + m_{h-1} + a + m_h \ (m_i \in M^+) ,$$

and

$$x \leqq n_1 + b + n_2 + \ldots + n_{k-1} + b + n_k \ (n_i \in M^+) .$$

Thus

$$x \leqq (m_1 + a + \ldots + m_h) \wedge (n_1 + b + \ldots + n_k) .$$

But for $u, v, w \in G^+$, $u \wedge (v + w) \leqq (u \wedge v) + (u \wedge w)$, and hence it follows that x is less than or equal to a sum of positive elements of the form $m_i \wedge b$, $a \wedge n_i$, $m_i \wedge n_j$ and $a \wedge b$. But each such element belongs to $C(M, a \wedge b)$ and so $C(M, a) \cap$ $\cap\, C(M, b) \subseteq C(M, a \wedge b)$. The other inclusion is trivial.

Corollary. *Let K be the intersection of all the non-zero convex l-subgroups of G. If $K \neq 0$, then G is an o-group and K is the convex subgroup of G that covers zero.*

Proof. If G is not an o-group, then there exists strictly positive elements a and b in G such that $a \wedge b = 0$. Thus $K \subseteq C(a) \cap C(b) = C(a \wedge b) = C(0) = 0$, a contradiction.

A convex l-subgroup M of G is called *regular* if there exists an element g in G such that M is maximal with respect to not containing g, and in this case M is said to be a *value* of g.

Proposition 3.5. *Each convex l-subgroup of G is the intersection of regular convex l-subgroups of G. Each $0 \neq g$ in G has at least one value.*

Proof. Let C be a convex l-subgroup of G and consider $g \in G \setminus C$. By Zorn's lemma there exists a convex l-subgroup M of G that is maximal with respect to $g \notin M \supseteq C$. In particular, M is regular and a value of g, and it follows that C is the intersection o f all such M.

Theorem 3.1. *For a convex l-subgroup M of G the following are equivalent.*

(1) *M is regular.*

(2) *There exists a convex l-subgroup M^* of G that properly contains M and is contained in every convex l-subgroup of G that properly contains M.*

(3) *M is meet irreducible in the lattice of all convex l-subgroups of G.*

If M is normal, then each of the above is equivalent to

(4) *G/M is an o-group with a convex subgroup that covers zero.*

Proof. Suppose that M is regular and let M be a value of $g \in G$. Let M^* be the intersection of all convex l-subgroups of G that properly contain M. Then $g \in M^* \setminus M$, and so (1) implies (2), and clearly (2) implies (3). By Proposition 3.5, M is the intersection of regular convex l-subgroups of G. Thus if M is meet irreducible, it must be regular.

Now suppose that M is normal. Then clearly (4) implies (2). Conversely if M satisfies (2), then by the Corollary to Proposition 3.4 it follows that G/M is an o-group and M^*/M is the convex subgroup of G/M that covers zero.

Corollary. If M is a regular convex l-subgroup of G and $a, b \in G^+ \setminus M$, then $a \wedge b \in G^+ \setminus M$.

Proof. Let M^* be as in (2), then

$$C(M, a \wedge b) = C(M, a) \cap C(M, b) \supseteq M^* .$$

Thus if $a \wedge b \in M$, then $M = C(M, a \wedge b) \supseteq M^*$, a contradiction.

The next theorem was first proven for abelian l-groups. The author wishes to thank A. H. Clifford for his help in translating it to the non-abelian case and in the process shortening the proof. Also, C. Holland [10] has shown that (4) and (6) are equivalent and that (6) implies (5).

Theorem 3.2. *For a convex l-subgroup M of G the following are equivalent*

(1) *If $M \supseteq A \cap B$, where A and B are convex l-subgroups of G, then $M \supseteq A$ or $M \supseteq B$.*

(2) *If $A \supset M$ and $B \supset M$, where A and B are convex l-subgroups of G, then $A \cap B \supset M$.*

(3) *If $a, b \in G^+ \setminus M$, then $a \wedge b \in G^+ \setminus M$.*

(4) *If $a, b \in G^+ \setminus M$, then $a \wedge b > 0$.*

(5) *The lattice $R(M)$ of right cosets of M is totally ordered.*

(6) *The convex l-subgroups of G that contain M form a chain.*

(7) *M is the intersection of a chain of regular convex l-subgroups.*

If M is normal, then each of the above is equivalent to

(8) *G/M is an o-group.*

Proof. Clearly (1) implies (2). If $a, b \in G^+ \setminus M$, then by (2) and Proposition 3.4, $C(M, a \wedge b) = C(M, a) \cap C(M, b) \supset M$. Thus $a \wedge b \notin M$ and hence (2) implies (3). Clearly (3) implies (4). Consider $M + a$ and $M + b$ with a, b in $G \setminus M$. Then $a = a' + a \wedge b$ and $b = b' + a \wedge b$ where $a' \wedge b' = 0$. By (4) either $a' \in M$ or $b' \in M$. But if $a' \in M$, then $M + a = M + a \wedge b \leqq M + b$, and if $b' \in M$, then $M + b = M + a \wedge b \leqq M + a$. Therefore (4) implies (5). Next assume that (5) is true and suppose (by way of contradiction) that there exist convex l-subgroups A and B of G such that $A \supset M$, $B \supset M$ and $A \parallel B$. Pick $0 < a \in A \setminus B$ and $0 < b \in B \setminus A$. Then $a = a \wedge b + a'$, $b = a \wedge b + b'$, $a' \wedge b' = 0$ and without loss of generality $M + a' \geqq M + b'$. Thus $M = M + a' \wedge b' = M + b'$ and hence $b' \in M \subset A$. But since $a \wedge b \in A$, it follows that $b \in A$, a contradiction. Therefore (5) implies (6). An immediate consequence of Proposition 3.5 is that (6) implies (7).

Assume that (7) is satisfied and that there exist convex l-subgroups A and B of G such that $M \supseteq A \cap B$, $M \not\supseteq A$ and $M \not\supseteq B$. Pick $0 < a \in A \setminus M$ and $0 < b \in B \setminus M$. By (7) $M = \bigcap M_\sigma$ $(\sigma \in \Sigma)$, where $\{M_\sigma : \sigma \in \Sigma\}$ is a chain of regular convex l-subgroups of G. Thus there exists $\sigma \in \Sigma$ such that $a, b \in M_\sigma$ and hence by the Corollary to Theorem 3.1, $a \wedge b \notin M_\sigma$. But $a \wedge b \in A \cap B \subseteq M \subseteq M_\sigma$, a contradiction. Therefore (7) implies (1). Finally if M is normal, then clearly (5) and (8) are equivalent.

We shall call a convex l-subgroup of G *prime* if it satisfies one of the equivalent conditions (1) through (7) in the last theorem. Note that each regular convex l-subgroup is prime, and that the prime convex l-subgroups are the finite meet irreducible convex l-subgroups. By (3) a prime convex l-subgroup is a prime x-ideal in the sense of K. Aubert [1] and conversely. Also the prime convex l-subgroups can be used to represent G as a group of o-permutations of a totally ordered set as in [10].

Let Γ be the set of all convex l-subgroups of G and let Γ_1 be the set of all regular convex l-subgroups of G. It follows from Proposition 3.2 that Γ is a complete sublattice of the lattice of all subgroups of G.

Theorem 3.3. *Γ_1 is the of meet irreducible elements of Γ, Γ_1 is a root system and Γ_1 generates Γ. Moreover the following are equivalent.*

(1) *Γ_1 freely generates Γ.*

(2) *Γ satisfies the generalized distributive law.*

(3) $B \vee (\bigwedge A_\sigma) = \bigwedge(B \vee A_\sigma)$ *for all* $A_\sigma, B \in \Gamma_1$ $(\sigma \in \Sigma)$.

Proof. It follows from part (3) of Theorem 3.1 that Γ_1 is the set of all meet irreducible elements of Γ, and it follows from part (6) of Theorem 3.2 that Γ_1 is a root system. By Proposition 3.5, Γ_1 generates Γ. The equivalence of (1), (2) and (3) is an immediate consequence of Theorem 2.1.

As in the last section for each A in Γ let

$$D(A) = \{X \in \Gamma_1 : X \supseteq A\}.$$

Then from (1) in Theorem 3.2 we have

$$D(A \cap B) = D(A) \cup D(B) \text{ for all } A, B \text{ in } \Gamma .$$

Proposition 3.6. *If $\{M_\sigma : \sigma \in \Sigma\}$ is a chain in Γ_1, then $D(\bigcap M_\sigma) = \bigcup D(M_\sigma)$.*

Proof. Clearly $D(\bigcap M_\sigma) \supseteq \bigcup D(M_\sigma)$. If $A \in D(\bigcap M_\sigma)$, then $A \supseteq \bigcap M_\sigma$, and $A \in \Gamma_1$. Thus by (6) and (7) of Theorem 3.2 applied to $M = \bigcap M_\sigma$, A is comparable with each of the M_σ. If $A \subset M_\sigma$ for all σ, then $A = \bigcap M_\sigma$, but this contradicts the fact that A is meet irreducible. Thus $A \supseteq M_{\sigma'}$ for some σ', and hence $A \in D(M_{\sigma'}) \subseteq \bigcup D(M_\sigma)$. Thus by Theorems 2.2 and 2.3 we have

Theorem 3.4. *Γ is finitely freely generated by Γ_1 and the mapping $A \to D(A)$ is an l-isomorphism of Γ into the lattice Γ_1' of all dual ideals of Γ_1. Thus Γ is isomorphic to a sublattice of Γ_1' which is complete, and such that arbitrary joins agree with those in Γ_1', but arbitrary meets agree with those in Γ_1' if and only if Γ is freely generated by Γ_1. Moreover Γ is distributive and*

$$B \wedge (\bigwedge A_\sigma) = \bigwedge (B \wedge A_\sigma) \quad \text{for all } A_\sigma, B \in \Gamma \ (\sigma \in \Sigma)$$

but the dual of this law holds if and only if Γ_1 freely generates Γ. Finally, if Γ_1 contains only a finite number of roots, then Γ_1 freely generates Γ.

Thus to within an l-embedding $\Gamma_1 \subseteq \Gamma \subseteq \Gamma_1'$, Γ is a distributive sublattice of Γ_1', Γ is a complete sublattice of Γ_1' if and only if $\Gamma = \Gamma_1'$, and if Γ_1 has only a finite number of roots, then $\Gamma = \Gamma_1'$. The following is an example of an abelian l-group for which Γ_1 does not freely generate Γ.

Example 3.1. For each positive integer n let I_n be the group of integers and let $\Pi(\Sigma)$ be the large (small) cardinal sum of the I_n. For each n let Π_n be the set of all vectors in Π with n-th coordinate zero. Then Σ and the Π_n are l-ideals of Π and

$$\Sigma \vee (\bigwedge \Pi_n) = \Sigma \vee \{0\} = \Sigma \dotplus \Pi = \bigwedge (\Sigma \vee \Pi_n) .$$

Let K_0 be the subgroup of G that is generated by $\{x \in G : x \parallel 0\}$ and let M_0 be the convex hull of K_0. A. Lavis [11] proves that if G is a po-group, then M_0 is a normal convex subgroup of G and G is a *lexico-extension* of M_0. That is, G/M_0 is an o-group and each positive element in $G \setminus M_0$ exceeds every element in M_0. Lavis also shows that the normal convex subgroups of G that contain M_0 form a chain and $G = \text{lex } M$ for a normal convex subgroup M of G if and only if $M \supseteq M_0$. Finally, if M is a normal convex subgroup of G and G/M is an o-group, but G is not a lexico-extension of M, then $M \subset M_0$. Now once again assume that G is an l-group. Then there is an alternate way of defining M_0. An element $0 < u \in G$ is a *non-unit* if $u \wedge v = 0$ for some $0 < v \in G$. Let N be the subgroup of G that is generated by all the non-units. Then N is an l-ideal of G, $G = \text{lex } N$ and N is not a proper lexico-extension of an l-ideal ([4] Theorem 9.1). Since G/N is an o-group it follows from Theorem 3.2 that N is a prime convex l-subgroup of G.

Proposition 3.7. *$M_0 = N$ and hence M_0 is an l-ideal of G and also a prime convex l-subgroup. A convex l-subgroup $M \neq 0$ of G contains M_0 if and only if $0 < g \in G \setminus M$ implies $g > M$. All other convex l-subgroups of G are contained in M_0.*

Proof. If $x \parallel 0$, then $x = x^+ + x^-$, where $0 < x^+ = x \vee 0$ and $0 > x^- = x \wedge 0$. Thus since $x^+ \wedge - x^- = 0$, x^+ and x^- belong to N and hence $M_0 \subseteq N$. If M_0 is an l-ideal of G, then $N = \operatorname{lex} M_0$ and hence $N = M_0$. Thus to prove that $M_0 = N$ it suffices to show that $y^+ \in M_0$ for all $y \in M_0$. For $z, w \in G$ define $z \approx w$ if there exist $t_1, \ldots, t_k \in G$ such that $z \parallel t_1 \parallel t_2 \parallel \ldots \parallel t_k \parallel w$. Lavis has shown that $M_0 = \{z \in G : z \approx 0\}$. Consider $a \parallel 0$. If $a \leqq - a^+$, then $a \leqq 0$ and if $- a^+ < a$

$$a^+ > - a \rightarrow a^+ \geqq - a \vee 0 = - a^- \rightarrow a = a^+ + a^- \geqq 0 .$$

Thus $- a^+ \parallel a \parallel 0$ and hence $a^+ \in M_0$. If $x \in K_0$, then $x = x_1 + \ldots + x_k$, where $x_i \parallel 0$ for $i = 1, \ldots, k$ and hence

$$0 \leqq x \vee 0 = (x_1 + \ldots + x_k) \vee 0 \leqq (x_1 \vee 0) + \ldots + (x_k \vee 0) \in M_0 .$$

Thus since M_0 is the convex hull of K_0, $x \vee 0 \in M_0$. Now consider $y \in M_0$. There exist a and b in K_0 such that $a \leqq y \leqq b$, and hence $a \vee 0 \leqq y \vee 0 \leqq b \vee 0$. Thus since $a \vee 0$ and $b \vee 0$ belong to M_0 and M_0 is convex, $y \vee 0 \in M_0$.

If M is a convex l-subgroup of G and $M \nsubseteq M_0$, then there exists $0 < g \in M \setminus M_0$ and since $g > M_0$ it follows that $M \supseteq M_0$. Thus each convex l-subgroup of G is comparable with M_0. Let M be a convex l-subgroup of G that contains M_0 and consider $0 < g \in G \setminus M$ and $m \in M$. Then M/M_0 is a convex subgroup of the o-group G/M_0 and hence $g + M_0 > m + M_0$. Thus by Lemma 9.1 in [4] $g - m + M_0$ consists of positive elements of G. Therefore $g > m$ and hence $g > M$. Conversely suppose that M is a convex l-subgroup of G such that $0 < g \in G \setminus M$ implies $g > M$ and consider strictly positive elements a and b in G such that $a \wedge b = 0$. If $a \notin M$, then either $b \in M$ and hence $a > b$ or $b \notin M$ and hence $a \wedge b > M$. Thus M contains all the non-units of G and hence $M \supseteq M_0$.

Proposition 3.8. *If $M \subset M_0$ is a convex l-subgroup of G, then there exists $N \in \Gamma_1$ such that $M \parallel N$.*

Proof. By Proposition 3.7 there exists $0 < g \in M_0 \setminus M$ such that $g \not> M$. Pick $0 < m \in M$ such that $g \parallel m$. Then $g = g \wedge m + g'$, $m = g \wedge m + m'$, $g' \wedge m' = 0$, $g' \notin M$ and $m' \in M$. By Proposition 3.4, $C(g') \cap C(m') = 0$. Pick $N \in \Gamma_1$ such that $m' \notin N$. Then $g' \in N \setminus M$ by the Corollary to Theorem 3.1, and $m' \in M \setminus N$, and hence $M \parallel N$.

Therefore M_0 is the smallest element in Γ that is comparable with every other element in Γ_1, and

$$\{M \in \Gamma_1 : M \supseteq M_0\} = \{M \in \Gamma_1 : M \subseteq N \text{ or } M \supseteq N \text{ for all } N \in \Gamma_1\} .$$

Thus we can think of M_0 as the "base of the trunk" of the root system Γ_1.

We recall that $M \in \Gamma_1$ is a value of $g \in G$ if M is maximal with respect to not containing g. By Proposition 3.5 each $0 \neq g \in G$ has at least one value. Let S be a convex l-subgroup of G. Then since $-|g| \leqq g \leqq |g|$, it follows that $g \in S$ if and only $|g| \in S$. In particular, M is a value of g if and only if M is a value of $|g|$. Clearly an element g in $G \setminus M_0$ has exactly one value M in Γ_1, and M is the maximal convex l-subgroup of $C(g)$. We next establish a one to one correspondence between the values of an arbitrary $g \in G$ and the maximal convex l-subgroups of $C(g)$.

Theorem 3.5. *Consider $0 \neq g \in G$ and let $M \in \Gamma_1$ be a value of g. Then $M \xrightarrow{\sigma} M \cap C(g)$ is a one to one mapping of the set of all values of g onto the set of all maximal convex l-subgroups of $C(g)$. Moreover, if N is a maximal convex l-subgroup of $C(g)$, then $N\sigma^{-1} = \{x \in G : |x| \wedge |g| \in N\}$.*

Proof. Clearly $C(g) = C(|g|)$, and since the values of g and $|g|$ coincide we may assume that $g > 0$. Let A and B be distinct values of g and pick $0 < a \in A \setminus B$. Then $a \wedge g \in A \cap C(g)$ and by (3) of Theorem 3.2, $a \wedge g \notin B \supseteq B \cap C(g)$. Thus it follows that $(A \cap C(g)) \parallel (B \cap C(g))$ and in particular, σ is one to one.

Suppose that A is a value of g and let $N = A \cap C(g)$. Then N is a proper convex l-subgroup of $C(g)$. Suppose (by way of contradiction) that $N \subset Q \subset C(g)$ for some convex l-subgroup Q of $C(g)$. Then Q is also a convex l-subgroup of G and hence by Zorn's lemma there exists a value B of g with $B \supseteq Q$. Clearly $A \neq B$ and hence $(A \cap C(g)) \parallel (B \cap C(g))$, but $B \cap C(g) \supseteq Q \supset N = A \cap C(g)$, a contradiction. Therefore N is a maximal convex l-subgroup of $C(g)$.

Let $N^* = \{x \in G : |x| \wedge |g| \in N\}$, where N is a maximal convex l-subgroup of $C(g)$. Then clearly $g \notin N^* \supseteq N$ and it follows by a straightforward argument that N^* is a convex l-subgroup of G. Suppose that Q is a convex l-subgroup of G that properly contains N^* and pick $0 < x \in Q \setminus N^*$. Then $0 < a = x \wedge g \in C(g) \setminus N$ and $a \in Q$. Thus $C(a) \subseteq Q$, and $C(g) = C(N, a) \subseteq Q$, but this means that $g \in Q$. Therefore N^* is a value of g and since $N^*\sigma = N^* \cap C(g) \supseteq N$, it follows that $N^*\sigma = N$.

A regular convex l-subgroup M of G is called *special* if there exists an element g in G such that M is the unique value of g. In this case g is also called special.

Theorem 3.6. *For $0 \neq g \in G$ the following are equivalent.*

(a) *$C(g)$ is a lexico-extension of a proper l-ideal.*

(b) *g is special in $C(g)$.*

(c) *g is special in G.*

If this is the case and if $M(N)$ the unique value of g in G $(C(g))$, then $N = M \cap C(g)$, $C(g) = \text{lex}\, N$, $C(g)/N$ is an Archimedean o-group (notation $C(g)/N \prec R$) and $M = N \oplus C'(g)$, where $C'(g)$ is the subgroup of G that is generated by $\{x \in G^+ : : x \wedge C(g)^+ = 0\}$. Thus $C'(g)$ is the polar of $C(g)$.

Proof. By Theorem 3.5, (b) and (c) are equivalent and since the values of g and $|g|$ coincide we may assume that $g > 0$. If $C(g) = \text{lex}\, I$, where I is a proper l-ideal, then since $C(g)/I$ is an o-group, it contains a unique maximal convex subgrojp $\mathcal{N}$ without $I + g$. But then $\mathcal{N} = N/I$, where N is the unique maxial convex l-subgroup of $C(g)$ without g. Therefore (a) implies (b).

Conversely if N is the unique value of g in $C(g)$, then by the first part of Proposition 3.7, to show that $C(g) = \operatorname{lex} N$ it suffices to show that if $a, b \in C(g)$, $a > 0$, $b > 0$ and $a \wedge b = 0$, then $a \in N$. But the subgroup S of $C(g)$ that is generated by $\{x \in C(g)^+ : x \wedge b = 0\}$ is a proper convex l-subgroup of $C(g)$ that contains a, but not b, and by Theorem 3.5, N is the greatest convex l-subgroup of $C(g)$. Therefore $a \in S \subseteq N$.

Now suppose that $M(N)$ is the unique value of g in $G\,(C(g))$. Then by Theorem 3.5, $N = M \cap C(g)$ and as we have shown $C(g) = \operatorname{lex} N$. Since N is the greatest convex l-subgroup of $C(g)$, $C(g)/N \prec R$. By lemma 6.1 in [4] $D = C(g) \oplus C'(g)$ is the group generated by $C(g)$ and $C'(g)$ and

$$D^+ = \{x \in G^+ : x \text{ does not exceed every element in } C(g)\}\,.$$

Thus clearly $M \subseteq D$. If $0 < x \in C'(g)$, then $x \wedge g = 0$ and hence by (4) of Theorem 3.2, $x \in M$. Therefore $N \oplus C'(g) \subseteq M \subseteq D$. If $0 < x \in M \setminus (N \oplus C'(g))$, then $x = a + b$, where $0 < a \in C(g) \setminus N$ and $0 \leqq b \in C'(g)$. But then since $C(g)/N \prec R$, $0 < g < na \leqq nx \in M$ for some $n > 0$ and so $g \in M$. Therefore $M = N \oplus C(g)$.

We next wish to investigate those elements in G that have only a finite number of values. In order to do this it will be useful to know just how Γ_1 determines the lattice operations in G. We shall call a subset Δ of Γ_1 *plenary* if each $0 \neq g \in G$ has at least one value in Δ, and if $g \notin M \in \Delta$, then there exists a value N of g in Δ such $N \supseteq M$. It is clear that $\Delta \subseteq \Gamma_1$ is plenary if and only if Δ is a dual ideal of Γ_1 and $\bigcap M = 0$ $(M \in \Delta)$. For each $0 \neq g \in G$ let Δ_g be the set of all values of g in Δ. Then Δ_g is a trivially ordered set. If $g \ngtr 0$ and $M \in \Delta_{g^-}$, then $g^+ \wedge -g^- = 0$ and by (4) of Theorem 3.2, $g^+ \in M$. Hence $M + g = M + g^- < M$ and $M \in \Delta_g$. Thus for an element $0 \neq g \in G$ the following are equivalent.

(a) $g > 0$.

(b) $M + g > M$ for all $M \in \Delta_g$.

Proposition 3.9. *If Δ is a plenary subset of Γ_1 and $g, h \in G$, then $h = g \vee 0$ if and only if the following conditions are satisfied.*

(1) $\Delta_h \subseteq \Delta_g$.

(2) *If $M + g < M$ for $M \in \Delta_g$, then $h \in N$ for all $M \supseteq N \in \Delta$.*

(3) *If $M + g > M$ for $M \in \Delta_g$, then $h - g \in N$ for all $M \supseteq N \in \Delta$.*

Proof. First suppose that g and h satisfy (1), (2) and (3). If $M \in \Delta_h$ and $M + {}+ g < M$, then by (1) and (2), $h \in M$, which is impossible. Thus if $M \in \Delta_h$, then $M + h = M + g > M$ and $N + g = N + h$ for all $M \supseteq N \in \Delta$. In particular $M + h > M$ for all $M \in \Delta_h$ and so $h \geqq 0$. Next consider $N \in \Delta_{h-g}$. If $N \in \Delta_h$, then $h - g \in N$. Thus either $h \in N$ or there exists $M \in \Delta_h \subseteq \Delta_g$ such that $M \supset N$, but in the latter case $h - g \in N$, a contradiction. If $h \in N$, then $N \in \Delta_g$ and $N + g < N$. Thus $N + h - g = N - g > N$ for all $N \in \Delta_{h-g}$ and hence $h \geqq g$.

Now consider $c \in G$ such that $c \geqq g$, $c \geqq 0$ and $c \neq h$, and $N \in \Delta_{c-h}$. If $N \in \Delta_h$, then $N + h = N + g$ and hence $N \neq N + h - c = N + g - c < N$. Thus $N +$

$+ c - h > N$. If $N \notin \Delta_h$, then either $h \in N$ or there exists $M \in \Delta_h$ such that $N \subset M$. If $h \in N$, then $N + h - c = N - c < N$ and hence $N + c - h > N$. If $N \subset M \in \Delta_h \subseteq \Delta_g$, then $h - g \in N$. Since $h - c = h - g + g - c$, we have $N + h - c = N + g - c < N$ and hence $N + c - h > N$. Therefore $c \geqq h$ and hence $h = g \vee 0$.

Conversely suppose that $h = g \vee 0$. Let $N \in \Delta_h$. If $g \in N$, then $h = g \vee 0 \in N$. Thus if $N \notin \Delta_g$, then there exists $M \in \Delta_g$ such that $N \subset M$. If $M + g > M$, then

$$M < M + g = M + g \vee M = M + h = M .$$

If $M + g < M$, then by Proposition 3.3, $N + g < N$ and hence $N = N + g \vee N = N + h$ which is impossible. Therefore $N \in \Delta_g$ and hence $\Delta_h \subseteq \Delta_g$.

Now consider $M \in \Delta_g$ and $M \supseteq N \in \Delta$. If $M + g < M$, then by Proposition 3.3 and the fact that $R(M)$ is totally ordered, $N + g < N$ and hence

$$N = N + g \vee N = N + g \vee 0 = N + h .$$

If $M + g > M$, then as above $N + g > N$ and hence

$$N + g = N + g \vee N = N + g \vee 0 = N + h .$$

Therefore (1), (2) and (3) are satisfied.

Proposition 3.10. *If Δ is a plenary subset of Γ_1 and $a, b \in G^+$, then $a \wedge b = 0$ if and only if $\Delta_a \cap \Delta_b = \Box$ and $\Delta_a \cup \Delta_b$ is trivially ordered. If $a \wedge b = 0$, then $\Delta_a \cup \Delta_b = \Delta_{a+b} = \Delta_{a-b}$. Thus for any $g \in G$*

$$\Delta_{|g|} = \Delta_g = \Delta_{g^+} \cup \Delta_{g^-} \quad \text{and} \quad \Delta_{g^+} \cap \Delta_{g^-} = \Box .$$

Proof. Suppose that $a \wedge b = 0$. if $M \subseteq N$, where $M \in \Delta_a$ and $N \in \Delta_b$, then $M + a$ and $M + b$ exceed M in the totally ordered set $R(M)$. Thus

$$M < M + a \wedge M + b = M + a \wedge b = M + 0 = M .$$

Therefore $\Delta_a \cap \Delta_b = \Box$ and $\Delta_a \cup \Delta_b$ is trivially ordered. Now consider $M \in \Delta_{a+b}$. If $a, b \in G \setminus M$, then by Theorem 3.2, $a \wedge b > 0$ which is a contradiction. Thus $a \in M$ or $b \in M$ and hence $M \in \Delta_b$ or $M \in \Delta_a$. If $N \in \Delta_a$, then by Theorem 3.2 $b \in N$ and hence $N \in \Delta_{a+b}$. Therefore $\Delta_{a+b} = \Delta_a \cup \Delta_b$ and similarly $\Delta_{a-b} = \Delta_a \cup \Delta_b$.

Conversely suppose that $\Delta_a \cap \Delta_b = \Box$ and $\Delta_a \cup \Delta_b$ is trivially ordered, and let $c = a \wedge b$ and consider $N \in \Delta$. If $N \subseteq M \in \Delta_a$, then $b \in N$. For otherwise there exists a value of b in Δ that exceeds N and hence by (6) of Theorem 3.2 is comparable with M. Thus $N + c = N + a \wedge N + b = N + a \wedge N = N$. If N is not contained in any element of $\Delta_a \cup \Delta_b$, then both a and b must belong to N and hence $c \in N$. Therefore $c \in \bigcap N = 0$ $(N \in \Delta)$. The remainder of the proposition follows from the fact that the values of g and $|g|$ coincide in Γ_1 and that $g^+ \wedge - g^- = 0$.

Proposition 3.11. *If Δ is a plenary subset of Γ_1, and if $g \in G$ has only a finite number of values $A_1, \ldots, A_n$ in Δ, then these are the only values of g in Γ_1.*

Proof. Since the values of g and $|g|$ coincide we may assume that $g \geqq 0$. If $n = 0$, then $g = 0$ and hence g has no values in Δ or in Γ_1. Suppose that the proposition is true for all $m < n$, where $n \geqq 1$, and suppose (by way of contradiction) that there exists a value A_0 of g in Γ_1 such that $A_0 \neq A_i$ for $i = 1, \ldots, n$. If $0 < x \in C(g)$ and A and B are values of x in Γ_1, then $A \cap C(g)$ and $B \cap C(g)$ are distinct values of x in $\Gamma_1(C(g))$ (see the proof of Theorem 3.5). It follows that $\{M \cap C(g) : M \in \Gamma_1$ and M is a value of some $0 < x \in C(g)\}$ is a plenary subset of $\Gamma_1(C(g))$. Hence, without loss of generality, we may assume that $G = C(g)$. In particular, $A_0, \ldots, A_n$ are maximal convex l-subgroups of G and if $A \in \Delta$, then $A \subseteq A_i$ for some $i = 1, \ldots, n$.

For each $i = 0, \ldots, n$ there exists a non-unit $a_i \in G \setminus A_i$. For otherwise $A_i \supseteq M_0$ and $G = \operatorname{lex} M_0$, where M_0 is the convex l-subgroup of G that is generated by the non-units (see Proposition 3.7). Thus A_i/M_0 is the maximal convex subgroup of the o-group G/M_0, hence A_i is normal in G and $G = \operatorname{lex} A_i$. But then by Theorem 3.5, A_i is the unique value of g in Γ_1 which is impossible. For each $i = 0, \ldots, n$ pick a non-unit $a_i \in G \setminus A_i$ in such a way that for each $j = 0, \ldots, n$ the elements in $A_j + a_0, A_j + a_1, \ldots, A_j + a_n$ that are different from A_j are distinct. Since if $a_i \notin A_j$, $A_j + a_i < A_j + 2a_i$ and $2a_i$ is a non-unit, this is always possible. Next pick a subset $b_1, \ldots, b_k$ of the a_i such that $b = b_1 \vee \ldots \vee b_k \notin A_j$ for $j = 0, \ldots, n$ and such that for each $i = 1, \ldots, k$ there exists a j such that $b_1 \vee \ldots \vee b_{i-1} \vee \vee b_{i+1} \vee \ldots \vee b_k \in A_j$. If $k = 1$, then there exist strictly positive elements x and y in G such that $x \wedge y = 0$ and $x \notin A_i$ for $i = 0, \ldots, n$, but then by part (4) of Theorem 3.2, $y \in A_i$ for $i = 0, \ldots, n$. In particular, $A_1, \ldots, A_n$ are the only values of $x - y$ in Δ, and $A_i + x - y > A_i$ for $i = 1, \ldots, n$. Thus $x > y$, which is a contradiction. Therefore $k > 1$, and by a permutation of the subscripts we may assume that $A_0 + + b_k < A_0 + b_i$ for some $i = 1, \ldots, k - 1$.

Let $c = (b_1 \vee \ldots \vee b_{k-1}) - b_k$. If $c \in A_j$, then $A_j + b_k = A_j + b_1 \vee \ldots \vee \vee b_{k-1} = A_j + b_t$ for some $1 \leqq t \leqq k - 1$. If $b_k \notin A_j$, then this is impossible by our choice of the a_i, and if $b_k \in A_j$, then it follows that $b \in A_j$ which is also impossible. Thus $c \in G \setminus A_j$ for $j = 0, \ldots, n$ and hence $A_1, \ldots, A_n$ are the only values of c in Δ. $b_1 \vee \ldots \vee b_{k-1} \in A_j$ for some $j = 1, \ldots, n$ and so $A_j + c < A_j$ and $A_0 + + c > A_0$. Thus by proposition 3.9 or 3.10, $c \vee 0$ has less than n values in Δ, and hence by induction these are the only values of $c \vee 0$ in Γ_1, but A_0 is also a value of $c \vee 0$ in Γ_1, a contradiction.

For each element g in G with only a finite number of values we have the following "local structure" theorem for G.

Theorem 3.7. *Suppose that Δ is a plenary subset of Γ_1 and that $g \in G$ has only a finite number of values $M_1, \ldots, M_n$ in Δ. Then these are the only values of g in Γ_1 and g has a unique representation $g = g_1 + \ldots + g_n = g_1 \vee \ldots \vee g_n$, where M_i is the only value of g_i in Γ_1 for $i = 1, \ldots, n$. Moreover*

$$C(g) = C(g_1) \oplus C(g_2) \oplus \ldots \oplus C(g_n),$$
$$C(g_i) = \operatorname{lex}(M_i \cap C(g_i)) \text{ and } C(g_i)/(M_i \cap C(g_i)) \prec R \quad (i = 1, \ldots, n).$$

Proof. By Proposition 3.11, $M_1, \ldots, M_n$ are the only values of g in Γ_1 and by Proposition 3.10 we may assume that $g > 0$. For each $i = 1, \ldots, n$ let $N_i = M_i \cap \cap C(g)$ and let $\hat{N}_i = \bigcap N_j$ (all $j \neq i$). Let $N = \bigcap N_i$ (all i). By Theorem 3.5 the N_i are the distinct maximal convex l-subgroups of $C(g)$.

(I) $\hat{N}_i \nsubseteq N_i$ for $i = 1, \ldots, n$.

For if we pick an element $0 < r_i \in M_i \setminus M_1$ for each $i = 2, \ldots, n$, then by (3) of Theorem 3.2, $r = \bigwedge r_i \in \hat{M}_1 \setminus M_1$. Thus $r \wedge g \in \hat{M}_1 \cap C(g) = \hat{N}_1$ and since r and g do not belong to M_1, $r \wedge g \notin M_1 \supseteq M_1 \cap C(g) = N_1$.

(II) N_i is an l-ideal of $C(g)$, $C(g)/N = N_i/N \oplus \hat{N}_i/N$ and $C(g)/N_i \prec R$ $(i = 1, \ldots, n)$. Any l-automorphism of $C(g)$ must permute the N_i and hence map N onto itself. Thus N is an l-ideal of $C(g)$, and since $\hat{N}_i \nsubseteq N_i$ and N_i is maximal, $C(g)$ is generated by N_i and $\hat{N}_i$, and hence $C(g)/N = N_i/N + \hat{N}_i/N$. If $N < X \in N_i/N$ and $N < Y \in \hat{N}_i/N$, then $X \wedge Y = (N + x) \wedge (N + y) = N + x \wedge y = N$. Therefore, $C(g) = N_i/N \oplus \hat{N}_i/N$. In particular, N_i/N is normal in $C(g)/N$ and so N_i is an l-ideal of $C(g)$. Finally, since N_i is a maximal convex l-subgroup of $C(g)$, it is regular. Thus $C(g)/N_i$ is an o-group with no convex subgroups, and hence $C(g)/N_i \prec R$.

(III) For each $i = 1, \ldots, n$ there exists an element $0 < g_i \in \hat{N}_i$ whose only

$$\text{value in } C(g) \text{ is } N_i, \text{ and } N_i + g_i = N_i + g\,.$$

Since $C(g)/N_i \prec R$ and $\hat{N}_i \nsubseteq N_i$, it is clear that there exists $0 < h_i \in \hat{N}_i$ such that $N_i + h_i > N_i + g$. For each $i \neq 1$ pick such an h_i and let $h = h_2 \vee \ldots \vee h_n$. Then $h \in N_1$ because all the h_i do, and $N_i + g - h$ is negative for all $i \neq 1$. Hence $g - h \notin N_i$ for all i, and thus the values of $g - h$ in $C(g)$ are $N_1, \ldots, N_n$. Thus by Proposition 3.10 the values of $g' = (g - h)^+ = (g - h) \vee 0$ in $C(g)$ are some of the N_i. But since for $i \neq 1$, $N_i + g' = N_i + g - h \vee N_i = N_i$, it follows that N_1 is the only value of g' in $C(g)$, and $N_1 + g' = N_1 + g - h \vee N_1 = N_1 + g \vee \vee N_1 = N_1 + g$.

(IV) M_i is the only value of g_i in Γ_1 and $M_i + g_i = M_i + g$ $(i = 1, \ldots, n)$. $C(g_i) \subseteq C(g) \subseteq G$. By Theorem 3.5 there is a $1 - 1$ correspondence between the values of g_i in $C(g)$ and the maximal convex l-subgroups of $C(g_i)$ and also a $1 - 1$ correspondence between the maximal convex l-subgroups of $C(g_i)$ and the values of g_i in Γ_1. Thus since N_i is the only value of g_i in $C(g)$, M_i is the only value of g_i in Γ_1. Also $N_i \subseteq M_i$ and $N_i + g_i = N_i + g$, and hence $M_i + g_i = M_i + g$.

As an immediate consequence of Theorem 3.6 we have

(V) $C(g_i) = \operatorname{lex}(M_i \cap C(g_i))$ and $C(g_i)/(M_i \cap C(g_i)) \prec R$ $(i = 1, \ldots, n)$.

By Proposition 3.10, $g_i \wedge g_j = 0$ for $i \neq j$, and hence by Proposition 3.4, $C(g_i) \cap \cap C(g_j) = 0$. Thus it follows that $C(g) \supseteq \Sigma \oplus C(g_i)$; see for example [4] Theorem 2.1.

$$2(g_1 + \ldots + g_n) + M_i = 2g_i + M_i > g_i + M_i \quad (i = 1, \ldots, n)\,.$$

It is easy to verify that the M_i are the only values of $2(g_1 + \ldots + g_n) - g$. Thus

$0 < g < 2(g_1 + \ldots + g_n) \in \Sigma \oplus C(g_i)$ which is convex, and hence g belongs to $\Sigma \oplus C(g_i)$. Therefore $C(g) = \Sigma \oplus C(g_i)$ and in particular, $g = \bar{g}_1 + \ldots + \bar{g}_n$, where $C(\bar{g}_i) = C(g_i)$ for $i = 1, \ldots, n$.

Corollary. *If $g \in G$ has only a finite number of values, then each of these values is special.*

In order to prove the converse of this result we need the following lemma.

Lemma 3.1. *For convex l-subgroups $M_1, \ldots, M_k$ of G let $G(M_1, \ldots, M_k) = \{g \in G :$ each value of g is a subgroup of one of the $M_i\} \cup \{0\}$,*

$$\delta(M_1, \ldots, M_k) = \{N \in \Gamma_1 : N \nsubseteq M_i \quad for \quad i = 1, \ldots, k\}\,.$$

Then $G(M_1, \ldots, M_k)$ is a convex l-subgroup of G that contains each of the $G(M_i)$ and $G(M_1, \ldots, M_k) = \bigcap N$ $(N \in \delta(M_1, \ldots, M_k))$. Moreover, $M \in \Gamma_1$ is special if and only if $G(M) \nsubseteq M$.

Proof. Let $X = G(M_1, \ldots, M_k)$ and $Y = \delta(M_1, \ldots, M_k)$ and consider $g \in X$ and $N \in Y$. If $g \notin N$, then there exists a value Q of g such that $Q \supseteq N$ and hence $Q \nsubseteq M_i$ for all i, which contradicts the fact that $g \in X$. Conversely consider $g \in \bigcap N$ $(N \in Y)$ and let Q be a value of g. If $Q \nsubseteq M_i$ for all i, then $Q \in Y$ and hence $g \in Q$, a contradiction. Therefore $X = \bigcap N$ $(N \in Y)$.

If $M \in \Gamma_1$ is special and g is an element in G whose only value is M, then $g \in G(M) \setminus M$, and conversely if $g \in G(M) \setminus M$, then M is the only value of g, and hence M is special.

Theorem 3.8. *For $0 \neq g \in G$ the following are equivalent.*

(a) *g has only a finite number of values in Γ_1.*
(b) *Each value of g is special.*

Proof. Since the values of g coincide with the values of $|g|$ we may assume that $g > 0$. We have already shown that (a) implies (b). Suppose (by way of contradiction) that (b) is satisfied but not (a). Let $\Delta = \{M_\sigma : \sigma \in \Sigma\}$ be the infinite set of values of g each of which is special. Let G^* be the subgroup of G that is generated by all the $G(M_\sigma)$. If $g \in G^*$, then $g = g_1 + \ldots + g_n$, where all the values of g_i are contained in M_{σ_i} $(i = 1, \ldots, n)$. Hence by Lemma 3.1 if Q is a value of g, then $Q \subseteq M_{\sigma_i}$ for some $i = 1, \ldots, n$, but this means that the set of values of g is finite, a contradiction. Therefore $g \notin G^*$. Now by Lemma 3.1 the $G(M_\sigma)$ are convex l-subgroups of G and hence by Proposition 3.2, G^* is a convex l-subgroup of G. Thus there exists a maximal convex l-sugroup M of G such that $g \notin M \supseteq G^*$. Clearly $M \in \Delta$ and hence M is special. Now let h be an element in G whose only value is M. Then $h \in G(M) \subseteq G^* \subseteq M$, which is impossible. Therefore (b) implies (a).

Corollary. *If Γ_1 contains only a finite number of roots, then each M in Γ_1 is special.*

Proof. In this case each trivially ordered subset of Γ_1 is necessarily finite. Thus each element of G has at most a finite number of values.

Theorem 3.9. *For an l-group G the following are equivalent.*

(1) *Γ_1 freely generates Γ.*

(2) *Γ satisfies the generalized distributive law.*

(3) $B \vee (\bigwedge A_\sigma) = \bigwedge(B \vee A_\sigma)$ *for all* $A_\sigma, B \in \Gamma_1$ $(\sigma \in \Sigma)$.

(4) *Each element in Γ_1 is special.*

(5) *Each element in G has at most a finite number of values in Γ_1.*

(6) *Each element in G has a unique representation as the sum of a finite number of pairwise disjoint special elements.*

Proof. (1), (2) and (3) are equivalent by Theorem 3.3, and (4) and (5) are equivalent by Theorem 3.8. The equivalence of (5) and (6) is an immediate consequence of Theorem 3.7 and Proposition 3.10. Suppose that $M \in \Gamma_1$ is not special, and consider $g \in \bigcap N$ $(N \in \delta(M))$. If Q is a value of g, then by Lemma 3.1, $Q \subseteq M$, If $Q = M$, then M is the only value of g and hence M is special, a contradiction. Thus $Q \subset M$ and $g \in M$. Let $\Delta_1 = \delta(M)$ and $\Delta_2 = \delta(M) \cup \{M\}$. Then $\Delta_1 \neq \Delta_2$ and both are dual ideals of Γ_1. Moreover

$$\bigcap_{N \in \Delta_1} N = \bigcap_{N \in \Delta_2} N$$

and hence (1) is false. Therefore (1) implies (4).

Conversely suppose that each element in Γ_1 is special and assume (by way of contradiction) that (1) is false. Thus without loss of generality $\bigcap N_\sigma \subseteq M$, where $\Delta_1 = \{N_\sigma : \sigma \in \Sigma\}$ is a dual ideal of Γ_1 and $M \in \Gamma_1 \setminus \Delta_1$. In particular, if $N_\sigma \in \Delta_1$, then $N_\sigma \nsubseteq M$. Let $\Delta_2 = \delta(M)$. Then $\Delta_1 \subseteq \Delta_2$, Δ_2 is a dual ideal of Γ_1 and by Lemma 3.1

$$\bigcap_{N \in \Delta_2} N = G(M) \subseteq \bigcap_{N \in \Delta_1} N \subseteq M .$$

Now pick an element $a \in G$ whose only value is M. Then $a \in G(M) \setminus M$, a contradiction. Thus (4) implies (1) and the theorem is proven.

Note that if Γ_1 contains only a finite number of roots, then (5) is clearly satisfied. Thus the last part of Theorem 3.4 is a corollary of Theorem 3.9.

4. The lattice of all l-ideals of an abelian l-group. Let G be an abelian l-group. If $\Gamma_1 = \Gamma_1(G)$ contains a minimal plenary subset, then that subset is unique ([5] Theorem 5.2). By combining Theorem 5.4 in [5] and the Theorem in [6] we have that Γ_1 contains a minimal plenary subset if and only if G is completely distributive. Thus whether or not G is completely distributive depends only on Γ_1. Clearly any plenary subset of Γ_1 must contain the special elements of Γ_1. Thus if the set S of special elements of Γ_1 is plenary, then S is the unique minimal plenary subset of Γ_1.

Let Λ be a root system and for each λ in Λ let $R_\lambda = R$. Let $V = V(\Lambda, R_\lambda)$ be the following subset of the large direct sum Π of the R_λ. An element $v = (\ldots, v_\lambda, \ldots)$ of Π

belongs to V if and only if $S_v = \{\lambda \in \Lambda : v_\lambda \neq 0\}$ contains no infinite ascending sequences. For each v in V let

$$\Lambda^v = \{\lambda \in \Lambda : v_\lambda \neq 0 \text{ and } v_\alpha = 0 \text{ for all } \alpha > \lambda\} .$$

The v_λ with $\lambda \in \Lambda^v$ are the *maximal components* of v. We define v in V to be positive if each maximal component v_λ of v is positive in R_λ. It is shown in [5] (Theorems 2.1 and 2.2) that V is an abelian l-group, and the main embedding theorem in [5] asserts that every abelian l-group can be embedded in an l-group of the form V.

We shall denote the small direct sum of the R_λ by $\Sigma = \Sigma(\Lambda, R_\lambda)$. As usual, let us define $\Sigma^+ = \Sigma \cap V^+$, then Σ is a subgroup and a sublattice of V. For each λ in Λ let

$$V_\lambda = \{v \in V : v_\alpha = 0 \text{ for all } \alpha \geqq \lambda\} .$$

Clearly each V_λ is an l-ideal of V, and it is shown in [5] that $\{V_\lambda : \lambda \in \Lambda\}$ is the minimal plenary subset of $\Gamma_1(V)$ and that each V_λ is special (Theorem 6.1).

Lemma 4.1. *If $\Sigma \subseteq G \subseteq V$, where G is a subgroup and a sublattice of V, then $\Delta = \{G \cap V_\lambda : \lambda \in \Lambda\}$ is the minimal plenary subset of $\Gamma_1 = \Gamma_1(G)$, every element of which is special in Γ_1 For each $g \in G$ there is a one to one correspondence between the maximal components of g and its values in Δ. Moreover, if g has only a finite number of maximal components, the the corresponding values in Δ are the only values of g.*

Proof. Consider $0 \neq g \in G$ and let g_σ be a maximal component of g. Let h be the element in G with $h_\sigma = |g_\sigma|$ and $h_\lambda = 0$ for all other λ in Λ. Since G is a sublattice of V it follows that

$$G \cap V_\sigma = \{x \in G : x_\alpha = 0 \text{ for all } \alpha \geqq \sigma\}$$

is an l-ideal of G. Let M be an l-ideal of G that properly contains $G \cap V_\sigma$, and consider $0 < x \in M \setminus (G \cap V_\sigma)$. Then x must have a maximal component $x_\beta > 0$, where $\beta \geqq \sigma$, and hence there exists a positive integer n such that $0 < h < nx \in M$. Therefore $h \in M$ and since $g - h$ or $g + h$ belongs to $G \cap V_\sigma \subseteq M$, $g \in M$. In particular, $G \cap V_\sigma$ is a value of g. If $k \in G \setminus (G \cap V_\sigma)$, then k has a maximal component k_α with $\alpha \geqq \sigma$, and hence $G \cap V_\alpha$ is a value of k. Thus Δ is a plenary subset of Γ_1 and since $G \cap V_\sigma$ is the only value of h in Δ, Δ is the minimal plenary subset of Γ_1. If $G \cap V_\alpha$ is a value of $g \in G$, then clearly $g_\alpha \neq 0$, and if $g_\beta \neq 0$ for some $\beta > \alpha$, then there exists a maximal component g_γ of g with $\gamma \geqq \beta > \alpha$. Thus $G \cap V_\gamma$ is a value of g and $G \cap V_\gamma$ properly contains $G \cap V_\alpha$ which is impossible. Therefore if $G \cap V_\alpha$ is a value of g, then g_α is a maximal component of g, and we have a $1-1$ correspondence between the maximal components of g and its values in Δ. The last statement in the lemma follows at once from Proposition 3.11, and also we have that every element in Δ is special in Γ_1.

Theorem 4.1. *If $\Sigma \subseteq G \subseteq V$. where G is a subgroup and a sublattice of V, then the following are equivalent.*

(1) *Each g in G has at most a finite number of maximal components.*

(2) $\Gamma_1(G) = \{G \cap V_\lambda : \lambda \in \Lambda\}$.

(3) $\Gamma_1(G)$ *freely generates* $\Gamma(G)$.

(4) *Each g in G has a unique representation as a finite sum of pairwise disjoint elements each of which has exactly one maximal component.*

Proof. Let $\Delta = \{G \cap V_\lambda : \lambda \in \Lambda\}$, $\Gamma_1 = \Gamma_1(G)$ and $\Gamma = \Gamma(G)$. By Lemma 4.1 and (1), $\Gamma_1 \subseteq \Delta$, hence (1) implies (2). If $\Gamma_1 = \Delta$, then by Lemma 4.1, each element in Γ_1 is special and hence by Theorem 3.9, Γ_1 freely generates Γ. Suppose that Γ_1 freely generates Γ. Then by Theorem 3.9 each element in Γ_1 is special and hence Γ_1 contains no proper plenary subsets. Therefore $\Gamma_1 = \Delta$. Also by Theorem 3.9 each $0 \neq g \in G$ has at most a finite number of values in $\Gamma_1 = \Delta$, and hence by Lemma 4.1, each $g \in G$ has at most a finite number of maximal components. Therefore (1), (2) and (3) are equivalent. The equivalence of (1) and (4) follows at once from Theorem 3.9 and Lemma 4.1.

Corollary I. $\Gamma_1(\Sigma) = \{\Sigma \cap V_\lambda : \lambda \in \Lambda\}$ *and* $\Gamma_1(\Sigma)$ *freely generates* $\Gamma(\Sigma)$. *Thus there exists a lattice isomorphism between $\Gamma(\Sigma)$ and the lattice Λ' of the dual ideals of Λ, where the l-ideal of Σ corresponding to $\lambda' \in \Lambda'$ is*

$$\{v \in \Sigma : v_\alpha = 0 \text{ for all } \alpha \in \lambda'\} .$$

Moreover $C \in \Gamma(\Sigma)$ is regular (prime) [minimal prime] if and only if the corresponding dual ideal is principal (a chain) [a root].

An element g in an l-group G is called *basic* if $g > 0$ and $C(g)$ is an o-group. A subset S of G is called a *basis* if S is a maximal set of disjoint elements and each s in S is basic.

Corollary II. *For $V = V(\Lambda, R_\lambda)$ the following are equivalent.*

(a) *Λ contains only a finite number of roots.*

(b) $\Gamma_1(V)$ *freely generates* $\Gamma(V)$.

(c) $\Gamma_1(V) = \{V_\lambda : \lambda \in \Lambda\}$.

(d) *V has a finite basis.*

Proof. The equivalence of (a), (b) and (c) follows at once from Theorem 4.1 and the fact that a root system that contains an infinite number of roots must contain an infinite trivially ordered subset. By Theorem 5.11 in [5], V has a finite basis if and only if $\Gamma_1(V)$ contains only a finite number of roots. Thus it follows that (d) and (a) are equivalent.

Theorem 4.2. *Let L be a lattice that is freely generated by its set Λ of meet irreducible elements. If Λ is a root system, then L is l-isomorphic to the lattice $\Gamma(\Sigma)$ of all l-ideals of the abelian l-group $\Sigma = \Sigma(\Lambda, R_\lambda)$ and under this isomorphism Λ corresponds to $\Gamma_1(\Sigma)$.*

Proof. By Corollary I of Theorem 4.1, $\Gamma_1(\Sigma) = \{\Sigma \cap V_\lambda : \lambda \in \Lambda\}$ and $\Gamma_1(\Sigma)$ freely generates $\Gamma(\Sigma)$. But clearly $\{\Sigma \cap V_\lambda : \lambda \in \Lambda\}$ and Λ are o-isomorphic. Thus since $\Gamma_1(\Sigma)$ freely generates $\Gamma(\Sigma)$ and Λ freely generates L, there exists an l-isomorphism of L onto $\Gamma(\Sigma)$.

Corollary. *Suppose that L is a lattice that is generated by its set S of meet irreducible elements, and suppose that S is a root system that contains only a finite number of roots. Then L is (l-isomorphic to) a lattice of all convex l-subgroups of an l-group if and only if S freely generates L. If in addition, each root of S contains a least element, then L is a lattice of convex l-subgroups of an l-group if and only if L is distributive.*

Proof. If S freely generates L, then by Theorem 4.2, L is l-isomorphic to $\Gamma(G)$, where G is an abelian l-group. Conversely suppose that π is an l-isomorphism of L onto $\Gamma(H)$ for some l-group H. Then $\Gamma_1(H)$ contains only a finite number of roots and hence by Theorem 3.4, $\Gamma_1(H)$ freely generates $\Gamma(H)$ and hence S freely generates L. If L is distributive and each root in S contains a least element, then by the Corollary to Theorem 2.3, S freely generates L and hence, as above L is l-isomorphic to $\Gamma(G)$ for some abelian l-group G.

Note that the lattice in Example 2.1 is not l-isomorphic to the lattice of all convex l-subgroups of an l-group.

Theorem 4.3. *A finite distributive lattice L is l-isomorphic to the lattice of all convex l-subgroups of an l-group if and only if the set Λ of proper meet irreducible elements of L is a root system. If this is the case, then L is freely generated by Λ.*

Proof. By the corollary to Theorem 2.1, L is freely generated by Λ. If Λ is a root system, then by Theorem 4.2, L is l-isomorphic to the lattice of all l-ideals of some abelian l-group. Conversely if L is l-isomorphic to the lattice $\Gamma(H)$ of all convex l-subgroups of some l-group H, then Λ is l-isomorphic to the set of meet irreducible elements of $\Gamma(H)$ which by Theorem 3.3 is a root system.

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Резюме

СТРУКТУРА, СОСТОЯЩАЯ ИЗ ВСЕХ ВЫПУКЛЫХ l-ПОДГРУПП СТРУКТУРНО УПОРЯДОЧЕННОЙ ГРУППЫ

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Большая часть известной теории строения и представлений для структурно упорядоченной группы („l-группы") G зависит от строения структуры Γ всех выпуклых l-подгрупп из G. В настоящей работе исследуется строение Γ. Показано, что Γ порождается ее множеством Γ_1 неприводимых по пересечению элементов и что Γ_1 является корневой системой. Таким образом, имеется естественный изоморфизм между Γ и структурой свободно порождаемой множеством Γ_1. Мы показываем, что Γ_1 свободно порождает Γ (и поэтому Γ однозначно определяется Γ_1) тогда и только тогда, если каждый элемент g из G имеет не более, чем конечное число выпуклых l-подгрупп M, являющихся максимальными по отношению к „не содержанию" g. Кроме того, Γ_1 свободно но порождает Γ тогда и только тогда, если Γ_1 удовлетворяет распределительному закону

$$B \vee (\bigwedge A_\sigma) = \bigwedge(B \vee A_\sigma),\ A_\sigma,\ B \in \Gamma_1 \text{ и } \sigma \in \Sigma\,.$$

Если Γ_1 свободно порождает Γ, то мы получаем достаточно точную теорему о локальном строении G.

Каждая структура, свободно порождаемая ее множеством Λ неприводимых по пересечению элементов, является структурой всех выпуклых l-подгрупп некоторой l-группы тогда и только тогда, если Λ есть корневая система. В частности, конечная дистрибутивная структура является структурой всех выпуклых l-подгрупп некоторой l-группы тогда и только тогда, если ее множество Λ неприводимых по пересечению элементов является корневой системой. Таким образом ясно, что подмножество Γ_1 из Γ весьма важно, и поэтому мы даем восемь эквивалентных характеризаций элементов из Γ_1.