Štefan Schwarz

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CONVOLUTION SEMIGROUP OF MEASURES ON COMPACT NON-COMMUTATIVE SEMIGROUPS[1])

ŠTEFAN SCHWARZ, Bratislava

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To every compact semigroup S we associate the semigroup $\mathfrak{M}(S)$ of all probability measures on S with convolution as multiplication. The purpose of this paper is the study of the structure of $\mathfrak{M}(S)$. Here the emphasis is on the non-commutative case.

Let S be a compact semigroup, i.e. a compact Hausdorff space with a jointly continuous binary operation (multiplication) under which it forms a semigroup.

Let $\mathfrak{A}$ be the set of all compact subsets of S and $\mathfrak{S}$ the σ-algebra generated by $\mathfrak{A}$. The elements of the σ-algebra $\mathfrak{S}$ are called the Borel subsets of S.

A probability measure on S is a non-negative, real-valued, regular Borel measure μ on S such that $\mu(S) = 1$. The set of all probability measures on S is denoted by $\mathfrak{M}(S)$.

Let $\omega(S)$ be the Banach space of real continuous functions on S. By the Riesz representation theorem (see P. R. HALMOS [2], p. 247–248) the set of all positive linear functionals Φ on $\omega(S)$ such that $\Phi(1) = 1$ is in a biunivoque correspondence with $\mathfrak{M}(S)$ under the mapping $\mu \to \Phi$, where $\Phi(f) = \int_S f \, d\mu$ for each $f \in \omega(S)$. Thus we may consider $\mathfrak{M}(S)$ as a subset of $\omega(S)^*$ (the first conjugate space of $\omega(S)$).

One readily verifies that $\mathfrak{M}(S)$ with the weak* – topology is compact (see J. G. WENDEL [11], B. M. KLOSS [4], I. GLICKSBERG [1]).

We introduce in $\mathfrak{M}(S)$ a multiplication. If $\mu, \nu \in \mathfrak{M}(S)$, the convolution $\mu\nu$ is the unique measure $\in \mathfrak{M}(S)$ such that

$$(1) \qquad \int_S f(z) \, d(\mu\nu)(z) = \int_S \int_S f(xy) \, d\mu(x) \, d\nu(y) ,$$

for each $f \in \omega(S)$. It is known that this multiplication is associative and jointly continuous in the variables μ,ν in $\mathfrak{M}(S)$. (See I. Glicksberg [1].) Thus $\mathfrak{M}(S)$ becomes a compact semigroup.

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[1]) The main results of this paper have been communicated on the International Symposium on general topology and its relations to analysis and algebra, Prague, 1961, September 1–8. (See General Topology and its Relations to Modern Analysis and Algebra. Proceedings of the Symposium, Prague 1961, pp. 307–310.)

For any element $x \in S$ we define the element $x' \in \mathfrak{M}(S)$ as the point mass at x. The corresponding functional sends the function f into the number $f(x)$ and the element xy goes over into the measure $(xy)' = x'y'$. Therefore the mapping $x \to x'$ of S into $\mathfrak{M}(S)$ is a homeomorphic isomorphism, so that henceforth we may regard S as embedded in $\mathfrak{M}(S)$ and omit primes.

Let be $\mu \in \mathfrak{M}(S)$. The support of μ, denoted by $C(\mu)$, is the set of all $x \in S$ such that for each neighborhood U of x we have $\mu(U) > 0$. It is well known that $C(\mu)$ is a closed subset of S, $\mu(C(\mu)) = 1$ and for every relatively open subset V of $C(\mu)$ we have $\mu(V) > 0$. Also if A is a closed subset of S such that $\mu(A) = 1$, we have $C(\mu) \subset A$.[2])

Finally we mention the important fact that if $\mu, \nu \in \mathfrak{M}(S)$ then $C(\mu\nu) = C(\mu)\, C(\nu)$ (B. M. Kloss [4], I. Glicksberg [1]).

The purpose of this paper is to study the structure of $\mathfrak{M}(S)$. The results obtained are extensions of those of N. N. Vorobjev [10], E. Hewitt and H. S. Zuckerman [3], J. G. Wendel [11], B. M. Kloss [4], I. Glicksberg [1] and K. Stromberg [8] the essential novelty being that we are going beyond the restriction of commutativity even in the non-group case (for S). The case that S is finite has been treated in detail in the paper [7]. Also in the present paper a sort of finiteness condition will be imposed at some places by supposing that some simple subsemigroups of S contain only a finite number of idempotents.

In section 1 we are dealing with the idempotents $\in \mathfrak{M}(S)$. In section 2 we describe the maximal subgroups contained in $\mathfrak{M}(S)$. In section 3 two limit theorems are given.

1. THE IDEMPOTENTS $\in \mathfrak{M}(S)$

If $\varepsilon = \varepsilon^2 \in \mathfrak{M}(S)$, then $C(\varepsilon) . C(\varepsilon) = C(\varepsilon)$ implies that $C(\varepsilon)$ is a semigroup. Moreover B. M. Kloss [4] proved that $C(\varepsilon)$ is a (closed) simple subsemigroup of S. We shall prove below that conversely every closed simple subsemigroup of S containing a finite number of idempotents is the support of some idempotent element $\in \mathfrak{M}(S)$.

A semigroup P is called simple if it does not contain a two-sided ideal $\neq P$. If P is compact it is known that P contains minimal right and left ideals. In fact, $P = \bigcup_{\alpha \in \Lambda_1} R_\alpha = \bigcup_{\beta \in \Lambda_2} L_\beta$, where $R_\alpha (L_\beta)$ runs through all (disjoint) minimal right (left) ideals of P. Also $R_\alpha \cap L_\beta = R_\alpha L_\beta = G_{\alpha\beta}$ is a closed (compact) group and P can be written as a union of closed topologically isomorphic groups: $P = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$. The $G_{\alpha\beta}$'s will be called group-components of P. The symbol $e_{\alpha\beta}$ will denote always the unit element of the group $G_{\alpha\beta}$.

Lemma 1,1. *Let S be compact, μ an idempotent $\in \mathfrak{M}(S)$, $P = C(\mu)$ and L an arbitrary fixed chosen minimal left ideal of P. If $f \in \omega(P)$, then $\int_P f(x\xi)\, \mathrm{d}\mu(x)$ has the same value for every $\xi \in L$.*

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[2]) $C(\mu)$ is simply the complement of the union of all open sets of μ-measure zero.

Remark. This Lemma is a natural generalization of Lemma 2,3 of the paper [7].[2a])

Proof. Since μ is an idempotent and $C(\mu) = P$, we have[3])

$$\int_P F(x)\,\mathrm{d}\mu(x) = \int_P\int_P F(xy)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \tag{2}$$

for every $F \in \omega(P)$.

Let be e an idempotent $\in L$. Denote (for $y \in P$) $\varphi(y) = \int_P f(xye)\,\mathrm{d}\mu(x)$. Since $xye \in P \,.\, P \,.\, L \subset L$, $f(xye)$ is defined. Put in (2) $F(x) = f(xye)$. We have

$$\varphi(y) = \int_P f(xye)\,\mathrm{d}\mu(x) = \int_P\int_P f(zxye)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(x) =$$

$$= \int_P\left[\int_P f(zxye\,\mathrm{d}\mu(z)\right]\mathrm{d}\mu(x) = \int_P \varphi(xy)\,\mathrm{d}\mu(x)\,.$$

Suppose that $\varphi(y)$ takes its greatest value in the point $y_0 \in P$. Hence $\varphi(y_0) = \int_P \varphi(xy_0)\,\mathrm{d}\mu(x)$, and since $\mu(P) = 1$, we have $\int_P [\varphi(y_0) - \varphi(xy_0)]\,\mathrm{d}\mu(x) = 0$. With respect to the continuity of φ the last relation implies $\varphi(y_0) = \varphi(xy_0)$ for every $x \in P$. This means: $\int_P f(xye)\,\mathrm{d}\mu(x)$ takes the same value for $y = y_0$ and for every $y \in Py_0$. In other words: $\int_P f(x\xi)\,\mathrm{d}\mu(x)$ takes the same value for every $\xi \in Py_0e$. Now $Py_0e \subset Py_0L \subset L$, and since L is a minimal left ideal of P, we have $Py_0e = L$. This proves Lemma 1,1.

In what follows we shall often suppose that $P = C(\mu)$ contains only a finite number of idempotents. In this case we shall write in the above sense $P = \bigcup_{i=1}^{s} R_i = \bigcup_{k=1}^{r} L_k = \bigcup_{i=1}^{s}\bigcup_{k=1}^{r} G_{ik}$, where $r \geqq 1$, $s \geqq 1$ are integers and $G_{ik} = R_iL_k = R_i \cap L_k$.

Theorem 1,1. *Let S be a compact semigroup, μ such an idempotent $\in \mathfrak{M}(S)$ that $C(\mu) = P$ contains a finite number of idempotents. Let $P = \bigcup_{i=1}^{s}\bigcup_{k=1}^{r} G_{ik}$ be the group-decomposition of P. Then μ restricted to G_{ik} is an invariant measure on the group G_{ik}.*

Remark. Of course the measure μ restricted to G_{ik} does not necessarily belong to $\mathfrak{M}(G_{ik})$ since $\mu(G_{ik}) \neq 1$ if $rs > 1$.

[2a]) (Added in proofs.) In the meantime Lemma 1,1 and some of its consequences have been proved also by H. S. Collins in the paper [13]. (See also the recent papers [14] and [15].)

[3]) We use tacitly the following Lemma: Let P be a closed subsemigroup of S and $\mathfrak{P} = \{\mu \mid \mu \in \mathfrak{M}(S),\ C(\mu) \subset P\}$. Then $\mathfrak{P}$ is a closed subsemigroup of $\mathfrak{M}(S)$ which is isomorphic and homeomorphic to $\mathfrak{M}(P)$ under the mapping $\mu \to \mu'$, where $\mu'(E) = \mu(E)$ for each Borel subset $E \subset P$.

Proof. It is sufficient to prove our statement for the group G_{11}. The idempotency of μ implies that

$$(3) \qquad \int_P \int_P f(zy)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y) = \int_P f(x)\,\mathrm{d}\mu(x)$$

for any $f \in \omega(P)$.

Choose for f a function $\Phi_{11}(x) \in \omega(P)$ which is zero outside of G_{11}. (This is possible since G_{11} and $P - G_{11}$ are closed subsets of P.) To the right hand of (3) we then have $\int_{G_{11}} \Phi_{11}(x)\,\mathrm{d}\mu(x)$.

By Lemma 1,1 the expression $\int_P f(zy)\,\mathrm{d}\mu(z) = \int_P \Phi_{11}(zy)\,\mathrm{d}\mu(z)$ has the same value for every $y \in L_1$. If $y \in P - L_1$ (and $P - L_1 \neq \emptyset$), we have $y \in L_i$ for some i, $2 \leq i \leq r$, and $zy \in zL_i \subset L_i$, hence $\Phi_{11}(zy) = 0$. Therefore the left hand side of (3) can be written in the form

$$\int_P \int_P f(zy)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y) = \int_{z \in P} \int_{y \in L_1} \Phi_{11}(zy)\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y) = \mu(L_1) \int_P \Phi_{11}(zy)\,\mathrm{d}\mu(z)\,.$$

The relation (3) implies

$$\mu(L_1) \int_P \Phi_{11}(zy)\,\mathrm{d}\mu(z) = \int_{G_{11}} \Phi_{11}(x)\,\mathrm{d}\mu(x)$$

for every $y \in L_1$.

Since $zy \in G_{11}$ if and only if $z \in R_1$, the last relation can be written in the form

$$(4) \qquad \mu(L_1) \int_{z \in R_1} \Phi_{11}(zy)\,\mathrm{d}\mu(z) = \int_{x \in G_{11}} \Phi_{11}(x)\,\mathrm{d}\mu(x)\,.$$

To prove that μ is translation invariant on G_{11} it is sufficient to show that for any $\Phi_{11} \in \omega(G_{11})$ the expression $\int_{G_{11}} \Phi_{11}(xu)\,\mathrm{d}\mu(x)$ is constant for $u \in G_{11}$.

Write in (4) instead of $\Phi_{11}(x)$ the function $\Psi_{11}(x)$ defined as follows: For a fixed chosen $u \in G_{11}$ let be

$$\Psi_{11}(x) = \begin{cases} \Phi_{11}(xu) & \text{for} \quad x \in G_{11}\,, \\ 0 & \text{for} \quad x \in P - G_{11}\,. \end{cases}$$

We then have

$$\mu(L_1) \int_{z \in R_1} \Phi_{11}(zyu)\,\mathrm{d}\mu(z) = \int_{G_{11}} \Phi_{11}(xu)\,\mathrm{d}\mu(x)$$

for any $y \in L_1$. Now since $yu \in L_1(R_1L_1) = L_1$, we have by (4)

$$\mu(L_1) \int_{R_1} \Phi_{11}[z(yu)]\,\mathrm{d}\mu(z) = \int_{G_{11}} \Phi_{11}(x)\,\mathrm{d}\mu(x)\,.$$

Hence

$$\int_{G_{11}} \Phi_{11}(xu)\,\mathrm{d}\mu(x) = \int_{G_{11}} \Phi_{11}(x)\,\mathrm{d}\mu(x)\,.$$

This completes the proof of Theorem 1,1.

R e m a r k. We return to the relation (4) and note again that for any $z \in R_1$ $zy \in G_{11}$. Hence taking for $\Phi_{11}(x)$ the characteristic function of G_{11} in P we obtain $\mu(L_1)\,\mu(R_1) = \mu(G_{11})$. By an analogous argument we prove:

Corollary. *If the suppositions of Theorem* 1,1 *are satisfied, and if we write (in the sense introduced above)* $P = \bigcup_{i=1}^{s} R_i = \bigcup_{k=1}^{r} L_k$, $G_{ik} = R_i L_k$, *we have* $\mu(R_i)\,\mu(L_k) = \mu(G_{ik})$.

For later purposes it is necessary to recall some relations concerning the intrinsic structure of a simple semigroup $P = \bigcup_{\alpha \in \Lambda_1} R_\alpha = \bigcup_{\beta \in \Lambda_2} L_\beta = \bigcup_{\alpha}\bigcup_{\beta} G_{\alpha\beta}$. The following facts will be freely used. (Hereby $g_{\alpha\beta}$ denotes an element $\in G_{\alpha\beta}$ and $e_{\alpha\beta}$ is the unit element of $G_{\alpha\beta}$.)

a) $L_\beta g_{\gamma\delta} = L_\delta$, $g_{\gamma\delta} R_\alpha = R_\gamma$.

b) $\{e_{\alpha\beta}, \alpha \in \Lambda_1\}$ is the set of all idempotents $\in L_\beta$. Each of them is a right unit of L_β. The set $\{e_{\alpha\beta}, \beta \in \Lambda_2\}$ is the set of all idempotents $\in R_\alpha$. Each of them is a left unit of R_α.

c) Any two minimal left ideals L_α, L_β are isomorphic. The corresponding mapping can be realized by $x \in L_\alpha \to x e_{\alpha\beta} \in L_\beta$. The inverse mapping is $y \in L_\beta \to y e_{\beta\alpha} \in L_\alpha$.

d) $g_{\alpha\beta} L_\gamma = G_{\alpha\gamma}$, $R_\gamma g_{\alpha\beta} = G_{\gamma\beta}$.

e) $G_{\alpha\beta} g_{\gamma\delta} = G_{\alpha\delta}$, $g_{\alpha\beta} G_{\gamma\delta} = G_{\alpha\delta}$.

f) $G_{\alpha\beta} G_{\gamma\delta} = G_{\alpha\delta}$. (Note that $e_{\alpha\beta} e_{\gamma\delta} \in G_{\alpha\delta}$ but – in general – $e_{\alpha\beta} e_{\gamma\delta} = e_{\alpha\delta}$ need not hold. Of course, we have $e_{\alpha\beta} e_{\alpha\gamma} = e_{\alpha\gamma}$ and $e_{\alpha\beta} e_{\gamma\beta} = e_{\alpha\beta}$.)

g) Any two groups $G_{\alpha\beta}$ and $G_{\gamma\delta}$ are topologically isomorphic. The corresponding mapping can be realized by [4])

$$a_{\gamma\delta} \in G_{\gamma\delta} \to e_{\alpha\beta} a_{\gamma\delta} e_{\gamma\beta} \in G_{\alpha\beta} . \tag{5}$$

The inverse mapping is given by

$$a_{\alpha\beta} \in G_{\alpha\beta} \to e_{\gamma\beta} a_{\alpha\beta} e_{\gamma\delta} \in G_{\gamma\delta} . \tag{6}$$

Denote by μ_{ik} the normalized Haar measure on the group G_{ik} and extend the definition of μ_{ik} to all Borel subsets E of S by putting $\mu_{ik}(E) = \mu_{ik}(E \cap G_{ik})$. If μ is an idempotent $\in \mathfrak{M}(S)$ and $C(\mu) = P$, then by Theorem 1,1 we have necessarily $\mu = \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} \mu_{ik}$ with positive numbers t_{ik} satisfying $\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} = 1$.

[4]) To prove that (5) is a homomorphism let be $a_{\gamma\delta} \to e_{\alpha\beta} a_{\gamma\delta} e_{\gamma\beta}$, $b_{\gamma\delta} \to e_{\alpha\beta} b_{\gamma\delta} e_{\gamma\beta}$. Then (since $e_{\gamma\beta} e_{\alpha\beta} = e_{\gamma\beta}$ and $e_{\gamma\beta}$ is a left unit of $b_{\gamma\delta} \in R_\gamma$) we have $(e_{\alpha\beta} a_{\gamma\delta} e_{\gamma\beta})(e_{\alpha\beta} b_{\gamma\delta} e_{\gamma\beta}) = e_{\alpha\beta} a_{\gamma\delta}(e_{\gamma\beta} e_{\alpha\beta} b_{\gamma\delta})\, e_{\gamma\beta} = e_{\alpha\beta} a_{\gamma\delta} b_{\gamma\delta} e_{\gamma\beta}$. Hence $a_{\gamma\delta} b_{\gamma\delta} \to e_{\alpha\beta}(a_{\gamma\delta} b_{\gamma\delta})\, e_{\gamma\beta}$. To prove that it is an isomorphism suppose that $e_{\alpha\beta} a_{\gamma\delta} e_{\gamma\beta} = e_{\alpha\beta} b_{\gamma\delta} e_{\gamma\beta}$. Multiplying by $e_{\gamma\delta}$ to the right and by $e_{\gamma\beta}$ to the left we have $e_{\gamma\beta} e_{\alpha\beta} a_{\gamma\delta} \cdot e_{\gamma\beta} e_{\gamma\delta} = e_{\gamma\beta} e_{\alpha\beta} b_{\gamma\delta} e_{\gamma\beta} e_{\gamma\delta}$ and successively $e_{\gamma\beta} a_{\gamma\delta} e_{\gamma\delta} = e_{\gamma\beta} b_{\gamma\delta} e_{\gamma\delta}$, $e_{\gamma\beta} e_{\gamma\delta} a_{\gamma\delta} = e_{\gamma\beta} e_{\gamma\delta} b_{\gamma\delta}$, $e_{\gamma\delta} a_{\gamma\delta} = e_{\gamma\delta} b_{\gamma\delta}$, hence $a_{\gamma\delta} = b_{\gamma\delta}$.

To prove the converse of Theorem 1,1 we first prove the following

Lemma 1,2. *Under the suppositions and notations introduced above we have:*

a) $g_{ik}\mu_{jl} = \mu_{ik}g_{jl} = \mu_{il}$ *for any point mass* g_{ik}, g_{jl}.

b) $\mu_{ik}\mu_{jl} = \mu_{il}$.

c) *If* $\nu \in \mathfrak{M}(S)$ *and* $C(\nu) \subset P$, *then* $\mu_{ik}\nu\mu_{jl} = \mu_{il}$.

Proof. a) We first prove that $e_{ik}\mu_{il} = \mu_{il}$. In fact (since e_{ik} is a left unit for every $z \in G_{il}$) we have:

$$\int_P f(x)\,\mathrm{d}(e_{ik}\mu_{il})(x) = \int_P\int_P f(yz)\,\mathrm{d}e_{ik}(y)\,.\,\mathrm{d}\mu_{il}(z) = \int_{G_{il}} f(e_{ik}z)\,\mathrm{d}\mu_{il}(z) =$$
$$= \int_{G_{il}} f(z)\,\mathrm{d}\mu_{il}(z) = \int_P f(z)\,\mathrm{d}\mu_{il}(z)\,.$$

This implies the required formula. Analogously we prove $e_{ik}\mu_{jk} = \mu_{ik}$ and $\mu_{ik}e_{il} = \mu_{il}e_{jl} = \mu_{il}$.

Now we have

$$g_{ik}\mu_{jl} = g_{ik}(e_{jl}\mu_{jl}) = (g_{ik}e_{jl})\,\mu_{jl}\,.$$

The measure $g_{ik}e_{jl}$ is the point mass at the point $g_{ik}e_{jl} = g'_{il} \in G_{il}$. Therefore

$$g_{ik}\mu_{jl} = g'_{il}\mu_{jl} = (g'_{il}e_{il})\,\mu_{jl} = g'_{il}(e_{il}\mu_{jl}) = g'_{il}\mu_{il}\,.$$

Since μ_{il} is the Haar measure on G_{il} and $g'_{il} \in G_{il}$, we have

$$\int_P f(x)\,\mathrm{d}(g'_{il}\mu_{il})(x) = \iint_{G_{il}} f(yz)\,\mathrm{d}g'_{il}(y)\,\mathrm{d}\mu_{il}(z) = \int_{G_{il}} f(g'_{il}z)\,\mathrm{d}\mu_{il}(z) =$$
$$= \int_{G_{il}} f(z)\,\mathrm{d}\mu_{il}(z)\,,$$

hence $g'_{il}\mu_{il} = \mu_{il}$, and finally $g_{ik}\mu_{jl} = \mu_{il}$, which proves the first relation. The second statement can be proved analogously.

b) By a) we have $\mu_{ik}\mu_{jl} = (\mu_{ik}e_{ik})\,(e_{jl}\mu_{jl}) = \mu_{ik}(e_{ik}e_{jl})\,\mu_{jl}$. Denoting $e_{ik}e_{jl} = g_{il}$ (point mass at a point $\in G_{il}$) we further have $\mu_{ik}\mu_{jl} = \mu_{ik}(g_{il}\mu_{jl}) = \mu_{ik}\mu_{il}$. Again by a) and noting that μ_{il} is an idempotent $\in \mathfrak{M}(S)$ we finally have $\mu_{ik}\mu_{jl} = \mu_{ik}(e_{il}\mu_{il}) = (\mu_{ik}e_{il})\,\mu_{il} = \mu_{il}\mu_{il} = \mu_{il}$, which proves our assertion.

c) Write first $\mu_{ik}\nu\mu_{jl} = \mu_{ik}e_{ik}\nu e_{jl}\mu_{jl} = \mu_{ik}\varrho\mu_{jl}$, where ϱ is a measure with the support $C(\varrho) = C(e_{ik}\nu e_{jl}) \subset e_{ik}Pe_{jl} \subset G_{ik}PG_{jl} = G_{il}$. Since $\varrho e_{il} = e_{il}\varrho = \varrho$, we further have

$$\mu_{ik}\varrho\mu_{jl} = (\mu_{ik}e_{il})\,\varrho(e_{il}\mu_{jl}) = \mu_{il}\varrho\mu_{il}\,.$$

Now (since in what follows $z \,.\, t \in G_{il}$ and μ_{il} is invariant on G_{il}) we have for $f \in \omega(S)$

$$\int_S f(x)\,\mathrm{d}(\mu_{il}\varrho\mu_{il})(x) = \iiint_{G_{il}} f(yzt)\,\mathrm{d}\mu_{il}(y)\,\mathrm{d}\varrho(z)\,\mathrm{d}\mu_{il}(t) =$$

$$= \iint_{G_{il}} \left[\int_{G_{il}} f(y)\,\mathrm{d}\mu_{il}(y)\right] \mathrm{d}\varrho(z)\,\mathrm{d}\mu_{il}(t) = \int_{G_{il}} f(y)\,\mathrm{d}\mu_{il}(y)\,,$$

whence $\mu_{ik}\nu\mu_{jl} = \mu_{il}\varrho\mu_{il} = \mu_{il}$.

Lemma 1,2 is completely proved.

Remark. The relation between the translates of a subset of a group-component into the various G_{ik} is clarified by the following result which is a consequence of the isomorphisms (5) and (6). By Lemma 1,1 we have $e_{ik}\mu_{jl}e_{jk} = (e_{ik}\mu_{jl})\,e_{jk} = \mu_{il}e_{jk} = \mu_{ik}$. Therefore, for any $f \in \omega(P)$,

$$\int_P f(x)\,\mathrm{d}\mu_{ik}(x) = \int_{G_{ik}} f(x)\,\mathrm{d}\mu_{ik}(x) = \iiint_P f(yzt)\,\mathrm{d}e_{ik}(y)\,\mathrm{d}\mu_{jl}(z)\,.\,\mathrm{d}e_{jk}(t) =$$

$$= \int_{G_{jl}} f(e_{ik}ze_{jk})\,\mathrm{d}\mu_{jl}(z)\,.$$

If E is a Borel subset of G_{ik} we have therefore

$$\mu_{ik}(E) = \mu_{jl}\{z \in G_{jl} \mid e_{ik}ze_{jk} \in E\}\,.$$

Now $e_{ik}ze_{jk} \in E$ implies $e_{jk}(e_{ik}ze_{jk})\,e_{jl} \in e_{jk}Ee_{jl}$, hence $e_{jk}ze_{jl} \in e_{jk}Ee_{jl}$ and (since $z \in G_{jl}$) $z \in e_{jk}Ee_{jl}$. This implies the remarkable result:

$$\mu_{ik}(E) = \mu_{jl}(e_{jk}Ee_{jl})\,. \tag{7}$$

Note also that the μ_{ik}'s are completely given by means of a fixed μ_{ij}, say μ_{11}, and the idempotents $\in P$, since we have $\mu_{ik}(E) = \mu_{11}(e_{1k}Ee_{11})$ for any Borel subset $E \subset G_{ik}$ or alternatively $\mu_{ik} = e_{1k}\mu_{11}e_{1k}$.

Write now $\mu = \mu^2 \in \mathfrak{M}(S)$ with $C(\mu) = P$ in the form $\mu = \sum_{i=1}^{s}\sum_{k=1}^{r} t_{ik}\mu_{ik}$ with $\sum_{i=1}^{s}\sum_{k=1}^{r} t_{ik} = 1$, $t_{ik} > 0$. We have

$$\Big(\sum_{i=1}^{s}\sum_{k=1}^{r} t_{ik}\mu_{ik}\Big)\Big(\sum_{j=1}^{s}\sum_{l=1}^{r} t_{jl}\mu_{jl}\Big) = \sum_{i=1}^{s}\sum_{l=1}^{r} t_{il}\mu_{il}\,,$$

and with respect to Lemma 1,2b

$$\sum_i\sum_k\sum_j\sum_l t_{ik}t_{jl}\mu_{il} = \sum_i\sum_l t_{il}\mu_{il}\,,$$

$$\sum_{k=1}^{r}\sum_{j=1}^{s} t_{ik}t_{jl} = t_{il}\,. \tag{8}$$

Put $\sum_{k=1}^{r} t_{ik} = \xi_i$, $\sum_{j=1}^{s} t_{jl} = \eta_l$. Then (8) implies $t_{il} = \xi_i\eta_l$.

Let conversely $\mu_1 = \sum_{i=1}^{s} \sum_{l=1}^{r} \xi_i \eta_l \mu_{il}$ be an element $\in \mathfrak{M}(S)$, where ξ_i, η_l are positive numbers satisfying $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$. We then have

$$\mu_1^2 = \sum_{i=1}^{s} \sum_{l=1}^{r} \xi_i \eta_l \mu_{il} \cdot \sum_{j=1}^{s} \sum_{k=1}^{r} \xi_j \eta_k \mu_{jk} = \sum_i \sum_l \sum_j \sum_k \xi_i \eta_l \xi_j \eta_k \mu_{ik} =$$

$$= \left(\sum_{l=1}^{r} \eta_l\right) \left(\sum_{j=1}^{s} \xi_j\right) \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k \mu_{ik} = \mu_1 .$$

We have proved:

Theorem 1,2. *Let S be compact and P such a closed simple subsemigroup of S that contains a finite number of idempotents. Let be $P = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ its decomposition into the union of groups. Let μ_{ik} denote the normalized Haar measure on G_{ik}. Then every idempotent $\varepsilon \in \mathfrak{M}(S)$ with $C(\varepsilon) = P$ is of the form*

$$\varepsilon = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k \mu_{ik} , \tag{9}$$

where ξ_i, η_k are positive numbers satisfying $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$.

Conversely, if ξ_i, η_k are positive numbers satisfying $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$, then $\sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k \mu_{ik}$ is an idempotent $\in \mathfrak{M}(S)$ whose support is exactly P.

Remark. If we admit in (9) some ξ_i, η_k to be zero the formula (9) gives again an idempotent $\in \mathfrak{M}(S)$ but the corresponding support is a proper (simple and closed) subsemigroup of P. Of course there can exist also other simple (closed) subsemigroups of P, the group-components of which are isomorphic with proper subgroups of G_{ik}.

We now proceed to the determination of primitive idempotents and the kernel (= minimal two-sided ideal) of $\mathfrak{M}(S)$. If S is finite the problem has been treated in detail in [7], so that we can be concise by only quoting the results that can be proved in the same manner as in [7].

The kernel of S will be denoted by N and the kernel of $\mathfrak{M}(S)$ by $\mathfrak{N}$.

An idempotent π of a semigroup T is said to be primitive if there does not exist an idempotent $\mu \in T$, $\mu \neq \pi$ such that $\pi\mu = \mu\pi = \mu$ holds. Those and only those idempotents of a compact semigroup T which are contained in the kernel K of T are primitive idempotents of T. (See [7], Lemma 3,1.)

The following two lemmas can be proved analogously as Theorems 3,1 and 3,2 in the paper [7].

Lemma 1,3. *Let S be a compact semigroup with the kernel N. Suppose that N contains a finite number of idempotents. Let P be a closed subsemigroup of N containing at least one maximal group of N.*[5]) *Then every idempotent the support of which is equal to P is a primitive idempotent $\in \mathfrak{M}(S)$.*

Lemma 1,4. *Let S be compact with the kernel N containing a finite number of idempotents. If π is a primitive idempotent $\in \mathfrak{M}(S)$, then $C(\pi) \subset N$.*

Lemma 1,5. *Let the suppositions of Lemma 1,4 be satisfied. If π is a primitive idempotent $\in \mathfrak{M}(S)$, then $C(\pi)$ is a union of some maximal groups contained in N.*

Proof. Let $N = \bigcup_{i=1}^{s} R_i = \bigcup_{k=1}^{r} L_k$ be the decomposition of N into its minimal right and left ideals respectively. Denote $C(\pi) = P'$ and let $P' = \bigcup_{i=1}^{\sigma} R'_i = \bigcup_{k=1}^{\varrho} L'_k$ be the decomposition of P' into the union of minimal right and left ideals of P' respectively. By Lemma 1,1 of the paper [6] to every L'_i there is a L_j, $1 \leqq j \leqq r$ such that $L'_i = P' \cap L_j$. Analogously for minimal right ideals R'_i. Without loss of generality let be $L'_i = P' \cap L_i$ $(i = 1, 2, \ldots, \varrho)$ and $R'_i = R_i \cap P'$ $(i = 1, 2, \ldots, \sigma)$. Consider the semigroup $P = (\bigcup_{i=1}^{\sigma} R_i) \cap (\bigcup_{k=1}^{\varrho} L_k)$. Denoting $G_{ik} = R_i L_k$ and $G'_{ik} = R'_i L'_k$ we have $P' = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$, $P = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik}$, and π can be written in the form

$$\pi = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k \mu'_{ik} \quad \left(0 < \xi_i \leqq 1,\ 0 < \eta_k \leqq 1,\ \sum_{i=1}^{\sigma} \xi_i = \sum_{k=1}^{\varrho} \eta_k = 1\right),$$

where μ'_{ik} is the normalized Haar measure on the group G'_{ik}.

Suppose now for an indirect proof that the group-components of P' are not maximal groups of N, i.e. $G'_{ik} \subset G_{ik}$ and $G'_{ik} \neq G_{ik}$. To prove that π is not a primitive idempotent $\in \mathfrak{M}(S)$ it is sufficient to find an idempotent ν such that $\pi \neq \nu$ and $\pi\nu = \nu\pi = \nu$. Construct the idempotent $\nu = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k \mu_{ik}$, where μ_{ik} is the normalized Haar measure on G_{ik}. Then $\nu \neq \pi$ since $C(\nu) \neq C(\pi)$.

We first prove that $\mu_{ik}\mu'_{jl} = \mu_{il}$. We have

$$\mu_{ik}\mu'_{jl} = (\mu_{ik}e_{ik})\,\mu'_{jl} = \mu_{ik}(e_{ik}\mu'_{jl}) = \mu_{ik}\mu'_{il} = \mu_{ik}(e_{il}\mu'_{il}) = (\mu_{ik}e_{il})\,\mu'_{il} = \mu_{il} \cdot \mu'_{il}\,.$$

Further, for $f \in \omega(P)$,

$$\int_P f(x)\, \mathrm{d}(\mu_{il}\mu'_{il})(x) = \int_{y \in G_{il}} \int_{z \in G'_{il}} f(yz)\, \mathrm{d}\mu_{il}(y)\, \mathrm{d}\mu'_{il}(z) =$$

$$= \int_{z \in G'_{il}} \left[\int_{y \in G_{il}} f(yz)\, \mathrm{d}\mu_{il}(y)\right] \mathrm{d}\mu'_{il}(z)\,.$$

[5]) P is then automatically a closed simple subsemigroup all group-components of which are maximal groups of N. (See [6].)

Since $z \in G_{il}$ and μ_{il} is invariant on G_{il}, the bracket is equal to $\int_{y \in G_{il}} f(y) \, \mathrm{d}\mu_{il}(y)$, so that

$$\int_P f(x) \, \mathrm{d}(\mu_{il}\mu'_{il})(x) = \left[\int_{z \in G'_{il}} \mathrm{d}\mu'_{il}(z)\right] \cdot \left[\int_{y \in G_{il}} f(y) \, \mathrm{d}\mu_{il}(y)\right] = \int_{y \in P} f(y) \, \mathrm{d}\mu_{il}(y) \,,$$

whence $\mu_{il} \cdot \mu'_{il} = \mu_{il}$ and finally $\mu_{ik}\mu'_{jl} = \mu_{il}$. Analogously we prove $\mu'_{ik}\mu_{jl} = \mu_{il}$.

Now

$$\nu\pi = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k\mu_{ik} \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_j\eta_l\mu'_{jl} = \left(\sum_{j=1}^{\sigma} \xi_j\right)\left(\sum_{k=1}^{\varrho} \eta_k\right) \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i\eta_l\mu_{il} = \nu \,.$$

Analogously $\pi\nu = \nu$. This proves Lemma 1,5.

Summarily we have

Theorem 1,3. *Let S be a compact semigroup the kernel N of which contains a finite number of idempotents. An idempotent $\pi \in \mathfrak{M}(S)$ is primitive if and only if $C(\pi)$ is a union of some maximal subgroups of N.*

The next two theorems clarify the structure of $\mathfrak{N}$.

Theorem 1,4. *Let S be a compact semigroup the kernel of which contains a finite number of idempotents. Then the kernel $\mathfrak{N}$ of $\mathfrak{M}(S)$ is identical with the set of primitive idempotents $\in \mathfrak{M}(S)$.*

Proof. Let be $\pi = \pi^2 \in \mathfrak{N}$. Since it is known that the maximal group $\mathfrak{G}(\pi) \subset \mathfrak{N}$ containing π as its unit element is given by the formula $\mathfrak{G}(\pi) = \pi\mathfrak{N}\pi$ it is sufficient to show that for any $\nu \in \mathfrak{N}$ we have $\pi\nu\pi = \pi$.

Note first: Since $\nu \in \mathfrak{N}$ and $\mathfrak{N}$ is a union of groups, there is a $\pi' \in \mathfrak{N}$ such that $\nu \in \mathfrak{G}(\pi')$, hence $\nu\pi' = \nu$. This implies $C(\nu) = C(\nu) \, C(\pi') \subset C(\nu) \, N \subset N$.

Write $N = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ and $\pi = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i\eta_k\mu_{ik}$ with non-negative ξ_i, η_k satisfying the usual conditions. Then

$$\pi\nu\pi = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i\eta_k\mu_{ik} \cdot \nu \cdot \sum_{j=1}^{s} \sum_{l=1}^{r} \xi_j\eta_l\mu_{jl} \,.$$

Now by Lemma 1,2 c) $\mu_{ik}\nu\mu_{jl} = \mu_{il}$. Hence

$$\pi\nu\pi = \left(\sum_{k=1}^{s} \eta_k\right)\left(\sum_{j=1}^{r} \xi_j\right) \sum_{i=1}^{s} \sum_{l=1}^{r} \xi_i\eta_l\mu_{il} = \pi \,,$$

which proves our theorem.

By means of Theorem 1,4 and an analogous argument as used in [7] (Theorem 3,6) we can now prove:

Theorem 1,5. *Let S be a compact semigroup containing s minimal right ideals and r minimal left ideals respectively. Let $\mathfrak{T}$ be the set of all $(s + r)$-tuples of* *non-negative real numbers* $(\xi_1, \dots, \xi_s, \ \eta_1, \dots, \eta_r)$ *satisfying* $\xi_1 + \dots + \xi_s = \eta_1 + \dots + \eta_r = 1$. *Define in* $\mathfrak{T}$ *a multiplication* $\circ$ *by*

$$(\xi'_1, \dots, \xi'_s, \ \eta'_1, \dots, \eta'_r) \circ (\xi''_1, \dots, \xi''_s, \ \eta''_1, \dots, \eta''_r) = (\xi'_1, \dots, \xi'_s, \ \eta''_1, \dots, \eta''_r) .$$

Then $\mathfrak{T}$ *is isomorphic with the kernel* $\mathfrak{N}$ *of the semigroup* $\mathfrak{M}(S)$.

2. THE MAXIMAL GROUPS OF $\mathfrak{M}(S)$

In this section we shall identify the maximal groups $\in \mathfrak{M}(S)$. To this end it is useful to make first some remarks concerning the location of simple subsemigroups of S.

The principal ideal generated by x (i.e. the set $x \cup Sx \cup xS \cup SxS$) will be denoted by $J(x)$. By an F_x-class we shall denote the set $F_x = \{y \mid y \in S, \ J(y) = J(x)\}$. Clearly S can be written as a union of disjoint F-classes: $S = \bigcup_x F_x$.

If H is a simple subsemigroup of S it is easy to see that all elements $\in H$ generate the same principal ideal which we shall denote by $J(H)$. Hence a simple subsemigroup cannot meet two different F-classes.

Let now be H a simple subsemigroup of S and F_H the F-class containing H, $J(H)$ the two-sided ideal as above. It is known that the set $K_H = J(H) - F_H$ is a two-sided ideal of $J(H)$. The difference semigroup $J(H)/K_H$ is a simple semigroup with zero. The elements of this semigroup are the elements $\in J(H) - K_H = F_H$ together with an adjoint zero element O_H and the product in $F_0 = F_H \cup \{O_H\}$ is defined in an obvious manner.

Suppose now that S is compact and H is closed. Then, since H contains an idempotent which is contained in F_H, we have $F_0^2 \neq O_H$, hence $F_0^2 = F_0$. Moreover (if S is compact) F_0 is known to be completely simple with zero. (See R. J. KOCH-A. D. WALLACE [5].)

We can now use Lemma 2,2 of the paper [6] by which under our hypotheses there exists a unique greatest simple subsemigroup H_1 of F_0 contained in F_H and having exactly the same idempotents as H.[6])

Returning to the semigroup S we have:

Lemma 2,1. *Let S be a compact semigroup and H a closed simple subsemigroup of S. Then there exists a unique greatest subsemigroup $H_1 \supset H$ having the same idempotents as H.*

[6]) The precise formulation of this Lemma is as follows: If S is a completely simple semigroup with zero 0 satisfying $S^2 \neq 0$ and T a simple subsemigroup of S containing an idempotent but not containing the zero element, then there exists a unique greatest simple subsemigroup $T_1 \supset T$ of S having (exactly) the same idempotents as T. The semigroup T_1 is completely simple and it can be written in the form $T_1 = [\{\bigcup_\alpha R_\alpha\} \cap \{\bigcup_\beta L_\beta\}] - \{0\}$ with suitably chosen minimal right and left ideals R_α, L_β of S respectively.

In the sequel we shall consequently use the following notations. ε will be an idempotent $\in \mathfrak{M}(S)$ with $C(\varepsilon) = H$. Further $H = \bigcup_{\alpha \in \Lambda_1} R'_\alpha = \bigcup_{\beta \in \Lambda_2} L'_\beta$ is the decomposition of H into the union of its minimal right and left ideals respectively and $H = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G'_{\alpha\beta}$ $[G'_{\alpha\beta} = R'_\alpha L'_\beta]$ is the group decomposition of H. H_1 will denote the largest simple subsemigroup of S having the same idempotents as H and $H_1 = \bigcup_{\alpha \in \Lambda_1} R_\alpha = \bigcup_{\beta \in \Lambda_2} L_\beta = \bigcup_\alpha \bigcup_\beta G_{\alpha\beta} [G_{\alpha\beta} = R_\alpha L_\beta]$ the corresponding decompositions of H_1. Without loss of generality we may suppose that $R'_\alpha = R_\alpha \cap H$ $(\alpha \in \Lambda_1)$, $L'_\beta = L_\beta \cap H (\beta \in \Lambda_2)$, so that $G'_{\alpha\beta} \subset G_{\alpha\beta}$. (See [6], Lemma 1,1.)

In [6] it has been proved also that H_1 admits a decomposition mod (H, H) into a union of pairwise disjoint classes

$$H_1 = H \cup HaH \cup HbH \cup \dots \tag{10}$$

with suitably chosen $a, b, \dots \in H_1$. In particular $HaH = H$ if and only if $a \in H$. Moreover $HaH \cap G_{\alpha\beta} = G'_{\alpha\beta} a G'_{\alpha\beta}$ for any $a \in H_1$. (See [6], Theorem 3,2.) Hence if $T_{\alpha\beta} = HaH \cap G_{\alpha\beta}$, then $HaH = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} T_{\alpha\beta} = \bigcup_\alpha \bigcup_\beta G'_{\alpha\beta} a G'_{\alpha\beta}$.

The following simple lemma will be used in computations.

Lemma 2,2. *If a is any element $\in H_1$, then $G'_{\alpha\beta} a G_{\gamma\delta} = G'_{\alpha\delta} a G'_{\alpha\delta}$.*

Proof. Suppose that $a \in G_{\sigma\varrho} \subset H_1$. Then $e_{\sigma\varrho} a e_{\sigma\varrho} = a$. Hence $G'_{\alpha\beta} a G'_{\gamma\delta} = (G'_{\alpha\beta} e_{\sigma\varrho})\, a (e_{\sigma\varrho} G'_{\gamma\delta}) = G'_{\alpha\varrho} a G'_{\sigma\delta}$. Since this is clearly independent of β and γ we may take $\beta = \delta$ and $\gamma = \alpha$, so that $G'_{\alpha\beta} a G'_{\gamma\delta} = G'_{\alpha\delta} a G'_{\alpha\delta}$.

If P is a compact semigroup and $a \in P$, then a is said to belong to the idempotent e if e is the (unique) idempotent contained in the closure of the sequence $\{a, a^2, a^3, \dots\}$. An element a is called m-regular if it is contained in some subgroup of P.

In the next two theorems we do not suppose that $C(\varepsilon)$ contains only a finite number of idempotents. The first of them can be proved by the same argument as Theorem 5,1 in the paper [7]. We omit the proof of it.

Theorem 2,1. *Let S be a compact semigroup and ε an idempotent $\in \mathfrak{M}(S)$ with $C(\varepsilon) = H$. Let H_1 denote the largest subsemigroup of S having the same idempotents as H. If v is an m-regular element belonging to ε, then $C(v) = HaH$ with a suitably chosen element $a \in H_1$.*

Theorem 2,2. *Let the suppositions of Theorem 2,1 be satisfied. Denote $H = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G'_{\alpha\beta}$, $H_1 = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$. Then $T_{\alpha\beta} = HaH \cap G_{\alpha\beta}$ is exactly one two-sided class of the decomposition of the group $G_{\alpha\beta}$ modulo the group $G'_{\alpha\beta}$ (i.e. $T_{\alpha\beta} = G'_{\alpha\beta} a_{\alpha\beta} = a_{\alpha\beta} G'_{\alpha\beta}$ with a suitably chosen $a_{\alpha\beta} \in G_{\alpha\beta}$).*

Proof. If v is m-regular, then there exists an m-regular $v^{(0)} \in \mathfrak{M}(S)$ belonging to ε such that $vv^{(0)} = v^{(0)}v = \varepsilon$. Denote $C(v^{(0)}) = HbH$ and $T^{(0)}_{\gamma\delta} = G'_{\gamma\delta} b G'_{\gamma\delta} \subset G_{\gamma\delta}$. Since

$C(v) = \bigcup_\alpha \bigcup_\beta T_{\alpha\beta} = \bigcup_\alpha \bigcup_\beta G'_{\alpha\beta} a G'_{\alpha\beta}$, $C(v^{(0)}) = \bigcup_{\gamma \in \Lambda_1} \bigcup_{\delta \in \Lambda_2} T_{\gamma\delta} = \bigcup_\gamma \bigcup_\delta G'_{\gamma\delta} b G'_{\gamma\delta}$, the relation $C(v)\, C(v^{(0)}) = H$ implies

$$\bigcup_\alpha \bigcup_\beta \bigcup_\gamma \bigcup_\delta G'_{\alpha\beta} a G'_{\alpha\beta} G'_{\gamma\delta} b G'_{\gamma\delta} = \bigcup_\alpha \bigcup_\delta G'_{\alpha\delta} .$$

By Lemma 2,2 we have

$$G'_{\alpha\beta} a G'_{\alpha\beta} G'_{\gamma\delta} b G'_{\gamma\delta} = (G'_{\alpha\beta} a G'_{\alpha\delta})\, b G'_{\gamma\delta} = (G'_{\alpha\delta} a G'_{\alpha\delta})\, b G'_{\gamma\delta} =$$
$$= (G'_{\alpha\delta} a)\,(G'_{\alpha\delta} b G'_{\gamma\delta}) = G'_{\alpha\delta} a G'_{\alpha\delta} b G'_{\alpha\delta} .$$

Therefore

$$\bigcup_\alpha \bigcup_\delta G'_{\alpha\delta} a G'_{\alpha\delta} b G'_{\alpha\delta} = \bigcup_\alpha \bigcup_\delta G'_{\alpha\delta} .$$

Now since

$$G'_{\alpha\delta}(a G'_{\alpha\delta} b)\, G'_{\alpha\delta} \subset G'_{\alpha\delta} H_1 G'_{\alpha\delta} \subset R_\alpha H_1 L_\delta = R_\alpha L_\delta = G_{\alpha\delta} ,$$

we have $G'_{\alpha\delta} a G'_{\alpha\delta} b G'_{\alpha\delta} = G'_{\alpha\delta}$ and $(G'_{\alpha\delta} a G'_{\alpha\delta})\,(G'_{\alpha\delta} b G'_{\alpha\delta}) = G'_{\alpha\delta}$, i.e. $T_{\alpha\delta} \,.\, T^{(0)}_{\alpha\delta} = G'_{\alpha\delta}$. Analogously $v^{(0)} v = \varepsilon$ implies $T^{(0)}_{\alpha\delta} T_{\alpha\delta} = G'_{\alpha\delta}$.

The expresion $T_{\alpha\delta} = G'_{\alpha\delta} a G'_{\alpha\delta}$ shows that we can write

$$T_{\alpha\delta} = a_1 G'_{\alpha\delta} \cup a_2 G'_{\alpha\delta} \cup \ldots \quad (a_1, a_2, \ldots \in G_{\alpha\delta})$$

and analogously

$$T^{(0)}_{\alpha\delta} = G'_{\alpha\delta} b_1 \cup G'_{\alpha\delta} b_2 \cup \ldots \quad (b_1, b_2, \ldots \in G_{\alpha\delta}) .$$

We prove that $T_{\alpha\delta}$ contains a unique left class of the decomposition of $G_{\alpha\delta}$ modulo $G'_{\alpha\delta}$. Suppose that $a_1 G'_{\alpha\delta} \neq a_2 G'_{\alpha\delta}$. The relation $T^{(0)}_{\alpha\delta} T_{\alpha\delta} = G'_{\alpha\delta}$ implies $G'_{\alpha\delta} b_1 a_1 G'_{\alpha\delta} \subset$ $\subset G'_{\alpha\delta}$, $G'_{\alpha\delta} b_1 a_2 G'_{\alpha\delta} \subset G'_{\alpha\delta}$. But then $g_{\alpha\delta} = b_1 a_1 \in G'_{\alpha\delta}$, $b_1 = g_{\alpha\delta} a_1^{-1}$, i.e. $G'_{\alpha\delta} b_1 =$ $= G'_{\alpha\delta} g_{\alpha\delta} a_1^{-1} = G'_{\alpha\delta} a_1^{-1}$ and $G'_{\alpha\delta} b_1 a_2 G'_{\alpha\delta} = G'_{\alpha\delta} a_1^{-1} a_2 G'_{\alpha\delta} \subset G'_{\alpha\delta}$ implies $a_1^{-1} a_2 =$ $= g^{(0)}_{\alpha\delta} \in G'_{\alpha\delta}$, $a_2 = a_1 g^{(0)}_{\alpha\delta}$ and $a_2 G'_{\alpha\delta} = a_1 g^{(0)}_{\alpha\delta} G'_{\alpha\delta} = a_1 G'_{\alpha\delta}$, which is a contradiction.

Hence, $T_{\alpha\beta} = a_1 G'_{\alpha\beta}$, and analogously $T_{\alpha\beta} = G'_{\alpha\beta} \bar{a}_1$, with $a_1, \bar{a}_1 \in G_{\alpha\beta}$. Now $a_1 G'_{\alpha\beta} =$ $= G'_{\alpha\beta} \bar{a}_1$ implies $a_1 = \bar{g}_{\alpha\beta} \bar{a}_1$ with $\bar{g}_{\alpha\beta} \in G'_{\alpha\beta}$. Therefore $a_1 G'_{\alpha\beta} = G'_{\alpha\beta} (\bar{g}_{\alpha\beta})^{-1} a_1 = G'_{\alpha\beta} a_1$ (and also $G'_{\alpha\beta} a_1 G'_{\alpha\beta} = a_1 G'_{\alpha\beta} = G'_{\alpha\beta} a_1$). This proves our Theorem.

Remark 1. It is necessary to remark that though for a m-regular v $T_{\alpha\beta} = G_{\alpha\beta} \cap$ $\cap\, C(v)$ is a two-sided class of $G_{\alpha\beta}$ mod $G'_{\alpha\beta}$ it is in general not true that $C(v) = HaH$ is a two-sided clas of the decomposition (10), i.e. $HaH = Ha = aH$. (See [7], Example 5,1.)

Remark 2. We prove the following assertion: *If for one couple, say* (α, β), $T_{\alpha\beta} = G_{\alpha\beta} \cap HaH = G'_{\alpha\beta} a G'_{\alpha\beta} = G'_{\alpha\beta}(e_{\alpha\beta} a e_{\alpha\beta})\, G'_{\alpha\beta} = G'_{\alpha\beta} a_{\alpha\beta} G'_{\alpha\beta}$ *is a two-sided class of the decomposition* $G_{\alpha\beta}(\mathrm{mod}\, G'_{\alpha\beta})$ *the same holds for every other couple* (σ, ϱ), $\sigma \in \Lambda_1$, $\varrho \in \Lambda_2$.

By supposition $a_{\alpha\beta}G'_{\alpha\beta} = G'_{\alpha\beta}a_{\alpha\beta}$. This implies $G'_{\sigma\varrho}a_{\alpha\beta}G'_{\alpha\beta}e_{\sigma\varrho} = G'_{\sigma\varrho}G'_{\alpha\beta}a_{\alpha\beta}e_{\sigma\varrho}$. The left hand side can be written in the following form:

$$G'_{\sigma\varrho}a_{\alpha\beta}G'_{\alpha\varrho} = G'_{\sigma\varrho}e_{\alpha\beta}ae_{\alpha\beta}G'_{\alpha\varrho} = G'_{\sigma\beta}aG'_{\alpha\varrho} = G'_{\sigma\varrho}aG'_{\sigma\varrho}\,.$$

For the right hand side we have

$$G'_{\sigma\beta}a_{\alpha\beta}e_{\sigma\varrho} = G'_{\sigma\beta}(e_{\sigma\beta}a_{\alpha\beta}e_{\sigma\varrho}) = G'_{\sigma\beta}\xi_{\sigma\varrho}\,,$$

where $\xi_{\sigma\varrho} = e_{\sigma\beta}a_{\alpha\beta}e_{\sigma\varrho} \in G_{\sigma\beta}G_{\alpha\beta}G_{\sigma\varrho} \in G_{\sigma\varrho}$. Therefore $G'_{\sigma\beta}\xi_{\sigma\varrho} = G'_{\sigma\beta}(e_{\sigma\varrho}\xi_{\sigma\varrho}) = G'_{\sigma\varrho}\xi_{\sigma\varrho}$. Finally we have $G'_{\sigma\varrho}aG'_{\sigma\varrho} = G'_{\sigma\varrho}\xi_{\sigma\varrho}$.

The relation $a_{\alpha\beta}G'_{\alpha\beta} = G'_{\alpha\beta}a_{\alpha\beta}$ implies also $e_{\sigma\varrho}a_{\alpha\beta}G'_{\alpha\beta}G'_{\sigma\varrho} = e_{\sigma\varrho}G'_{\alpha\beta}a_{\alpha\beta}G'_{\sigma\varrho}$, which can be transformed by an analogous argument into the relation $G'_{\sigma\varrho}aG'_{\sigma\varrho} = \eta_{\sigma\varrho}G'_{\sigma\varrho}$, where $\eta_{\sigma\varrho} = e_{\sigma\varrho}a_{\alpha\beta}e_{\alpha\varrho} \in G_{\sigma\varrho}$.

Now $G'_{\sigma\varrho}aG'_{\sigma\varrho} = \eta_{\sigma\varrho}G'_{\sigma\varrho} = G'_{\sigma\varrho}\xi_{\sigma\varrho}$ implies that $\eta_{\sigma\varrho} = \bar{g}_{\sigma\varrho}\xi_{\sigma\varrho}$ with $\bar{g}_{\sigma\varrho} \in G'_{\sigma\varrho}$. Hence $\eta_{\sigma\varrho}G'_{\sigma\varrho} = G'_{\sigma\varrho}(\bar{g}_{\sigma\varrho}^{-1}\eta_{\sigma\varrho}) = G'_{\sigma\varrho}\eta_{\sigma\varrho}$. This says that $T_{\sigma\varrho}$ is a two-sided class in the decomposition of $G_{\sigma\varrho}$ modulo $G'_{\sigma\varrho}$ which completes the proof of our assertion.

(Of course, since $e_{\sigma\varrho}ae_{\sigma\varrho} \in \eta_{\sigma\varrho}G'_{\sigma\varrho}$, we can write $e_{\sigma\varrho}ae_{\sigma\varrho} = \eta_{\sigma\varrho}\hat{g}_{\sigma\varrho}$ and $\eta_{\sigma\varrho}G'_{\sigma\varrho} = e_{\sigma\varrho}ae_{\sigma\varrho}\hat{g}_{\sigma\varrho}^{-1}G'_{\sigma\varrho} = e_{\sigma\varrho}aG'_{\sigma\varrho}$. Hence $T_{\sigma\varrho} = e_{\sigma\varrho}aG'_{\sigma\varrho}$, and analogously $T_{\sigma\varrho} = G'_{\sigma\varrho}ae_{\sigma\varrho}$.)

For the rest of this section we shall again make the restriction as to the finiteness of the number of idempotents in H (and a fortiori in H_1). We shall therefore write $H = \bigcup_{i=1}^{s}\bigcup_{k=1}^{r} G'_{ik}$, $H_1 = \bigcup_{i=1}^{s}\bigcup_{k=1}^{r} G_{ik}$.

Lemma 2,3. *If a is a point mass at any element $\in H_1$ and μ'_{ik} the normalized Haar measure on G'_{ik}, then $\mu'_{ik}a\mu'_{jl} = \mu'_{il}a\mu'_{il}$.*

Proof. Suppose that $a \in G_{uv} \subset H_1$, then $\mu'_{ik}a\mu'_{jl} = (\mu'_{ik}e_{uv})\,a(e_{uv}\mu'_{jl}) = \mu'_{iv}a\mu'_{ul}$. Since the last element is clearly independent of k and j we can take $k = l$ and $j = i$ so that $\mu'_{ik}a\mu'_{jl} = \mu'_{il}a\mu'_{il}$.

We shall now identify the m-regular measures ν with $C(\nu) = HaH$ that belong to the idempotent $\varepsilon = \sum_{i=1}^{s}\sum_{k=1}^{r}\xi_i\eta_k\mu'_{ik}$. It will turn out that there exists exactly one such measure.

Since ν is m-regular, we have $\nu = \varepsilon\nu\varepsilon$ and $C(\nu) = HaH$. This implies

$$\nu = \sum_i\sum_k\sum_j\sum_l \xi_i\eta_k\xi_j\eta_l\mu'_{ik}\nu\mu'_{jl}\,. \tag{11}$$

Our next (and main) goal is to show that $\mu'_{ik}\nu\mu'_{jl} = \mu'_{il}a\mu'_{il}$.

Denote $\varrho = e_{ik}\nu e_{jl}$, then $e_{il}\varrho = \varrho e_{il} = \varrho$ and

$$C(\varrho) = e_{ik}HaHe_{jl} = e_{ik}\{[\bigcup_{\alpha=1}^{s}\bigcup_{\beta=1}^{r} G'_{\alpha\beta}]\,a[\bigcup_{\gamma=1}^{s}\bigcup_{\delta=1}^{r} G'_{\gamma\delta}]\}\,e_{jl} =$$
$$= [\bigcup_{\beta} G'_{i\beta}]\,a[\bigcup_{\gamma} G'_{\gamma l}] = G'_{il}aG'_{il}\,.$$

(The last relation follows by Lemma 2,2.) Further

$$\mu'_{ik}\nu\mu'_{jl} = (\mu'_{ik}e_{ik})\,\nu(e_{jl}\mu'_{jl}) = \mu'_{ik}\varrho\mu'_{jl} = \mu'_{ik}(e_{il}\varrho e_{il})\,\mu'_{jl} =$$
$$= (\mu'_{ik}e_{il})\,\varrho(e_{il}\mu'_{jl}) = \mu'_{il}\varrho\mu'_{il}\,,$$

and

$$C(\mu'_{il}\varrho\mu'_{il}) = G'_{il}\,C(\varrho)\,G'_{il} = G'_{il}aG'_{il}\,.$$

We have

$$T_{il} = G'_{il}aG'_{il} = G'_{il}(e_{il}ae_{il})\,G'_{il} = G'_{il}a_{il}G'_{il}$$

with $a_{il} = e_{il}ae_{il} \in G_{il}$. Now since ν is m-regular, we also have (by Theorem 2,2) $T_{il} = G'_{il}a_{il} = a_{il}G'_{il}$ (and this is very essential in the following).

Put $\sigma = \mu'_{il}\varrho$. Then $\sigma a_{il}^{-1} = \mu'_{il}\varrho a_{il}^{-1}$ is a measure with the support

$$C(\sigma a_{il}^{-1}) = G'_{il}\,C(\varrho)\,a_{il}^{-1} = G'_{il}(G'_{il}a_{il}G'_{il})\,a_{il}^{-1} = G'_{il}(a_{il}G'_{il})\,a_{il}^{-1} =$$
$$= G'_{il}(G'_{il}a_{il})\,a_{il}^{-1} = G'_{il}e_{il} = G'_{il}\,.$$

Now it is known (and easy to prove) that every measure with the support G'_{il} is annihilated by μ'_{il}, hence, in particular, $\mu'_{il}(\sigma a_{il}^{-1}) = \mu'_{il}$. This implies successively $\mu'_{il}(\mu'_{il}\varrho a_{il}^{-1}) = \mu'_{il}$, $\mu'_{il}\varrho_{il}^{2} = \mu'_{il}a_{il}$, $\mu'_{il}\varrho = \mu'_{il}a_{il}$, and $\mu'_{il}\varrho\mu'_{il} = \mu'_{il}a_{il}\mu'_{il}$. Therefore we finally have

$$\mu'_{ik}\nu\mu'_{jl} = \mu'_{il}\varrho\mu'_{il} = \mu'_{il}a_{il}\mu'_{il} = \mu'_{il}a\mu'_{il}\,.$$

Returning to (11) we get

$$\nu = \Big(\sum_{k=1}^{r}\eta_k\Big)\Big(\sum_{j=1}^{s}\xi_j\Big)\sum_{i=1}^{s}\sum_{l=1}^{r}\xi_i\eta_l\mu'_{il}a\mu'_{il} = \sum_{i=1}^{s}\sum_{l=1}^{r}\xi_i\eta_l\mu'_{il}a\mu'_{il}\,.$$

We have proved:

Theorem 2,3. *Let S be a compact semigroup and $\varepsilon = \sum_{i=1}^{s}\sum_{k=1}^{r}\xi_i\eta_k\mu'_{ik}$ an idempotent $\in \mathfrak{M}(S)$ with $C(\varepsilon) = H$ containing a finite number of idempotents. If ν is a m-regular element $\in \mathfrak{M}(S)$ belonging to ε with $C(\nu) = HaH$, then $\nu = \sum_{i=1}^{s}\sum_{k=1}^{r}\xi_i\eta_k\tau'_{ik}$, where $\tau'_{ik} = \mu'_{ik}a\mu'_{ik}$.*

Note that ν is uniquely determined by $C(\nu)$ and ε.

Conversely:

Theorem 2,4. *Let $\varepsilon = \sum_{i=1}^{s}\sum_{k=1}^{r}\xi_i\eta_k\mu'_{ik}$ be an idempotent $\in \mathfrak{M}(S)$ with $C(\varepsilon) = H = \bigcup_{i=1}^{s}\bigcup_{k=1}^{r}G'_{ik}$ containing a finite number of idempotents $\in S$. Let $H_1 = \bigcup_{i=1}^{s}\bigcup_{k=1}^{r}G_{ik}$ be as in Theorem 2,1. Let HaH be a class of the decomposition*

$$H_1 = H \cup HaH \cup HbH \cup \ldots$$

such that $HaH \cap G_{ik}$ is exactly one two-sided class of G_{ik} modulo G'_{ik}. Denote $\tau'_{ik} = \mu'_{ik}a\mu'_{ik}$. Then $\nu = \sum_{i=1}^{s}\sum_{k=1}^{r}\xi_i\eta_k\tau'_{ik}$ is a m-regular element $\in \mathfrak{M}(S)$ belonging to ε with $C(\nu) = HaH$.

Proof. It is sufficient to prove that 1) $\nu\varepsilon = \varepsilon\nu = \nu$, 2) there is a ν_0 with $\nu\nu_0 = \nu_0\nu = \varepsilon$, 3) $\nu_0\varepsilon = \varepsilon\nu_0 = \nu_0$. For then ν is contained in the cyclic group generated by ν and ν_0.

1) Since

$$\mu'_{ik}\tau'_{jl} = \mu'_{ik}(\mu'_{jl}a\mu'_{jl}) = \mu'_{il}a\mu'_{jl} = \mu'_{il}a\mu'_{il},$$

we have

$$\varepsilon\nu = \sum_i\sum_k \xi_i\eta_k\mu'_{ik}\sum_j\sum_l \xi_j\eta_l\tau'_{jl} = \left(\sum_j \xi_j\right)\left(\sum_k \eta_k\right)\sum_i\sum_l \xi_i\eta_l(\mu'_{il}a\mu'_{il}) = \nu,$$

and analogously $\nu\varepsilon = \nu$.

2) The element $a \in H_1$ is contained in a group, say $G_{\alpha\beta} \subset H_1$. Denote by $\bar{a}$ the element $\in G_{\alpha\beta}$ such that $a\bar{a} = \bar{a}a = e_{\alpha\beta}$ and construct the measure $\nu_0 = \sum_{j=1}^{s}\sum_{l=1}^{r}\xi_j\eta_l\bar{\tau}_{jl}$ with $\bar{\tau}_{jl} = \mu'_{jl}\bar{a}\mu'_{jl}$. We then have

$$\nu\nu_0 = \sum_{i=1}^{s}\sum_{k=1}^{r}\sum_{j=1}^{s}\sum_{l=1}^{r}\xi_i\eta_k\xi_j\eta_l\mu'_{ik}a\mu'_{ik}\mu'_{jl}\bar{a}\mu'_{jl}. \tag{12}$$

Now

$$\mu'_{ik}a\mu'_{ik}\mu'_{jl}\bar{a}\mu'_{jl} = \mu'_{ik}a\mu'_{il}\bar{a}\mu'_{jl} = \mu'_{ik}(ae_{\alpha\beta}\mu'_{il}e_{\alpha\beta}\bar{a})\,\mu'_{jl} = \mu'_{ik}(a\mu'_{\alpha\beta}\bar{a})\,\mu'_{jl}.$$

The measure $\varrho = a\mu'_{\alpha\beta}\bar{a}$ is an idempotent since $\varrho^2 = a\mu'_{\alpha\beta}\bar{a}\,a\mu_{\alpha\beta}\bar{a} = a(\mu'_{\alpha\beta}e_{\alpha\beta}\mu'_{\alpha\beta})\,\bar{a} = a\mu'_{\alpha\beta}\bar{a}$. Further $C(\varrho) = aC(\mu'_{\alpha\beta})\,\bar{a} = aG'_{\alpha\beta}\bar{a}$. Now by supposition (and this is essential) $aG'_{\alpha\beta} = G'_{\alpha\beta}a$ so that $C(\varrho) = G'_{\alpha\beta}a\bar{a} = G'_{\alpha\beta}e_{\alpha\beta} = G'_{\alpha\beta}$. But the unique idempotent measure with the support $G'_{\alpha\beta}$ is the normalized Haar measure on $G'_{\alpha\beta}$, i.e. $\mu'_{\alpha\beta}$. Therefore $a\mu'_{\alpha\beta}\bar{a} = \mu'_{\alpha\beta}$.

The relation $\mu'_{ik}(a\mu'_{\alpha\beta}\bar{a})\,\mu'_{jl} = \mu'_{ik}\mu'_{\alpha\beta}\mu'_{jl} = \mu'_{il}$ and (12) imply (by the usual argument) $\nu\nu_0 = \varepsilon$. Analogously $\nu_0\nu = \varepsilon$.

3) Since (by Lemma 2,3) $\mu'_{jl}\bar{a}\mu'_{jl}\,\mu'_{ik} = \mu'_{jl}\bar{a}\mu'_{jk} = \mu'_{jk}\bar{a}\mu'_{jk}$, we have

$$\nu_0\varepsilon = \sum_{j=1}^{s}\sum_{l=1}^{r}\sum_{i=1}^{s}\sum_{k=1}^{r}\xi_j\eta_l\xi_i\eta_k\mu'_{jl}\bar{a}\mu'_{jl}\,\mu'_{ik} = \sum_j\sum_k \xi_j\eta_k\mu'_{jk}\bar{a}\mu'_{jk} = \nu_0,$$

and analogously $\varepsilon\nu_0 = \nu_0$. This proves Theorem 2,4.

Theorems 2,1-2,4 give a clear insight into the group $\mathfrak{G}(\varepsilon)$ of all m-regular elements $\in \mathfrak{M}(S)$ belonging to the idempotent ε (at least in the case when $C(\varepsilon)$ contains a finite number of idempotents).

With the same notations as above write again

$$H_1 = H \cup HaH \cup HbH \cup \dots .$$

Take an arbitrary fixed group, say G_{11}, and consider the double coset decomposition

$$G_{11} = G'_{11} \cup G'_{11}aG'_{11} \cup G'_{11}bG'_{11} \cup \dots . \tag{13}$$

The totality of all classes in (13) which are two-sided constitutes the normalizer $G_{11}^{(0)}$ of G'_{11} in G_{11}.

Let μ_1, μ_2 be two m-regular elements (belonging to the same ε) with $C(\mu_1) = HaH$, $C(\mu_2) = HbH$. Consider the correspondence

$$\mu_1 \to HaH \cap G_{11} = G'_{11}aG'_{11}\,, \quad \mu_2 \to HbH \cap G_{11} = G'_{11}bG'_{11}\,.$$

Theorem 2,3 and 2,4 imply that this correspondence is a one-to-one. Since the product $\mu_1\mu_2$ is a m-regular measure (belonging to ε) and $C(\mu_1\mu_2) = HaHHbH$, there is necessarily a c such that $HaHbH = HcH$. Hence in our correspondence we have

$$\mu_1\mu_2 \to HcH \cap G_{11} = G'_{11}cG'_{11}\,.$$

To prove that our correspondence is an (algebraic) isomorphism it is sufficient to show that $G'_{11}aG'_{11}\,G'_{11}bG'_{11} = G'_{11}cG'_{11}$. This is an immediate consequence of $HaHbH = HcH$. Multiplying this relation to both sides by G'_{11}, taking account of $G'_{11}H = G'_{11}\bigcup\limits_{\alpha}\bigcup\limits_{\beta} G'_{\alpha\beta} = \bigcup\limits_{\beta} G'_{1\beta}$ and $HG'_{11} = \bigcup\limits_{\alpha} G'_{\alpha 1}$, we have

$$\Big(\bigcup_{\beta} G'_{1\beta}\Big)\, a\Big(\bigcup_{\gamma}\bigcup_{\delta} G'_{\gamma\delta}\Big)\Big(\bigcup_{\sigma}\bigcup_{\varrho} G'_{\sigma\varrho}\Big)\, b\Big(\bigcup_{\alpha} G'_{\alpha 1}\Big) = \Big(\bigcup_{\beta} G'_{1\beta}\Big)\, c\Big(\bigcup_{\alpha} G'_{\alpha 1}\Big)\,.$$

By Lemma 2,2 the right hand side is clearly euqal to $G'_{11}cG'_{11}$. The left hand side can be simplified (again by Lemma 2,2) as follows:

$$\Big(\bigcup_{\delta} G'_{11}aG'_{1\delta}\Big)\Big(\bigcup_{\sigma} G'_{\sigma 1}bG_{11}\Big) = G'_{11}a\Big(\bigcup_{\delta} G'_{1\delta}\bigcup_{\sigma} G'_{\sigma 1}\Big)\, bG'_{11} = G'_{11}aG'_{11}bG'_{11}\,.$$

This proves our assertion.

We have proved:

Theorem 2,5. *Let ε be an idempotent $\in \mathfrak{M}(S)$ with $C(\varepsilon) = H = \bigcup\limits_{i=1}^{s}\bigcup\limits_{k=1}^{r} G'_{ik}$ containing a finite number of idempotents, and $H_1 = \bigcup\limits_{i=1}^{s}\bigcup\limits_{k=1}^{r} G_{ik}$ the greatest simple subsemigroup containing the same idempotents as H. Denote by $G_{11}^{(0)}$ the normalizer of G'_{11} in G_{11}. Then the group $\mathfrak{G}(\varepsilon)$ of all m-regular elements belonging to ε (i.e. the maximal group $\in \mathfrak{M}(S)$ belonging to ε) is algebraically isomorphic to the factor group $G_{11}^{(0)}/G'_{11}$.*

3. TWO LIMIT THEOREMS

Recall first that in accordance with our earlier considerations we shall use the following notation. If $\{\mu_1, \mu_2, \mu_3, \ldots\}$ is a sequence of elements $\in \mathfrak{M}(S)$ we shall say that μ_n converges to $\mu \in \mathfrak{M}(S)$ if $\int f\, d\mu_n \to \int f\, d\mu$ for every $f \in \omega(S)$.

Let μ belong to the idempotent ε. It is known and easy to prove that $\lim\limits_{n=\infty} \mu^n$ exists if and only if $\varepsilon\mu = \mu\varepsilon = \varepsilon$. An alternative answer to this question (in the case treated above) is given by the following theorem:

Theorem 3,1. *Let $\mu \in \mathfrak{M}(S)$ belong to ε and suppose that $H = C(\varepsilon)$ contains a finite number of idempotents. Then $\lim\limits_{n=\infty} \mu^n$ exists if and only if $H\,C(\mu)H = H$.*

Proof. a) If $\lim_{n=\infty} \mu^n$ exists, we have $\varepsilon\mu = \varepsilon$, hence $H\, C(\mu) = H$ and $H\, C(\mu)\, H = H$.

b) Write (in our usual notations) $\varepsilon = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k \mu'_{ik}$, $H = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G'_{ik}$ and consider the measure $\varrho = \varepsilon\mu\varepsilon$. Since $H\, C(\mu)\, H = H$, we have $C(\varrho) = H$ and

$$\varrho = \varepsilon\varrho\varepsilon = \sum_i \sum_k \sum_j \sum_l \xi_i \eta_k \xi_j \eta_l \mu'_{ik} \varrho \mu'_{jl} .$$

Further (by Lemma 1,2 c) $\mu'_{ik}\varrho\mu'_{jl} = \mu'_{il}$, hence $\varrho = \varepsilon$. This implies $\varepsilon\mu\varepsilon = \varepsilon$, $(\varepsilon\mu)^2 = \varepsilon\mu$ and since $\varepsilon\mu$ is an idempotent and at the same time an element belonging to ε we have $\varepsilon\mu = \varepsilon$. Analogously $\mu\varepsilon = \varepsilon$. This proves our theorem.

Before proving a second limit theorem in which no finiteness assumption as to the number of idempotents is required we shall prove Lemma 3,1 formulated below.

If $\mathfrak{N}$ is a subset of $\mathfrak{M}(S)$ we shall call the closure of $\bigcup_{\mu \in \mathfrak{N}} C(\mu)$ the support of $\mathfrak{N}$ and we shall denote it by $C(\mathfrak{N})$.

If $\mathfrak{N}$ is a subsemigroup of $\mathfrak{M}(S)$, then $C(\mathfrak{N})$ is a (closed) subsemigroup of S. Moreover it can be easily seen that $C(\mathfrak{N}) = C(\overline{\mathfrak{N}})$ (see I. Glicksberg [1]).

Let $\mathfrak{P}_\mu = \{\mu, \mu^2, \mu^3, \ldots\}$ be the cyclic subsemigroup generated by μ, $\mathfrak{G}_\mu$ the maximal group contained in $\overline{\mathfrak{P}}_\mu$. If μ belongs to ε, we have of course $\varepsilon \in \mathfrak{G}_\mu \subset \overline{\mathfrak{P}}_\mu$ and $C(\varepsilon) = H \subset C(\mathfrak{G}_\mu) \subset C(\overline{\mathfrak{P}}_\mu) = C(\mathfrak{P}_\mu)$.[7]) If H_1 is the largest simple subsemigroup containing the same idempotents as H, we have (by Theorem 2,1) $C(\varrho) \subset H_1$ for every $\varrho \in \mathfrak{G}_\mu$. Therefore $C(\mathfrak{G}_\mu) \subset \overline{H}_1$. Now it is easy to prove that the closure of a simple semigroup is itself simple. Consider the relation $C(\mathfrak{G}_\mu) \subset \overline{H}_1$. Since $\overline{H}_1$ is a compact simple semigroup and $C(\mathfrak{G}_\mu)$ a closed subsemigroup, we may use a result of [6] (Theorem 1,1) which implies: $C(\mathfrak{G}_\mu)$ is a closed simple subsemigroup of S (contained in $\overline{H}_1$).

We next show that $C(\mathfrak{G}_\mu)$ is exactly the minimal two-sided ideal of $C(\mathfrak{P}_\mu)$. Denote for brevity $C(\mathfrak{P}_\mu) = P$, $C(\mathfrak{G}_\mu) = K$ and let be J the minimal two-sided ideal of P. Denote $P_0 = C(\mu) \cup C(\mu^2) \cup \ldots$. This is a subsemigroup of P which is dense in P.

Let be $x \in P_0$, i.e. $x \in C(\mu^l)$ for some $l > 0$. We have $C(\varepsilon)\, x\, C(\varepsilon) \subset C(\varepsilon)\, C(\mu^l)\, C(\varepsilon) = C(\varepsilon\mu^l\varepsilon)$, and since $\varepsilon\mu^l\varepsilon \in \mathfrak{G}_\mu$, we have $C(\varepsilon)\, x\, C(\varepsilon) \subset K$. Therefore $PxP \cap K \neq \emptyset$ for every $x \in P_0$. Now since $PxyP \subset PxP \cap PyP$ (for any $x, y \in P_0$) it follows from the compactness of K that $[\bigcap_{x \in P_0} PxP] \cap K \neq \emptyset$. Now it can be proved (in the same manner as in I. Glicksberg [1], 1,11, for the abelian case) that $\bigcap_{x \in P_0} PxP$ is the minimal two-sided ideal J of P (i.e. it is equal to $\bigcap_{x \in P} PxP$). Hence $J \cap K \neq \emptyset$. Since K is a simple subsemigroup of P we have necessarily $K \subset J$ (for if $a \in K \cap J$, the relation $KaK = K$ implies $K \subset KJK \subset J$).

Let be again $x \in P_0$ and $x \in C(\mu^l)$ for an integer $l > 0$. Let further ν be any

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[7]) $C(\mathfrak{P}_\mu)$ is the closure of $C(\mu) \cup C(\mu^2) \cup C(\mu^3) \cup \ldots$, i.e. the closure of the algebraic subsemigroup of S generated by $C(\mu)$.

element $\in \mathfrak{G}_\mu$. Then $x\, C(\nu) \subset C(\mu^l)\, C(\nu) = C(\mu^l \nu) \subset C(\mathfrak{G}_\mu) = K$. Hence $x \bigcup_{\nu \in \mathfrak{G}_\mu} C(\nu) \subset$ $\subset K$. Since K is closed $\overline{x \bigcup_{\nu \in \mathfrak{G}_\mu} C(\nu)} \subset K$ and by continuity of the multiplication

$$x\, C(\mathfrak{G}_\mu) = x\, \overline{\bigcup_{\nu \in \mathfrak{G}_\mu} C(\nu)} \subset \overline{x \bigcup_{\nu \in \mathfrak{G}_\mu} C(\nu)} \subset K\,,$$

i.e. $xK \subset K$ for any $x \in P_0$. This implies $yK \subset K$ for any $y \in P$ and analogously $Ky \subset K$. Therefore K is a two-sided ideal of P. Since $K \subset J$, and J is minimal, we have $K = J$.

We have proved:

Lemma 3,1. *Let S be a compact semigroup, $\mu \in \mathfrak{M}(S)$ and $\mathfrak{P}_\mu = \{\mu, \mu^2, \mu^3, \ldots\}$. If $\mathfrak{G}_\mu$ is the maximal group ($=$ minimal idel) contained in $\overline{\mathfrak{P}}_\mu$, and J is the minimal two-sided ideal of $C(\mathfrak{P}_\mu)$, then $C(\mathfrak{G}_\mu) = J$.*

Theorem 3,2. *Let S be a compact semigroup and $\mu \in \mathfrak{M}(S)$. Denote $\sigma_n = (1/n) \sum_{k=1}^{n} \mu^k$. Then $\lim_{n=\infty} \sigma_n$ exists and it is equal to an idempotent $\sigma \in \mathfrak{M}(S)$. If P is the closed subsemigroup generated by $C(\mu)$ and J the minimal two-sided ideal of P, then $C(\sigma) = J$.*

Proof. If $\mathfrak{H}_\mu$ is the closed convex hull of the subsemigroup $\mathfrak{P}_\mu = \{\mu, \mu^2, \mu^3, \ldots,\}$ then $C(\mathfrak{H}_\mu) = C(\mathfrak{P}_\mu) = P$.

Let σ be any cluster point of the sequence $\{\sigma_n\}$. Clearly $\sigma \in \mathfrak{H}_\mu$. Since $\mu\sigma_n - \sigma_n = 1/n(\mu^{n+1} - \mu)$ it is easily seen that $\mu\sigma = \sigma$. Since this implies $\sigma = \mu\sigma = \mu^2\sigma = \ldots$, we also have $\sigma = (t_1\mu + t_2\mu^2 + t_3\mu^3 + \ldots)\,\sigma$ for any $t_i \geqq 0$ with $\sum_i t_i = 1$. Consequently (with respect to the continuity) $\sigma = \lambda\sigma$ for every $\lambda \in \mathfrak{H}_\mu$. This means that $\mathfrak{H}_\mu$ (an abelian subsemigroup of $\mathfrak{M}(S)$) contains σ as its zero element. But any semigroup contains at most one zero element. Therefore there is a unique cluster point of $\{\sigma_n\}$ and $\lim_{n=\infty} \sigma_n = \sigma$ follows by compactness. Moreover σ is an idempotent (and a trivial minimal two-sided ideal of $\mathfrak{H}_\mu$).

Now if $\lambda \in \mathfrak{H}_\mu$, then $\sigma\mathfrak{H}_\mu = \mathfrak{H}_\mu\sigma = \sigma$ implies $C(\sigma)\, C(\lambda) = C(\lambda)\, C(\sigma) = C(\sigma)$ and $C(\sigma) \bigcup_{\lambda \in \mathfrak{H}_\mu} C(\lambda) = \bigcup_{\lambda \in \mathfrak{H}_\mu} C(\lambda)\, C(\sigma) = C(\sigma)$. Further

$$C(\sigma) = C(\sigma) \bigcup_{\lambda \in \mathfrak{H}_\mu} C(\lambda) \subset C(\sigma)\overline{\bigcup_{\lambda \in \mathfrak{H}_\mu} C(\lambda)} = C(\sigma)\, P$$

and analogously $C(\sigma) \subset P\, C(\sigma)$. This says that $C(\sigma)$ is a two-sided ideal of P. Since $J\, C(\sigma) \subset J \cap C(\sigma)$, $J \cap C(\sigma) \neq \emptyset$, and since $C(\sigma)$ is a simple subsemigroup, we have $C(\sigma) \subset J$. Finally with respect to the minimality of J we have $C(\sigma) = J$. This completes the proof of our theorem.[8])

[8]) After this paper has been finished for publication prof. E. Hewitt has drawn my attention to the fact that a part of Theorem 3,2 is proved in a recent paper of M. Rosenblatt [12]. Our proof differs essentially from that of [12].

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Резюме

ПОЛУГРУППА МЕР НА БИКОМПАКТНЫХ НЕКОММУТАТИВНЫХ ПОЛУГРУППАХ

ШТЕФАН ШВАРЦ (Štefan Schwarz), Братислава

Пусть S — бикомпактная хаусдорфова полугруппа. Под мерой μ мы будем подразумеванть σ-аддитивную неотрицательную регулярную множественную функцию, определенную на борелевских множествах из S такую, что $\mu(S) = 1$. Обозначим символом $\mathfrak{M}(S)$ множество всех мер полугруппы S.

Пусть $\omega(S)$ — банахово пространство непрерывных действительных функций $f(x)$, определенных на S. Известно, что $\mathfrak{M}(S)$ можно погрузить в $\omega(S)^*$ (сопряженное пространство к $\omega(S)$) и если задать в $\omega(S)^*$ слабую топологию, то $\mathfrak{M}(S)$ образует бикомпактное хаусдорфово пространство. Если определить произведение мер μ, ν с помощью уравнения (1), $\mathfrak{M}(S)$ превращается в бикомпактную топологическую полугруппу. Цель работы-изучение строения полугруппы $\mathfrak{M}(S)$.

1. Пусть $\varepsilon = \varepsilon^2 \in \mathfrak{M}(S)$ — идемпотентная мера, носителем которой являетая множество $C(\varepsilon) \subset S$. $C(\varepsilon)$ — простая замкнутая полугруппа и, следовательно, вида $C(\varepsilon) = \bigcup\limits_{\alpha}\bigcup\limits_{\beta} G_{\alpha\beta}$, где $G_{\alpha\beta}$ — изоморфные между собою бикомпактные группы. Предположим что $C(\varepsilon)$ имеем конечное число идемпотентов и $\alpha = 1, \ldots, s$, $\beta = 1, \ldots, r$. (Известно, что s, r — число минимальных правых, соотвественно левых, идеалов из $C(\varepsilon)$.)

В теоремах 1,1 и 1,2 доказаны следующие утверждения. Идемпотент ε индуцирует на каждой из групп $G_{\alpha\beta}$ инвариантную меру. Если $\mu_{\alpha\beta}$ — нормализированная мера Хаара на $G_{\alpha\beta}$, то ε имеем вид $\varepsilon = \sum\limits_{\alpha=1}^{s} \sum\limits_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu_{\alpha\beta}$, где ξ_α, η_β — положительные числа, удовлетворяющие соотношениям $\sum\limits_{\alpha=1}^{s} \xi_\alpha = \sum\limits_{\beta=1}^{r} \eta_\beta = 1$. Каждая из мер такого вида-идемпотент $\in \mathfrak{M}(S)$, и всякая замкнутая простая подполугруппа из S, имеющая конечное число идемпотентов-носитель некоторой идемпотентной меры из $\mathfrak{M}(S)$. (Если носитель не является группой, то число таких мер бесконечно.)

В теоремах 1,3–1,5 характеризуются примитивные идемптотенты полугруппы $\mathfrak{M}(S)$ и дается строение ядра полугруппы $\mathfrak{M}(S)$.

2. В разделе 2 изучаются максималные подгруппы $\mathfrak{G}(\varepsilon) \subset \mathfrak{M}(S)$, имееющие ε в качестве единичного элемента.

Пусть $C(\varepsilon) = H$ и H_1 — наибольшая простая полугруппа из S, имеющая те же идемпотенты как H. Рассмотрим разложение $H_1 = H \cup HaH \cup HbH \cup \ldots$ $(a, b, \ldots \in H_1)$. Такое разложение в дизъюнктные слагаемые существует. Обозначим $H = \bigcup\limits_{\alpha}\bigcup\limits_{\beta} G'_{\alpha\beta}$, $H_1 = \bigcup\limits_{\alpha}\bigcup\limits_{\beta} G_{\alpha\beta}$. Если $\mu \in \mathfrak{G}(\varepsilon)$, то имеет место $C(\mu) = HaH$ (где a — удобно выбранный элемент $\in H_1$). Далее, $C(\mu) \cap G_{\alpha\beta}$ — двусторонний класс смежности разложения группы $G_{\alpha\beta}$ модуло $G'_{\alpha\beta}$.

Если H имеет конечное число идемпотентов, $\mu \in \mathfrak{G}(\varepsilon)$ и $C(\mu) = HaH$, то μ определено однозначно. Именно, если $\varepsilon = \sum\limits_{\alpha=1}^{s} \sum\limits_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu'_{\alpha\beta}$ ($\mu'_{\alpha\beta}$ — нормализированная мера Хаара на $G'_{\alpha\beta}$), то имеет место

$$\mu = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu'_{\alpha\beta} a \mu'_{\alpha\beta} \, .$$

Класс HbH есть носитель некоторой меры $\in \mathfrak{G}(\varepsilon)$ тогда и только тогда, если $HbH \cap G_{\alpha\beta}$ лежит в нормализаторе $G^{(0)}_{\alpha\beta}$ группы $G'_{\alpha\beta}$ в группе $G_{\alpha\beta}$. Кроме того, $\mathfrak{G}(\varepsilon) \cong G^{(0)}_{\alpha\beta} \mid G'_{\alpha\beta}$.

3. В разделе 3 доказывается следующая теорема:

Пусть $\mu \in \mathfrak{M}(S)$. Обозначим $\sigma_n = 1/n\,(\mu + \mu^2 + \ldots + \mu^n)$. Тогда $\lim\limits_{n=\infty} \sigma_n$ существует и равняется некоторому идемпотенту $\sigma \in \mathfrak{M}(S)$. Если P-замкнутая подполугруппа из S, порожденная $C(\mu)$, J-минимальный двусторонний идеал из P, то $C(\sigma) = J$.