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EIGENVALUES OF OPERATORS IN l_p-SPACES IN DENUMERABLE MARKOV CHAINS

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Let p_{jk} be transition probabilities of a Markov chain in discrete time, with a denumerable state space, and with a sub-invariant measure μ. Let $l_p(\mu)$ $(1 \leqq p < \infty)$ be the space of functions f on the state space for which $\|f\|_p = [\sum_k |f_k|^p \mu_k]^{1/p} < \infty$, and let the operator T_p in $l_p(\mu)$ be defined by $(T_p f)_j = \sum_k f_k p_{jk}$. In the paper the eigenvalues of T_p on the unit circle are found.

1. INTRODUCTION AND NOTATION

Consider a Markov chain in discrete time, with a denumerable state space and stationary transition probabilities in one step p_{jk}, in n steps $p_{jk}^{(n)}$. In the whole paper it is supposed that the chain is irreducible.

Dealing with a function f on the state space, we shall denote by f_k its value at the point k. Furthermore, $|f|$ denotes the function whose values are $|f_k|$; $f \neq 0$ means that f is not identically equal to zero; $f \geqq 0$ means $f_k \geqq 0$ for every k, etc.

A function μ is called a sub-invariant measure (for the given Markov chain) if $\mu \geqq 0$, $\mu \neq 0$, and

$$\sum_j p_{jk} \mu_j \leqq \mu_k \tag{1}$$

for every k. D. G. KENDALL [2] has shown that for every irreducible Markov chain there is at least one sub-invariant measure. In the following the letter μ will always denote some fixed sub-invariant measure.

Let $l_p(\mu)$ $(1 \leqq p < \infty)$ be the Banach space of all complex functions f on the state space for which $\|f\|_p = [\sum_k |f_k|^p \mu_k]^{1/p}$ is finite; $\|f\|_p$ is the norm of the function f.

Finally denote by T_p the operator in $l_p(\mu)$ defined by the formula

$$(T_p f)_j = \sum_k f_k p_{jk} \tag{2}$$

for every $f \in l_p(\mu)$. It is well known (see e.g. E. NELSON [3], Z. ŠIDÁK [5]) that T_p is in fact a linear continuous operator in the space $l_p(\mu)$ with the norm $\|T_p\|_p \leqq 1$. A complex number α for which $|\alpha| = 1$ and $T_p f = \alpha f$ for some $f \neq 0$, $f \in l_p(\mu)$, is called an eigenvalue of T_p on the unit circle.

In the present paper we prove that for a positive-recurrent chain with a period d, the eigenvalues of T_p on the unit circle are precisely the d-th roots of unity, while for a null-recurrent or a transient chain T_p has no eigenvalues on the unit circle.

The same results for the operator T_2 in the Hilbert space $l_2(\mu)$ have been obtained by a somewhat complicated method in [4], so that the present paper generalizes the results in [4]. This generalization has been achieved by some strengthening of certain methods in [4] and [5]; at the same time our present proofs are considerably more elementary and more direct. Although some lemmas and methods can also be found in [4] or [5], we prefer to present here the whole logical development of proofs in order to clarify the steps of our present approach and to facilitate the reading.

Let us observe also that the eigenvalues on the unit circle of an analogous operator in l_∞, the space of all bounded complex functions, were found in [4].

2. SEVERAL LEMMAS

Lemma 1. *If μ is a sub-invariant measure then $\mu_k > 0$ for every k.*

Proof. By definition of sub-invariant measures, there is a j for which $\mu_j > 0$. For every k irreducibility of the chain implies the existence of a positive integer n such that $p_{jk}^{(n)} > 0$. Now, by induction we obtain

$$\mu_k \geqq \sum_r p_{rk}\mu_r \geqq \dots \geqq \sum_r p_{rk}^{(n)}\mu_r ,$$

and clearly

$$\sum_r p_{rk}^{(n)}\mu_r \geqq p_{jk}^{(n)}\mu_j > 0 .$$

This lemma 1 was used in [4] also.

Lemma 2. *Let h be a function on the state space such that $h \geqq 0$, $\sum_k h_k p_{jk} \leqq h_j$ for every j. Then either $h_j > 0$ for all j or $h_j = 0$ for all j.*

Proof. This lemma follows also from the assumption of irreducibility. If $h_j = 0$ for all j, the assertion is obvious. Consider the contrary case, when $h_r > 0$ for some r. Then for every j there is a positive integer n such that $p_{jr}^{(n)} > 0$, and we obtain easily

$$h_j \geqq \sum_k h_k p_{jk} \geqq \dots \sum_k h_k p_{jk}^{(n)} \geqq h_r p_{jr}^{(n)} > 0 .$$

This lemma can also be found in [4].

Lemma 3. *If $T_p h \geqq h$ for a function $h \geqq 0$, $h \in l_p(\mu)$, then $T_p h = h$.*

Proof. By our assumptions and by the well-known Hölder's inequality,

$$\sum_j h_j^p \mu_j \leqq \sum_j (T_p h)_j^p \mu_j = \sum_j (\sum_k h_k p_{jk})^p \mu_j \leqq \sum_j \sum_k h_k^p p_{jk} \mu_j \leqq \sum_k h_k^p \mu_k .$$

Since the two extreme expressions in this chain of inequalities are equal to $\|h\|_p^p$, we must have in particular, that

$$\sum_j h_j^p \mu_j = \sum_j (T_p h)_j^p \mu_j .$$

By lemma 1, $\mu_j > 0$ for every j so that $h_j^p = (T_p h)_j^p$ for every j, and the assertion of the lemma follows.

Some related lemmas for recurrent chains were proved in [4] and [5].

Lemma 4. *If $T_p h = h$ for a function $h \in l_p(\mu)$, then h is identically equal to some constant.*

Proof. Clearly it is sufficient to give the proof only for h real. Let $f^{(a)}$ be the function identically equal to a non-negative constant a. Then $\sum_k f_k^{(a)} p_{jk} = f_j^{(a)}$ for every j. Setting $g = h - f^{(a)}$ we have $g \leqq h$ and also $\sum_k g_k p_{jk} = g_j$ for every j. Finally, denote by g^+ the function defined by $g_k^+ = g_k$ whenever $g_k \geqq 0$, and by $g_k^+ = 0$ whenever $g_k < 0$. It follows that $0 \leqq g^+ \leqq |h|$, so that $g^+ \in l_p(\mu)$, and it is easy to verify that $g_j^+ \leqq \sum_k g_k^+ p_{jk}$ for every j. Lemma 3 now implies $g_j^+ = \sum_k g_k^+ p_{jk}$ for every j, and by lemma 2 either $g_j^+ = 0$ for all j or $g_j^+ > 0$ for all j. The first case yields, for all j, the inequalities $g_j \leqq 0$, $h_j - f_j^{(a)} \leqq 0$, $h_j \leqq a$. The second case yields, for all j, $g_j > 0$, $h_j - f_j^{(a)} > 0$, $h_j > a$. Choosing firstly $a = 0$ we see that the function h is either non-positive or positive. However, if h is positive then it must be constant, because a is an arbitrary non-negative number; if h is non-positive it suffices to carry the proof through with the function $-h$.

Some related results for recurrent chains were proved also in [4] and [5].

3. POSITIVE-RECURRENT CHAINS

In the present section let us suppose that the chain is positive-recurrent and has period d (for an aperiodic chain this means, of course, $d = 1$). More explicitly, let $S_0, S_1, \ldots, S_{d-1}$ be the cyclical classes of states of the chain, i.e. whenever $j \in S_m$ $(m = 0, 1, \ldots, d-1)$ and $p_{jk} > 0$, then $k \in S_{m+1}$ (where we have put $S_d = S_0$). Finally, define the following functions on the state space: $e^{(0)}$ is the function which is identically equal to 1; $e^{(k)}$ (for $k = 1, \ldots, d-1$) is the function which is equal to $e_j^{(k)} = \exp[2\pi imk/d]$ for every $j \in S_m$ $(m = 0, 1, \ldots, d-1)$. Since for a positive-recurrent chain $\sum_j \mu_j < \infty$, clearly $e^{(k)} \in l_p(\mu)$.

Theorem 1. *Let the chain be positive-recurrent with the period d. Then the set of all eigenvalues of the operator T_p $(1 \leqq p < \infty)$ on the unit circle consists precisely of the numbers $e^{2\pi ik/d}$, $k = 0, 1, \ldots, d-1$. Every eigenfunction $f \in l_p(\mu)$ for which*

$$(3) \qquad T_p f = e^{2\pi ik/d} f$$

is equal to some multiple of $e^{(k)}$.

Proof. First, we shall verify that each number $e^{2\pi ik/d}$ is an eigenvalue of T_p. In fact, if k is one of the numbers $0, 1, \ldots, d-1$, and if $j \in S_m$, $m = 0, 1, \ldots, d-1$, then

$$(T_p e^{(k)})_j = \sum_r e_r^{(k)} p_{jr} = \sum_{r \in S_{m+1}} e_r^{(k)} p_{jr} = e^{2\pi i(m+1)k/d} = e^{2\pi ik/d} e_j^{(k)}.$$

For the opposite assertion we must show that every eigenvalue of T_p on the unit circle is of the form $e^{2\pi ik/d}$. That is, we must prove that $T_p f = \alpha f$, $|\alpha| = 1$, $f \neq 0$, $f \in l_p(\mu)$ implies $\alpha^d = 1$. Obviously these assumptions imply $T_p|f| \geqq |\alpha|\,|f| = |f|$, hence by lemma 3 $T_p|f| = |f|$. Furthermore, by lemma 4 we see that $|f|$ is identically equal to some constant c, and clearly $c > 0$.

In the sequel let l be some fixed point from the cyclical class S_0. From the equality $T_p f = \alpha f$ there follows $T_p^n f = \alpha^n f$ for every $n = 1, 2, \ldots$, and in other notation in particular

$$(4) \qquad \sum_k f_k p_{lk}^{(n)} = \alpha^n f_l \quad \text{for} \quad n = 1, 2, \ldots .$$

The sum on the left side of (4) may be viewed as the Stieltjes integral of the function f with respect to the measure given by $p_{lk}^{(n)}$ for one-element set $\{k\}$. By (4), and using Schwarz's inequality, we obtain

$$c^2 = |f_l|^2 = |\alpha^n f_l|^2 = |\sum_k f_k p_{lk}^{(n)}|^2 \leqq \sum_k |f_k|^2\, p_{lk}^{(n)} = c^2 .$$

Thus all expressions in the previous line are equal; however, if equality occurs in Schwarz's inequality then the integrated function f must be constant almost everywhere with respect to the measure in the integral. For our case this shows that there exist constants c_n, $n = 1, 2, \ldots$, such that

$$(5) \qquad p_{lk}^{(n)} > 0 \quad \text{implies} \quad f_k = c_n .$$

Clearly, by (4) and (5) we have

$$\alpha^n f_l = \sum_k f_k p_{lk}^{(n)} = c_n \sum_k p_{lk}^{(n)} = c_n ,$$

and therefore (5) may be stated in a more precise form

$$(6) \qquad p_{lk}^{(n)} > 0 \quad \text{implies} \quad f_k = \alpha^n f_l .$$

It is well known that there exists a sufficiently large r such that $p_{ll}^{(rd)} > 0$, $p_{ll}^{((r+1)d)} > > 0$. Hence, by (6),

$$f_l = \alpha^{rd} f_l = \alpha^{(r+1)d} f_l .$$

Since $f_l \neq 0$, $\alpha \neq 0$, we obtain the desired conclusion $\alpha^d = 1$.

It remains to prove the second assertion of the theorem (which is, of course, for $d = 1$ already contained in lemma 4). Supposing that (3) holds we may use the previous part of our proof with $\alpha = e^{2\pi ik/d}$; in particular, (6) is true with this α. Now, if $j \in S_m$ $(m = 0, 1, \ldots, d - 1)$, there exists a positive integer r such that $p_{lj}^{(rd+m)} > 0$. Thus (6) yields

$$f_j = \left(e^{2\pi ik/d}\right)^{rd+m} f_l = e^{2\pi ikr} e^{2\pi ikm/d} f_l = f_l e_j^{(k)} ,$$

or, more concisely, $f = f_l e^{(k)}$.

4. NULL-RECURRENT AND TRANSIENT CHAINS

Lemma 5. *If μ is a sub-invariant measure for a null-recurrent or transient chain, then $\sum_j \mu_j = \infty$.*

Proof. Suppose the contrary, that $\sum_j \mu_j < \infty$. In this case, if there were $\mu_k > > \sum_j p_{jk}\mu_j$ for some k, then

$$\sum_k \mu_k > \sum_k \sum_j p_{jk}\mu_j = \sum_j \mu_j \sum_k p_{jk} = \sum_j \mu_j ,$$

which is a contradiction. Hence $\mu_k = \sum_j p_{jk}\mu_j$ for all k, and then it is easily seen that $\mu_k = \sum_j p_{jk}^{(n)}\mu_j$ for all k and all $n = 1, 2, \ldots$. Using $|p_{jk}^{(n)}| \leqq 1$, $\sum_j \mu_j < \infty$, we obtain

$$\mu_k = \lim_{n\to\infty} \sum_j p_{jk}^{(n)}\mu_j = \sum_j \lim_{n\to\infty} p_{jk}^{(n)}\mu_j = 0 .$$

However, by definition, μ cannot be identically zero.

For a null-recurrent chain this lemma follows from the results in [4] and C. Derman [1]; for a transient chain it appears as a corollary to the results on eigenvalues of T_2 in [4].

Theorem 2. *Let the chain be null-recurrent or transient. Then the operator T_p $(1 \leqq p < \infty)$ has no eigenvalues on the unit circle.*

Proof. Suppose $T_p f = \alpha f$ for some $f \in l_p(\mu)$, $|\alpha| = 1$. Then $T_p|f| \geqq |\alpha f| = |f|$, which by lemma 3 gives $T_p|f| = |f|$. By lemma 4 it follows that $|f|$ is identically equal to some constant c. Since $f \in l_p(\mu)$, $\sum_k |f_k|^p \mu_k = c^p \sum_k \mu_k < \infty$, by lemma 5 we must have $c = 0$, that is $f = 0$.

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Резюме

СОБСТВЕННЫЕ ЗНАЧЕНИЯ ОПЕРАТОРОВ В ПРОСТРАНСТВАХ l_p ДЛЯ СЧЕТНЫХ ЦЕПЕЙ МАРКОВА

ЗБЫНЕК ШИДАК (Zbyněk Šidák), Прага

Рассматривается неприводимая цепь Маркова в дискретном времени, со счетной системой состояний, стационарными вероятностями перехода p_{jk} и субинвариантной мерой μ. Обозначим через $l_p(\mu)$ (для $1 \leqq p < \infty$) пространство всех комплексных функций f на пространстве состояний, для которых норма $\|f\|_p = [\sum_k |f_k|^p \mu_k]^{1/p}$ конечна. Определим оператор T_p в пространстве $l_p(\mu)$ так, что значение $T_p f$ в точке j равно $(T_p f)_j = \sum_k f_k p_{jk}$.

Доказывается, что для положительной возвратной цепи с периодом d множество всех собственных значений T_p на единичной окружности совпадает с множеством $\{e^{2\pi ik/d};\ k = 0, 1, \ldots, d-1\}$; собственные подпространства, принадлежащие к этим значениям, одномерны. Для нулевых возвратных и для невозвратных цепей операторы T_p не имеют никаких собственных значений на единичной окружности.