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ON SOME PROBLEMS CONCERNING MULTIVALUED CONVERGENCES

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M. FRÉCHET [4] has axiomatically introduced the notion of convergence on a point set L. Fréchet's convergence is a (onevalued) map of the system of (not necessarily all) sequences $\{x_n\}$ of points $x_n \in L$ onto L fulfilling axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$. Some authors ([2], [3], [6] and others) are interested in multivalued convergences. In this paper some topological properties of multivalued convergences are investigated.

Each multivalued convergence space is a closure space fulfilling axioms (C_0), (C_1) and (C_2). In section 1 a statement on successive closures of a given subset in a closure space is proved (Theorem 1).

In section 2 the relation between multivalued convergences and multivalued convergence topologies on the same point set is determined by means of the following equivalence: two multivalued convergences are equivalent if they induce the same closure topology (Theorem 5). In each class of equivalent multivalued convergences there is a largest multivalued convergence (Theorems 2 and 4) which can be topologically characterized by means of neighbourhoods of points in a multivalued convergence space (Theorems 3 and 4). In section 3 the operation α^*) is defined in such a way that it is possible to get from a given system of sequences fulfilling axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ the smallest from the largest multivalued convergences containing the given system (Statement 1).

In section 3 some problems of M. Dolcher concerning certain multivalued convergence spaces are solved (Statements 2, 3 and 4).

1.

A point set P and a map w of the system of all subsets of P into the same system is called a closure space and denoted by (P, w) provided that the following axioms are fulfilled:

(C_0) $w\emptyset = \emptyset$

(C_1) $A \subset wA$

(C_2) $w(A \cup B) = wA \cup wB$.

If the axiom

(F) $w(wA) = wA$

is true, the closure space P is called a topological space.

The map w is called the closure topology and if w has property (F), we say that it is a topology.[1]) The set wA is a w-closure or simply a closure[1]) of the set A. If $A = wA$ then the set A is called closed and its complement $P - A$ open.

The axiom (C_2) shows that the closure topology is an isotone map, that is $A \subset B$ implies $wA \subset wB$.

Neighbourhoods of points in a closure space are defined in such a way that the following statement holds true:

A point x belongs to the closure of a set A if and only if every neighbourhood of x contains at least one point of A.

This postulate leads to the following definition of neighbourhoods in a closure space:

A set $U(x)$ is a w-neighbourhood or simply a neighbourhood[1]) of a point $x \in P$, if $x \in P - w(P - U(x))$.

By means of (C_1) one easily verifies that the point x belongs to each neighbourhood of x. Using the De Morgan formulae and in view of (C_2) it is easy to prove that the intersection of any two w-neighbourhoods of x is a w-neighbourhood of x as well.

A closure space (P, w) is T_i-closure space if the respective condition (T_i) ($i = 0, 1, 2$) is satisfied

(T_0) If $x, y \in P$ and $x \in wy$, $y \in wx$ then $x = y$

(T_1) If $x \in P$ then $wx = x$

(T_2) If $x, y \in P$ and $x \neq y$ then x and y can be separated by two disjoint w-neighbourhoods.

Evidently, each T_{i+1}-closure space is also a T_i-closure space ($i = 0, 1$). The converse assertion, however, is false as the examples on p. 552 show.

Let w' and w'' be two closure topologies on the same point set P. We say that w' is weaker[2]) than w'' or that w'' is stronger than w', written $w' < w''$ or $w'' > w'$, if $w'A \subset w''A$ for each $A \subset P$. The binary relation $<$ orders the system of all closure topologies in the point set P.

In a closure space (P, w) it is possible to form the successive closures of a set A as follows [7]:

$$w^0A \subset w^1A \subset \ldots \subset w^\xi A \subset \ldots,$$

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[1]) The topology will usually be denoted by letters u or v. When no confusion is possible we may suppress the signs of topologies and convergences.

[2]) Let u and v be topologies on a set P. It is easy to see that $uA \subset vA$ for each $A \subset P$ if and only if $\mathfrak{U} \supset \mathfrak{V}$, $\mathfrak{U}$ and $\mathfrak{V}$ being the systems of all u-closures resp. of all v-closures in P. For this reason, in the literature the topology u is sometimes said to be a stronger than the topology v.

where

$$w^0 A = A,\ w^1 A = wA \quad \text{and} \quad w^\xi A = \bigcup_{\eta < \xi} w(w^\eta A).$$

The map w^ξ fulfills axioms (C_0) and (C_1). By means of transfinite induction it can be easily shown that axiom (C_2) is also true. Consequently (P, w^ξ) is a closure space for each ordinal ξ.

Theorem 1. *Let (P, w) be a closure space. Let ω_α be the least ordinal of regular power $\aleph_\alpha$. If $A \subset P$ and $x \in wA$, let there be a subset $B \subset A$ of power $< \aleph_\alpha$ such that $x \in wB$. Then w^{ω_α} is the weakest topology on P among all topologies which are stronger than w.*

Proof. Let $x \in w(w^{\omega_\alpha} A)$. Then there is a subset $B \subset w^{\omega_\alpha} A$ of power $< \aleph_\alpha$ such that $x \in wB$. Since $w^{\omega_\alpha} A = \bigcup_{\xi < \omega_\alpha} w^\xi A$ then there is an ordinal $\beta < \omega_\alpha$ such that $B \subset w^\beta A$ $\aleph_\alpha$ being regular. Consequently $x \in w^{\omega_\alpha} A$. Therefore $w^{\omega_\alpha} A$ is a w-closed set and $w^{\omega_\alpha}(w^{\omega_\alpha} A) = w^{\omega_\alpha} A$. Thus axiom (F) is fulfilled.

Now, if a topology u is stronger than w then $w^\xi A \subset u^\xi A \subset uA$ for each ξ. Hence $w^{\omega_\alpha} < u$.

Corollary 1. *Let (P, w) be a closure space. Let $\aleph_\alpha > \mathfrak{m}$ where $\aleph_\alpha$ is a regular power and $\mathfrak{m}$ the power of P. Then w^{ω_α} is the weakest topology on P among all topologies which are stronger than w.*

According to Corollary 1, to each closure topology w it is possible to assign in a unique way the topology w^{ω_α} which is the weakest of all topologies v such that $v > w$; it will be denoted by $u(w)$.

Corollary 2. *Let (P, w) be a closure space. Let $A \subset P$ and $x \in wA$ imply that there is a countable subset $B \subset A$ such that $x \in wB$. Then $u(w) = w^{\omega_1}$.*

Example. Let ω_β be the least ordinal of power $\aleph_\beta$ which fails to be regular. Let P be the set of all ordinals $\xi \leqq \omega_\beta$ and Q the subset of all $\xi < \omega_\beta$. Then the power of P is $\aleph_\beta$. Now put $w(\xi) = (\xi) \cup (\xi + 1)$ for $\xi \in P$ and for $A \subset P$ define $wA = \bar{A} \cup \bigcup_{\xi \in A} w(\xi)$ where $\bar{A}$ denotes the closure of A in the usual order topology. Then w is a closure topology on P such that $w^{\omega_\beta}(0) = Q$ and $w^{\omega_\beta}(Q) = P$. Consequently $w^{\omega_\beta}(0) \neq w^{\omega_\beta}(w^{\omega_\beta}(0))$ and w^{ω_β} fails to be a topology.

This example shows that the assumption in Theorem 1 that $\aleph_\alpha$ is regular cannot be omitted.

Now we shall define the notion of continuity of a map and the notion of homeomorphism on a closure space. Let φ be a map on a closure space (P, w_1) into a closure space (Q, w_2). Let $x_0 \in P$. The map φ is continuous [7] at the point x_0 whenever for each $A \subset P$

$$x_0 \in w_1 A \quad \text{implies} \quad \varphi(x) \in w_2 \varphi(A).$$

The map on P to Q is defined to be continuous if it is continuous at each point $x \in P$.

It is easy to show that φ is continuous at the point x_0 if and only if for each w_2-neighbourhood $V(\varphi(x_0))$ of the point $\varphi(x_0) \in Q$ there is a w_1-neighbourhood $U(x_0)$ of x_0 such that $\varphi(U(x_0)) \subset V(\varphi(x_0))$.

A one-to-one continuous map φ on a closure space P onto a closure space Q is a homeomorphism provided that φ^{-1} is also continuous.

2.

Let L be a point set. Let N be the set of all naturals. We say that $\{x_n\}$ is a sequence of points $x_n \in L$ if there is a map φ on N into L such that $\varphi(n) = x_n$. If $n_1 < n_2 < \ldots$ then $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$. We say that $\{x_{mn}\}$ is a double sequence of points $x_{mn} \in L$ if there is a map ψ on $N \times N$ into L such that $\psi(m, n) = x_{mn}$. A double sequence of points may also be denoted by $\{x_n^m\}$. A (simple) sequence $\{x_{n_m}^m\}_{m=1}^{\infty}$ will be called a cross-sequence of $\{x_n^m\}$ if $n_m = f(m)$, where f is a function on N into N. Each subsequence of a cross-sequence will be called a cross-subsequence and denoted by $\{x_{m_{n_i}}^{m_i}\}$ or simply $\{x_{n_i}^{m_i}\}$.

Let L be a point set. Let $\mathfrak{L}$ be the set of pairs $(\{x_n\}, x)$ where $\{x_n\}$ is a sequence of points $x_n \in L$ and x a point of L. The set $\mathfrak{L}$ is called a multivalued convergence on L (abbreviated: the m-convergence on L), if the following axioms are true:

($\mathscr{L}_1$) If $x_n = x$ for each n then $(\{x_n\}, x) \in \mathfrak{L}$

($\mathscr{L}_2$) If $(\{x_n\}, x) \in \mathfrak{L}$ and $n_1 < n_2 < \ldots$ then $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

Instead of $(\{x_n\}, x) \in \mathfrak{L}$ we shall sometimes write symbolically $\mathfrak{L} - \lim x_n = x$ or simply[1]) $\lim x_n = x$ and we shall say that the sequence $\{x_n\}$ $\mathfrak{L}$-converges[1]) to the point x; in this case $\{x_n\}$ is called $\mathfrak{L}$-convergent and the corresponding point x the limes of the sequence $\{x_n\}$. The constant sequence $\{x_n\}$, where $x_n = x$ for each natural n, will be sometimes denoted by $\{x\}$. If a sequence $\{x_n\}$ is $\mathfrak{L}$-convergent, let us denote by $\operatorname{Lim} x_n$ the set of all $x \in L$ such that $x = \mathfrak{L} - \lim x_n$. If $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ then, according to axiom ($\mathscr{L}_2$), the set $\operatorname{Lim} x_{n_i} \supset \operatorname{Lim} x_n$.

Let $\mathfrak{L}$ be an m-convergence on a point set L. The closure λA of a set $A \subset L$ is defined to be the set of all $x \in L$ such that $(\{x_n\}, x) \in \mathfrak{L}$ where $\bigcup x_n \subset A$.

The condition ($\mathscr{L}_1$) shows that $A \subset \lambda A$ for each $A \subset P$; from axiom ($\mathscr{L}_2$) it follows that $\lambda(A \cup B) = \lambda A \cup \lambda B$. Therefore λ is a closure topology. We say that λ is induced by the m-covergence $\mathfrak{L}$. The closure space L will be called a multivalued convergence space[3]) (abbreviated: the m-convergence space) and will be denoted by $(L, \mathfrak{L}, \lambda)$ or (L, λ) and sometimes only by L. If condition (T_i) is fulfilled we speak of T_i m-convergence space.

[3]) Multivalued convergence topologies induced by m-convergences $\mathfrak{L}$, $\mathfrak{L}^*$, $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{T}$ will usually be denoted by the Greek letters λ, λ^*, μ, ν, τ.

From the definition of neighbourhoods in closure spaces it follows that a set $U(x)$ is a λ-neighbourhood of a point x in an m-convergence space $(L, \mathfrak{L}, \lambda)$ if and only if

$$(\{x_n\}, x) \in \mathfrak{L} \quad \textit{implies} \quad x_n \in U(x) \quad \textit{for nearly all } n .$$

Examples. Let L_0 be a set consisting of two distinct points a, b. Let the multivalued convergence $\mathfrak{L}_0$ on L_0 contain elements of two kinds: $(\{x_n\}, a)$ and $(\{x_n\}, b)$ where $x_n = a$ or $x_n = b$ for nearly all n. The induced closure topology will be denoted by λ_0.

Let L_1 contain two distinct points a, b and $\mathfrak{L}_1$ four elements $(\{a\}, a)$, $(\{a\}, b)$, $(\{b\}, a)$, $(\{b\}, b)$. Then $(L_1, \mathfrak{L}_1, \lambda_1)$ is an m-convergence space which fails to be a T_i-closure space ($i = 0, 1, 2$).

Let L_2 contain two distinct points a, b; let $\mathfrak{L}_2$ consist of three elements $(\{a\}, a)$, $(\{a\}, b)$, $(\{b\}, b)$. Then $(L_2, \mathfrak{L}_2, \lambda_2)$ is a T_0 m-convergence space which does not satisfy either condition (T_1) or condition (T_2).

Let L_3 contain points a, b and x_n, $n = 1, 2, \ldots$. Let $\mathfrak{L}_3$ be a multivalued convergence on L_3 consisting of elements $(\{x\}, x)$, $(\{x_{n_i}\}, a)$ and $(\{x_{n_i}\}, b)$ for each $x \in L$ and each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Then $(L_3, \mathfrak{L}_3, \lambda_3)$ is T_1- but not T_2-multivalued convergence space.

It is easy to prove that each multivalued convergence $\mathfrak{L}$ on a T_2 m-convergence space $(L, \mathfrak{L}, \lambda)$ has the following property

$(\mathscr{L}_0)$ If $(\{x_n\}, x) \in \mathfrak{L}$ and $(\{x_n\}, y) \in \mathfrak{L}$, then $x = y$.

In such a case $\mathfrak{L}$ is a onevalued convergence (or simply a convergence) on L in the sense of M. Fréchet [4]. Consequently

Each T_2 multivalued convergence space $(L, \mathfrak{L}, \lambda)$ is a onevalued convergence space.

Notice that $\mathfrak{L}_3$ is a multivalued convergence on the T_1 m-convergence space $(L_3, \mathfrak{L}_3, \lambda_3)$ and that $\mathfrak{L}_3$ does not satisfy axiom $(\mathscr{L}_0)$. On the other hand, there exists a onevalued convergence on a convergence space, in which any two distinct points cannot be separated by disjoint neighbourhoods [9].

If $(L, \mathfrak{L}, \lambda)$ and $(L, \mathfrak{M}, \mu)$ are m-convergence spaces and if $\mathfrak{L} = \mathfrak{M}$ then evidently $\lambda = \mu$. The converse assertion, however, is not true. As a matter of fact, $\lambda_0 = \lambda_1$ whereas $\mathfrak{L}_0 \neq \mathfrak{L}_1$. In order to find the relation between m-convergences and m-convergence topologies, let us define m-convergences to a point x. Let $(L, \mathfrak{L}, \lambda)$ be an m-convergence space and $x_0 \in L$. Denote by $\mathfrak{L}(x_0)$ the set of all elements $(\{x_n\}, x) \in \mathfrak{L}$ such that $x = x_0$. The subset $\mathfrak{L}(x_0)$ of $\mathfrak{L}$ will be called the m-convergence to the point x_0. Clearly $\mathfrak{L} = \bigcup_{x_0 \in L} \mathfrak{L}(x_0)$ and $x \neq y$ implies $\mathfrak{L}(x) \neq \mathfrak{L}(y)$.

Definition. If $(L, \mathfrak{L}, \lambda)$ and $(L, \mathfrak{M}, \mu)$ are m-convergence spaces, x_0 a point of L and $\mathfrak{L}(x_0) \subset \mathfrak{L}$, $\mathfrak{M}(x_0) \subset \mathfrak{M}$, then we define $\mathfrak{L}(x_0) \sim \mathfrak{M}(x_0)$ whenever $x_0 \in L - (\lambda A \div \mu A)$

for each $A \subset L$. Since for any subsets E, F, G of L the symmetrical differences have the properties

$$E \div E = \emptyset, \; E \div F = F \div E \quad \text{and} \quad E \div G \subset (E \div F) \cup (F \div G)$$

it follows that $\sim$ is an equivalence relation on the system $\mathfrak{L}(x_0) \subset \mathfrak{L}$ of all m-convergences to the point x_0, where $\mathfrak{L}$ denotes the system of all m-convergences on the same point set L. In such a way we get a system of classes $[\mathfrak{L}(x_0)]$ on $\mathfrak{L}(x_0)$ each of which is partially ordered with respect to the inclusion $\subset$.

Theorem 2. *Let x_0 be a point in a multivalued convergence space $(L, \mathfrak{L}, \lambda)$. Then the class $[\mathfrak{L}(x_0)]$ of m-convergences to x_0 contains the largest element $\bigcup \mathfrak{P}(x_0)$, $\mathfrak{P}(x_0) \in [\mathfrak{L}(x_0)]$.*

Proof. Denote by $\mathfrak{L}^*$ the union of all m-convergences $\mathfrak{N}$ on L such that the induced m-convergence topology $\nu = \lambda$. Then evidently both axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ are fulfilled and consequently $\mathfrak{L}^*$ is an m-convergence on L. Now we shall prove that $\mathfrak{L}^*(x_0) \in [\mathfrak{L}(x_0)]$. Let $A \subset L$; then $x \in \lambda^* A$ implies the existence of an element $(\{x_n\}, x) \in \mathfrak{L}^*$, $\bigcup x_n \subset A$, so that $(\{x_n\}, x) \in \mathfrak{N}$, where $\mathfrak{N}$ denotes a suitable m-convergence in the union $\mathfrak{L}^*$. Since $\nu = \lambda$ it follows that $x \in \lambda A$. On the other hand, $\mathfrak{L} \subset \mathfrak{L}^*$ implies $\lambda A \subset \lambda^* A$. Hence $\lambda A = \lambda^* A$ and $\lambda A \div \lambda^* A = \emptyset$. Therefore $\mathfrak{L}^*(x_0) \sim \mathfrak{L}(x_0)$.

Now let us prove that $\mathfrak{L}^*(x_0)$ is the largest element in the class $[\mathfrak{L}(x_0)]$, i.e. that $\mathfrak{P}(x_0) \subset \mathfrak{L}^*(x_0)$ for each $\mathfrak{P}(x_0) \in [\mathfrak{L}(x_0)]$. Choose $\mathfrak{P}(x_0) \in [\mathfrak{L}(x_0)]$ and put $\mathfrak{M} = \mathfrak{P}(x_0) \cup (\mathfrak{L} - \mathfrak{L}(x_0))$. Since $\mathfrak{P}(x_0) \subset \mathfrak{M}$ it suffices to show that $\mathfrak{M}$ is an m-convergence on L and that the induced m-convergence topology $\mu = \lambda$. First $\mathfrak{M}$ is an m-convergence both axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ being fulfilled. Now, let A be a subset of L and let $x \in \lambda A$. Then $(\{x_n\}, x) \in \mathfrak{L}$ for a suitable sequence of points $x_n \in A$. If $(\{x_n\}, x) \in \mathfrak{L} - \mathfrak{L}(x_0)$, then $x \in \mu A$. If $(\{x_n\}, x) \in \mathfrak{L}(x_0)$ then $x = x_0$ and also $x \in \mu A$, because $\mathfrak{L}(x_0) \sim \mathfrak{P}(x_0)$, so that $x_0 \in L - (\lambda A \div \mu A)$ and therefore $x_0 \in \mu A \cap \lambda A$. Thus $\lambda < \mu$. On the other hand if $y \in \mu A$ then $(\{y_n\}, y) \in \mathfrak{M}$ and $\bigcup y_n \subset A$ for a suitable sequence $\{y_n\}$. If $y \neq x_0$ then $(\{y_n\}, y) \in \mathfrak{L}$ and $y \in \lambda A$. If $y = x_0$ then $\mathfrak{L}(x_0) \sim \mathfrak{P}(x_0)$ implies $y \in \lambda A$. Therefore $\mu < \lambda$ and so $\mu = \lambda$.

Since $\mathfrak{P}(x_0) \subset \mathfrak{L}^*(x_0)$ for each $\mathfrak{P}(x_0) \in [\mathfrak{L}(x_0)]$ then $\bigcup_{\mathfrak{P}(x_0) \in [\mathfrak{L}(x_0)]} \mathfrak{P}(x_0) \subset \mathfrak{L}^*(x_0)$. On the other hand $\mathfrak{L}^*(x_0) \in [\mathfrak{L}(x_0)]$. Therefore $\mathfrak{L}^*(x_0) = \bigcup_{\mathfrak{P}(x_0) \in [\mathfrak{L}(x_0)]} \mathfrak{P}(x_0)$.

Now, we are going to characterize topologically the largest m-convergence to a point.

Theorem 3. *Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. Let $\mathfrak{L}^*(x_0)$ be the largest multivalued convergence to a point $x_0 \in L$. Then two following statements are equivalent*

(1) $(\{x_n\}, x_0) \in \mathfrak{L}^*(x_0)$

(2) *Each λ-neighbourhood of x_0 contains nearly all x_n of $\{x_n\}$.*

Proof. (1) evidently implies (2). Now, denote by $\mathfrak{T}_0$ the set of all elements $(\{x_n\}, x_0)$ such that the sequences $\{x_n\}$ have property (2) and put $\mathfrak{T} = \mathfrak{T}_0 \cup \mathfrak{L}$. Since $\mathfrak{L} \subset \mathfrak{T}$ the set $\mathfrak{T}$ satisfies both axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ so that $\mathfrak{T}$ is a multivalued convergence on L. Denote by τ the induced m-convergence topology and prove that $\tau = \lambda$. Because $\mathfrak{L} \subset \mathfrak{T}$ we have $\lambda < \tau$. Now, if $A \subset L$ and $x \in \tau A$, then there exists a sequence of points $x_n \in A$ and an element $(\{x_n\}, x)$ in $\mathfrak{T}_0$ or in $\mathfrak{L}$; hence $x \in \lambda A$, by (2), so that $\lambda > \tau$. Therefore $\lambda = \tau$ and $\lambda A \div \tau A = 0$; hence $\mathfrak{T}(x_0) \sim \mathfrak{L}(x_0)$. Since, by Theorem 2, $\mathfrak{T}(x_0) \subset \mathfrak{L}^*(x_0)$ and because $\mathfrak{T}_0 \subset \mathfrak{T}(x_0)$ we have $\mathfrak{T}_0 \subset \mathfrak{L}^*(x_0)$. Thus we have proved that (2) implies (1).

Corollary 3. *Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. Let $\mathfrak{L}^*(x_0)$ be the largest multivalued convergence to a point $x_0 \in L$. Then $(\{x_n\}, x_0) \in \mathfrak{L}^*(x_0)$ if and only if each subsequence $\{x_{n_i}\}$ of a sequence $\{x_n\}$ contains a subsequence $\{x_{n_{i_k}}\}$ $\mathfrak{L}$-converging to x_0.*

Proof. If $(\{x_n\}, x_0)$ does not belong to $\mathfrak{L}^*(x_0)$ then by Theorem 3 – there is a λ-neighbourhood $U(x_0)$ of x_0 and a subsequence of points $x_{n_i} \in L - U(x_0)$. Consequently no subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ $\mathfrak{L}$-converges to x_0.

If $(\{x_n\}, x_0) \in \mathfrak{L}^*(x_0)$ then $x_0 \in \lambda \bigcup x_{n_i}$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Therefore there is a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ which $\mathfrak{L}$-converges to x_0.

It is possible to classify all multivalued convergences on a given point set L by means of the following equivalence relation:

$$\mathfrak{L} \sim \mathfrak{M} \quad \text{whenever} \quad \lambda = \mu\,.$$

Because the inclusion $\subset$ orders the system $\mathfrak{L}$ of all m-convergences on L each class $[\mathfrak{L}]$ is an ordered class containing all m-convergences which induce the same closure topology λ.

Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. In the proof of Theorem 2 we have shown that the union $\bigcup_{\mathfrak{N} \in [\mathfrak{L}]} \mathfrak{N}$ is a multivalued convergence $\mathfrak{L}^*$ on L such that $\mathfrak{L}^*(x_0)$ is the largest m-convergence to the point x_0 in the class $[\mathfrak{L}(x_0)]$. Therefore $\mathfrak{L}^*(x_0) \sim \mathfrak{L}(x_0)$; consequently $x_0 \in L - (\lambda^* A \div \lambda A)$ for each $x_0 \in L$ and each $A \subset L$. Hence $L \subset L - (\lambda^* A \div \lambda A)$ so that $\lambda^* A = \lambda A$ and $\lambda^* = \lambda$. Thus $\mathfrak{L}^* \sim \mathfrak{L}$.

From this result from Theorem 3 and Corollary 3 we have the following [4])

Theorem 4. *Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. Then in the class $[\mathfrak{L}]$ there is a largest multivalued convergence $\mathfrak{L}^* = \bigcup_{\mathfrak{N} \in [\mathfrak{L}]} \mathfrak{N}$ and the following three statements are equivalent:*

(1) $\mathfrak{L} = \mathfrak{L}^*$.

(2) *If $x \in L$ and if $\{x_n\}$ is a sequence of points $x_n \in L$ such that each λ-neighbourhood of x contains nearly all x_n, then $(\{x_n\}, x) \in \mathfrak{L}$.*

[4]) In the case of onevalued convergences the implication (1) $\Rightarrow$ $(\mathfrak{L}_3)$ was proved in [7]. As to he condition $(\mathfrak{L}_3)$ see [1], [10] and [8].

($\mathscr{L}_3$) *If $x \in L$ and if $\{x_n\}$ is a sequence of points $x_n \in L$ such that in each subsequence $\{x_{n_i}\}$ there is a subsequence $\mathfrak{L}$-converging to x, then $(\{x_n\}, x) \in \mathfrak{L}$.*

The statement ($\mathscr{L}_3$) will be called the Urysohn's axiom.

According to Theorem 4 each multivalued nonlargest convergence $\mathfrak{L}$ can be completed with new elements such that we get the largest multivalued convergence the induced m-convergence topology of which is $\lambda^* = \lambda$.

Theorem 5. *Let $(L, \mathfrak{L}, \lambda)$ and $(L, \mathfrak{M}, \mu)$ be multivalued convergence spaces. Let $\mathfrak{L}^*$ and $\mathfrak{M}^*$ be the largest multivalued convergences such that $\lambda^* = \lambda$ and $\mu^* = \mu$. Then $\lambda < \mu$ if and only if $\mathfrak{L}^* \subset \mathfrak{M}^*$.*

Proof. If $\lambda < \mu$ and if $(\{x_n\}, x) \in \mathfrak{L}^*$ then, in view of Theorem 2, outside of each λ-neighbourhood of x there is at most a finite number of x_n. The same holds true for each μ-neighbourhood of x. Hence, by Theorem 2, $(\{x_n\}, x) \in \mathfrak{M}^*$ and so $\mathfrak{L}^* \subset \mathfrak{M}^*$. The converse assertion is clear.

From Theorem 5 it follows that the ordered system of all multivalued convergence topologies on L is isomorphic to the system of all largest multivalued convergences ordered by the inclusion $\subset$.

The convergence of a sequence of points $x_n \in L$ to a point x is not a topological property. As a matter of fact, let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space such that $\mathfrak{L}^* \neq \mathfrak{L}$ (for instance $\mathfrak{L}_0$ and $\mathfrak{L}_1$) choose $(\{x_n\}, x) \in \mathfrak{L}^* - \mathfrak{L}$; then $\mathfrak{L}^* - \lim x_n = x$. The identical map j on $(L, \mathfrak{L}^*, \lambda)$ onto $(L, \mathfrak{L}, \lambda)$ is a homeomorphism because $\lambda^* = \lambda$. The sequence $\{j(x_n)\}$, however, does not $\mathfrak{L}$-converge to $j(x)$.

Lemma 1. *Let φ be a map on a multivalued convergence space $(L, \mathfrak{L}, \lambda)$ into a multivalued convergence space $(M, \mathfrak{M}, \mu)$. Then φ is continuous [5]) if and only if the following condition is satisfied for each point $x \in L$:*

If $\lim x_n = x$ *then* $\lim \varphi(x_{n_i}) = \varphi(x)$ *for a suitable subsequence $\{x_{n_i}\}$ of $\{x_n\}$.*

Proof. Let $\mathfrak{L}$-$\lim x_n = x$. Then $x \in \lambda \bigcup x_n$. If φ is continuous then $\varphi(x) \in \mu\varphi(\bigcup x_n)$. Since $\varphi(\bigcup x_n) = \bigcup \varphi(x_n)$ the condition holds. Now, let $A \subset L$ and $x \in \lambda A$. Then there is a sequence of points $x_n \in A$ $\mathfrak{L}$-converging to x. If the condition is fulfilled then $\lim \varphi(x_{n_i}) = \varphi(x)$ for a suitable subsequence $\{x_{n_i}\}$ of $\{x_n\}$ so that $\varphi(x) \in \mu A$.

Some mathematicians are interested in the characterization of the largest convergence by means of certain operations in a given system of sequences of points. M. Fréchet [5] introduced two such operations[6]) which do not change the (onevalued)

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[5]) For onevalued convergence spaces see footnote on page 85 in [8].

[6]) Si une suite S converge vers A, il en est de même de toute suite obtenue en ajoutant à S un nombre fini d'éléments (distincts ou non).

Si un nombre fini de suites $S_1, S_2, \ldots, S_n$ convergent vers A, il en est de même de toute suite obtenue en rangeant en une seule suite les éléments (distincts ou non) de $S_1, S_2, \ldots, S_n$.

topology λ. M. DOLCHER [3] defined three operations[7]) and V. K. BALACHANDRAN [2] considered several operations by means of which it is possible to get new sequences. However it is not possible to characterize the largest convergence by means of operations mentioned above. M. Dolcher [3] has raised the following problem[8]):

Define an operation (necessarily more general then that one fulfilling α, β, γ) such that it is possible to get from a given system $\mathfrak{B}_x$ of sequences (such that $\{x\} \in \mathfrak{B}_x$) the smallest system containing $\mathfrak{B}_x$ and satisfying axioms $(\mathscr{L}_1)$, $(\mathscr{L}_2)$ and $(\mathscr{L}_3)$.

Before solving this problem let us define the operation α^*) as follows:

Let L be a point set. Denote by $\mathfrak{U}$ the system of all sequences of points of L. Let x be a point of L and $\mathfrak{B}_x$ a system of sequences $\{x_n\} \in \mathfrak{U}$ such that $\{x\} \in \mathfrak{B}_x$ and such that $\{x_n\} \in \mathfrak{B}_x$ implies $\{x_{n_i}\} \in \mathfrak{B}_x$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Now define a subsystem $\mathfrak{B}_x^* \subset \mathfrak{U}$: $\{y_n\} \in \mathfrak{B}_x^*$ whenever the condition α^* is true:

α^*) there is a non-negative integer q and sequences $\{x_n^m\}_{n=1}^{\infty} \in \mathfrak{B}_x$, $m = 1, 2, \ldots$, such that each cross-subsequence $\{x_{n_i}^{m_i}\}_{i=1}^{\infty}$, contains a subsequence belonging to $\mathfrak{B}_x$ and such that the double sequence $\{x_n^m\}_{m,n=1}^{\infty}$ can be arranged into the simple sequence $\{y_{q+n}\}_{n=1}^{\infty}$.

Statement 1. *Let L be a set and x a point of L. Let $\mathfrak{B}_x$ be a system of sequences $\{x_n\} \in \mathfrak{U}$ containing the constant sequence $\{x\}$ and each subssequence of any sequence belonging to $\mathfrak{B}_x$. Then the smallest system $\mathfrak{S} \subset \mathfrak{U}$ containing $\mathfrak{B}_x$ and satisfying axioms $(\mathscr{L}_1)$, $(\mathscr{L}_2)$ and $(\mathscr{L}_3)$ with respect to the point x is generated by the operation α^*).*

Proof. First prove that $\mathfrak{B}_x^*$ fulfills axioms $(\mathscr{L}_1)$, $(\mathscr{L}_2)$ and $(\mathscr{L}_3)$.

Let $\{x_n\} \in \mathfrak{B}_x$. Arrange $\{x_n\}$ in any manner into a double sequence $\{x_n^m\}_{m,n=1}^{\infty}$. Since each subsequence of $\{x_n\}$ belongs to $\mathfrak{B}_x$, from α^*) it follows that $\{x_n\} \in \mathfrak{B}_x^*$. Consequently $\mathfrak{B}_x \subset \mathfrak{B}_x^*$; especially $\{x\} \in \mathfrak{B}^*$. Therefore $(\mathscr{L}_1)$ holds in $\mathfrak{B}_x^*$.

In order to prove the validity of axiom $(\mathscr{L}_3)$ for the system $\mathfrak{B}_x^*$ let us assume that $\{y_n\}$ is a sequence of points $y_n \in L$ such that each subsequence $\{y_{n_i}\}$ of $\{y_n\}$ contains a subsequence belonging to $\mathfrak{B}_x^*$. Notice that each sequence of $\mathfrak{B}_x^*$ contains a subsequence belonging to $\mathfrak{B}_x$. Consequently each subsequence $\{y_{n_i}\}$ of $\{y_n\}$ contains a subsequence $\{y_{n_{i_k}}\} \in \mathfrak{B}_x$. Now, show that $\{y_n\} \in \mathfrak{B}_x^*$. For this purpose, use the method of transfinite induction:

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[7]) (α) Data una $\{p_n\}$, ne consideriamo dedotta ogni successione della quale la data è un resto.

(β) Data la $\{p_n\}$, ne consideriamo dedotta ogni successione $\{p_{r_n}\}$ con $\lim r_n = \infty$ (ossia: ogni $\{p_{r_n}\}$ tale che per ogni intero s sia $r_i = s$ al più per un numero finito di valori di i); in particolare dunque, ogni sottosuccessione.

(γ) Date un numero finito di successioni $S^i = \{p_n^i\}$ $(i = 1, 2, \ldots, k)$, consideriamo dedotta dalle S^i ogni successione S la quale ammetta $h(\leqq k)$ sottosuccessioni $S'^j = \{p_{r_n^j}\}$ rispettivamente uguali ad h delle S^i e tali che gl'interi r_n^j $(j = 1, 2, \ldots, h;\ n = 1, 2, \ldots)$ siano tutti distinti ed ogni intero vi compaia.

[8]) It is presented here in a slightly modified form.

Suppose we have just chosen the subsequences $\{z_n^\xi\}_{n=1}^\infty$ of $\{y_n\}$ for all $\xi < \alpha$, such that

$$(\xi, n) \neq (\eta, m),\ \xi, \eta < \alpha \quad \text{and} \quad z_n^\xi = y_k,\ z_m^\eta = y_l \quad \text{implies} \quad k \neq l$$

then – if it is possible – find a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that no member y_{n_i} of $\{y_{n_i}\}$ occurs in any subsequence $\{z_n^\xi\}_{n=1}^\infty$, $\xi < \alpha$; by our supposition, there is a subsequence $\{y_{n_{i_k}}\}$ of $\{y_{n_i}\}$ belonging to $\mathfrak{B}_x$. In this case put $z_k^\alpha = y_{n_{i_k}}$, $k \in N$. If it is not possible then there remains at most a finite number (0 incl.) of y_n which occur in no subsequence $\{z_n^\xi\}_{n=1}^\infty$, $\xi < \alpha$; in this case denote by q the greatest index of these y_n or put $q = 0$ if there are no such y_n. Leaving out each y_n, $1 \leqq n \leqq q$, from subsequences $\{z_n^\xi\}_{n=1}^\infty$, $\xi < \alpha$, we get subsequences $\{z_n'^\xi\}_{n=1}^\infty$, $\xi < \alpha$; if $q = 0$ we put $z_n'^\xi = z_n^\xi$. In this case we do not continue to choose further subsequences of $\{y_n\}$.

Thus we have proved that there is a (countable) ordinal β, a non-negative integer q and subsequences $\{z_n'^\xi\}_{n=1}^\infty$ of $\{y_n\}$, $\xi < \beta$. It is always possible to choose an infinite number of subsequences $\{z_n^\xi\}$. Let φ be a one-to-one map of the set of all ordinals $\xi < \beta$ onto N. Put $y_n^m = z_n'^\xi$, where $m = \varphi(\xi)$. It is easy to see that the double sequence $\{y_n^m\}_{m,n=1}^\infty$ can be arranged into the simple sequence $\{y_{q+n}\}_{n=1}^\infty$. Since $\{y_n^m\}_{n=1}^\infty \in \mathfrak{B}_x$ for each $m \in N$ and in each subsequence of $\{y_{q+n}\}$ there is a subsequence belonging to $\mathfrak{B}_x$, from α^*) it follows that $\{y_n\}_{n=1}^\infty \in \mathfrak{B}_x^*$.

Also axiom $(\mathscr{L}_2)$ is fulfilled in $\mathfrak{B}_x^*$. As a matter of fact, it is easy to see that $(\mathscr{L}_3)$ and α^*) implies $(\mathscr{L}_2)$.

Now, prove that $\mathfrak{B}_x^* = \mathfrak{S}$. Let $\mathfrak{S}'$ be a subsystem of $\mathfrak{U}$ containing $\mathfrak{B}_x$ and fulfilling all three axioms $(\mathscr{L}_1)$, $(\mathscr{L}_2)$ and $(\mathscr{L}_3)$ with respect to the point x. Let $\{z_n\}$ be a sequence of $\mathfrak{B}_x^*$. Since each subsequence of $\{z_n\}$ contains a subsequence of $\mathfrak{B}_x \subset \mathfrak{S}'$ and because $\mathfrak{S}'$ fulfills axiom $(\mathscr{L}_3)$, then $\{z_n\} \in \mathfrak{S}'$. Hence $\mathfrak{B}_x^* \subset \mathfrak{S}'$ so that $\mathfrak{B}_x^* = \mathfrak{S}$.

Let us notice that from Statement 1 the solution of the Fréchet's problem mentioned above follows:

The smallest system of sequences containing a onevalued convergence $\mathfrak{B}$ on a given set L and fulfilling axiom $(\mathscr{L}_3)$ is generated by the operation α^*) for each $x \in L$, $\mathfrak{B}_x$ being the system of all sequences $\mathfrak{B}$-converging to x.

3.

Let (P, w) be a closure space. Denote by $\mathfrak{T}$ the set of all elements $(\{x_n\}, x)$ such that each w-neighbourhood of x contains nearly all points x_n. It can be easily proved that $\mathfrak{T}$ satisfies all three axioms $(\mathscr{L}_1)$, $(\mathscr{L}_2)$ and $(\mathscr{L}_3)$. Therefore $\mathfrak{T}$ is the largest multivalued convergence on P and $(P, \mathfrak{T}, \tau)$ is a multivalued convergence space such that $\tau < w$. If $x \in wA$, $A \subset P$, implies $(\{x_n\}, x) \in \mathfrak{T}$ for a suitable sequence of points $x_n \in A$, then evidently $\tau = w$ and (P, w) is a multivalued convergence space.

In such a way it is possible to assign to each closure topology w the multivalued convergence topology τ which will be denoted by $\lambda(w)$. It is easy to show that $\lambda(w)$ is the strongest among all multivalued convergence topologies which are weaker than w.

Definition. Let (P, w) be a closure space and let A be a subset of P. We say that a point $x \in P$ has order 0 with respect to the set A whenever $x \in A$. Under the asumption that the orders ξ with respect to A are defined for all isolated ordinals $\xi < \alpha$, where α is an isolated ordinal, we say that a point $x \in P$ has order α with respect to A if it fails to have order $\leqq \alpha - 1$ and if there is a sequence of points $x_n \in P$ of orders $\leqq \alpha - 1$ such that each w-neighbourhood of x contains nearly all x_n. In this case the subset A will be called a convergence basis of the point x. The order of x with respect to A will be denoted by $o(x, A)$.

If the set A consists of one point z, we speak of the order of a point $x \in P$ with respect to the point z; in this case the convergence basis is the point z.

Using the method of transfinite induction it can easily be proved that $o(x, A) = \xi$ in (P, w) if and only if $x \in \lambda^{\xi}(w) A - \lambda^{\xi-1}(w) A$. Therefore the set $\bigcup_{\xi < \omega_1} \lambda^{\xi}(w) A$, which equals $u(\lambda(w)) A$, by Corollary 2, consists of all points having an order with respect to the set A.

Lemma 2. *Let (P, w) be a closure space. Let A be a subset of P and $o(x, A)$ an order of a point $x \in P$ with respect to the set A. Then there is a countable convergence basis $A_0 \subset A$ such that $o(x, A) = o(x, A_0)$.*

Proof. Assume that the assertion is true for all isolated $\xi < \beta$ where β is an isolated ordinal less than $o(x, A)$. Let $y \in \lambda^{\beta}(w) A - \lambda^{\beta-1}(w) A$. Then $o(y, A) = \beta$ and there is a sequence of points $y_n \in \lambda^{\beta-1}(w) A$ such that $o(y_n, A) \leqq \beta - 1$, $n \in N$, and such that each $\lambda(w)$-neighbourhood of y contains nearly all y_n. Denote $\xi_n = o(y_n, A)$. According to our supposition $\xi_n = o(y_n, A_n)$ for suitable countable subsets $A_n \subset A$, $n \in N$. From this it follows that $y_n \in \lambda^{\xi_n}(w) A_n \subset \lambda^{\xi_n}(w) A_0 \subset \lambda^{\beta-1}(w) A_0$, $n \in N$, where $A_0 = \bigcup A_n$ is a countable set. Therefore $y \in \lambda^{\beta}(w) A_0 - \lambda^{\beta-1}(w) A_0$ i.e. $o(y, A_0) = \beta$.

Now we are going to solve the following problem[9]) of M. Dolcher [3]:

Let λ be a multivalued convergence topology on L, let $u(\lambda)$ be the weakest topology on L such that $\lambda < u(\lambda)$ and let $\mu(u(\lambda))$ be the strongest multivalued convergence topology on L such that $u(\lambda) > \mu(u(\lambda))$. What is the necessary and sufficient condition that $\mu(u(\lambda)) = \lambda$?

The solution of this problem is given in the following Lemma 3 and Statement 2:

Lemma 3. *Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. Then $\mu(u(\lambda)) = \lambda$ if and only if $(\{x_n\}, x) \in \mathfrak{L}$ implies that* Lim x_n *is a closed set.*

[9]) Il problema di caratterizzare, in termini di struttura di convergenza, le convergenze prive di unicita del limite lequali sono deducibili da topologie.

Proof. Suppose $\operatorname{Lim} x_n$ is not closed. Then there is a point $y \in \lambda \operatorname{Lim} x_n - \operatorname{Lim} x_n$. Consequently there is a sequence of points $y_m \in \operatorname{Lim} x_n$ such that $(\{y_m\}, y) \in \mathfrak{L}$. Since $(\{x_n\}, y_m) \in \mathfrak{L}$ for each $m \in N$, then every $u(\lambda)$-neighbourhood of y contains nearly all x_n; the sequence $\{x_n\}$ however, fails to converge to y. Hence $\mu(u(\lambda)) \neq \lambda$.

Now, assume that $\mu(u(\lambda)) \neq \lambda$. Then there is a set $A \subset L$ and a point $z \in \mu(u(\lambda)) A - \lambda A$. Consequently there is a sequence of points $z_n \neq z$ of A such that each $u(\lambda)$-neighbourhood of z contains nearly all z_n. The set $\bigcup z_n$ cannot be λ-closed; otherwise $L - \bigcup z_n$ would be a $u(\lambda)$-neighbourhood of z containing no z_n. From this it follows that there is a point $x \in L$ and a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $x \in \operatorname{Lim} z_{n_i}$. The set $\operatorname{Lim} z_{n_i}$ cannot be λ-closed; otherwise $\operatorname{Lim} z_{n_i} = u(\lambda) \operatorname{Lim} z_{n_i}$; z would belong to $\operatorname{Lim} z_{n_i}$ and consequently to λA. Thus we have proved that $\mathfrak{L} - \lim z_{n_i} = x$ and that $\operatorname{Lim} z_{n_i}$ fails to be a closed set.

Statement 2. *Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. Then $\mu(u(\lambda)) = \lambda$ if and only if the following condition is fulfilled:*

If $(\{x_n\}, y_m) \in \mathfrak{L}$, $m \in N$, and if $(\{y_m\}, y) \in \mathfrak{L}$, then $(\{x_n\}, y) \in \mathfrak{L}$.

Proof. It is easy to show that the condition is fulfilled if and only if each $\operatorname{Lim} x_n$ is closed in L. Consequently the proof of the statement follows immediately from Lemma 3.

Now, let us mention two other problems of M. Dolcher [3]: Let (P, u) be a topological space. What are the necessary and sufficient conditions that $u = v(\lambda(u))$? Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. What is the necessary and sufficient condition (expressed in terms of convergence) that $u(\lambda)$ is a T_0-topology?

The solutions of these problems are given in the following statements (3 and 4):

Statement 3. *Let (P, u) be a topological space. Then $u = v(\lambda(u))$ if and only if for each subset $A \subset P$ and each point $x \in uA$ there is a countable convergence basis $A_0 \subset A$ of the point x.*

Proof. Let $u = v(\lambda(u))$ and let $x \in uA$. Then $x \in v(\lambda(u)) A$. Since $v(\lambda(u)) A = \bigcup_{\xi < \omega_1} \lambda^\xi(u) A$, by Corollary 2, then either $x \in A$ or there is an ordinal $\alpha > 0$ such that $x \in \lambda^\alpha(u) A - \lambda^{\alpha-1}(u) A$. Therefore the point x has an order with respect to set A. In view of Lemma 2 there is a countable convergence basis $A_0 \subset A$ of x.

Conversely, suppose that we are given a set $B \subset P$ and a point $y \in uB$ which has an order β with respect to a countable subset $B_0 \subset B$. Then $y \in \lambda^\beta(u) B$ so that $y \in \bigcup_{\xi < \omega_1} \lambda^\xi(u) B$, β being an isolated countable ordinal. Hence $uB \subset v(\lambda(u)) B$. On the other hand, evidently $\lambda(u) B \subset uB$ so that $v(\lambda(u)) B \subset uB$. Therefore $u = v(\lambda(u))$.

Statement 4. *Let $(L, \mathfrak{L}, \lambda)$ be a multivalued convergence space. Then the two following statements are equivalent*

(1) *$u(\lambda)$ is a T_0-topology*

(2) *if x and y are points of L each of them having an order with respect to the other then $x = y$.*

Proof. Since the $u(\lambda)$-closure of any point $z \in L$ consists of all points $z' \in L$ which have an order with respect to the point z, the proof instantly follows from the fact that $\lambda(\lambda) = \lambda$ and consequently $u(\lambda)\, z = \bigcup_{\xi < \omega_1} \lambda^\xi z = \bigcup_{\xi < \omega_1} \lambda^\xi(\lambda)\, z$.

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Резюме

О НЕКОТОРЫХ ПРОБЛЕМАХ МНОГОЗНАЧНОЙ СХОДИМОСТИ

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М. ФРЕШЕ ввел аксиоматическим образом понятие (однозначной) сходимости на множестве L как однозначное отображение системы (не обязательно всех) последовательностей $\{x_n\}$ точек $x_n \in L$ на L, удовлетворяющее аксиомам $(\mathscr{L}_1)$ и $(\mathscr{L}_2)$. В этой статье изучается многозначная сходимость (вкратце m-сходимость), удовлетворяющая аксиомам $(\mathscr{L}_1)$ и $(\mathscr{L}_2)$ и отличающаяся от однозначной сходимости тем, что одна и та же последовательность точек может сходиться к различным точкам. Если определить замыкание λA подмножества

$A \subset L$ обычным образом как множество всех $\lim x_n$, где $\bigcup x_n \subset A$, то λ – топология замыкания, выполняющая аксиомы (C_0), (C_1) и (C_2). Таким образом мы получаем топологическое пространство с m-сходимостью.

В системе всех m-сходимостей на данном множестве L определяется эквивалетность и доказывается, что в каждом классе существует наибольшая m-сходимость, содержащая каждую эквивалентную m-сходимость как m-подсходимость. Эту наибольшую m-сходимость можно охарактеризовать как с помощью окрестностей точек, так и с помощью аксиомы $(\mathscr{L}_3)$.

В работе решены некоторые проблемы *М*. ДОЛЬХЕРА [3]. Одна из них заключается в характеризации наибольшей m-сходимости при помощи некоторых операций в m-сходимости, удовлетворяющих аксиомам $(\mathscr{L}_1)$ и $(\mathscr{L}_2)$. Была найдена операция α^*, решающая эту проблему. В случае однозначной сходимости это представляет также решение проблемы М. Фреше [5].