Czechoslovak Mathematical Journal

Štefan Schwarz

Subsemigroups of simple semigroups

Czechoslovak Mathematical Journal, Vol. 13 (1963), No. 2, 226–239

Persistent URL: http://dml.cz/dmlcz/100564

Чехословацкий математический журнал, т. 13 (88) 1963, Прага

SUBSEMIGROUPS OF SIMPLE SEMIGROUPS

ŠTEFAN SCHWARZ, Bratislava

(Received March 13, 1961)

The purpose of this paper is to study the structure of subsemigroups of a simple semigroup S, especially in the case when S is completely simple. Also coset decompositions of S modulo some subsemigroups are studied.

Let S be a semigroup. A left ideal of S is a subset $L \subset S$ with $SL \subset L$. A right ideal of S is a subset R with $RS \subset R$. A subset which is both a left and right ideal of S is called a two-sided ideal of S.

The semigroup S itself and the zero element 0 (if S contains a zero element) are always two-sided ideals.

A minimal left ideal of S is a left ideal of S which does not contain any proper left ideal of S with the eventually exception of (0) (if S contains a zero element). Minimal right and two-sided ideals are defined analogously.

A semigroup S without zero and containing at least two elements is called *simple* if it does not contain any two-sided ideal $\neq S$.

A semigroup S with a zero element 0 containing at least two elements is said to be simple if it does not contain any two-sided ideal different from (0) and S itself.

Also a semigroup consisting of a single element is called simple. Of course, in this case this element can be considered as a zero element of S. Hence to avoid confusion by a semigroup with zero we shall usually mean a semigroup containing at least two elements (one of them being a zero element).

The purpose of this paper is to study subsemigroups of a simple semigroup. The question arises whether any subsemigroup of a simple semigroup is simple. In general the answer to this question is certainly negative since it is known (see R. H. BRUCK [1], p. 48 and G. B. PRESTON [6]): Any semigroup T can be embedded in a simple semigroup (with or without zero) containing a unit element. On the other hand every finite simple semigroup without zero possesses the mentioned property.

In section 1 of this paper we show that the answer to this question is positive if S is a compact simple semigroup without zero and we restrict ourselves to closed subsemigroups.

In section 2 we treat an analogous problem for simple semigroups with a zero.

Some corollaries and special cases of these results are needed in a forthcomming paper on convolution semigroups of measures on non-commutative semigroups (see [9]). Also the results of section 3 dealing with some coset decompositions of completely simple semigroups are proved for this purpose. All these results seem to be also of an independent interest. For this reason I have found it convenient to publish them separately.

1

A simple semigroup S is called *completely simple* if it contains at least one minimal left and at least one minimal right ideal of S.

Recall that if S is a completely simple semigroup without zero it can be written in the form $S = \bigcup_{\alpha \in \Lambda_1} R_\alpha = \bigcup_{\beta \in \Lambda_2} L_\beta$, where R_α and L_β run through all minimal right and left ideals of S respectively. Moreover, every minimal left ideal L_α is generated by an idempotent, i.e. there is an idempotent $e \in L_\alpha$ such that $L_\alpha = Se = L_\alpha . e$. Clearly e (and every idempotent $\in L_\alpha$) is a right unit of L_α. Analogously for minimal right ideals. Also $R_\alpha L_\beta = R_\alpha \cap L_\beta$ is a group. Denoting $R_\alpha L_\beta = G_{\alpha\beta}$ we can write $S = \bigcup_\alpha \bigcup_\beta G_{\alpha\beta}$ as a union of pairwise disjoint maximal isomorphic groups $\subset S$. The $G'_{\alpha\beta}$s will be called the group-components of S.

Recall further that if S is completely simple with zero 0 (and S contains at least two elements) we have either $S^2 = 0$ or $S^2 = S$. In the first case S is of the form $S = \{0, a\}$ with $a^2 = a . 0 = 0 . a = 0^2 = 0$. In the second case S contains at least one idempotent $e \neq 0$ and it can be written in the form $S = \bigcup_\beta L_\beta$, where L_β runs through all minimal left ideals of S. For any two minimal left ideals $L_\alpha \neq L_\beta$ we have $L_\alpha \cap L_\beta = (0)$. Moreover, every minimal left ideal L_β contains at least one idempotent $\neq 0$ and it is generated by any such idempotent. Any minimal left ideal L_β of S can be written as a union of disjoit sets $L_\beta = \{\bigcup_\gamma G_{\gamma\beta}\} \cup P_\beta$, where $G_{\gamma\beta}$ are (isomorphic) groups and P_β a semigroup with $P_\beta^2 = 0$. Analogous statements hold for minimal right ideals. (The proofs of all these statements can be found f.i. in the recent book of E. C. Пляпин [4].)

R e m a r k. The restriction to completely simple semigroups though an essential one is itself not sufficient to obtain results of the kind mentioned in the introduction. If S is completely simple and T a subsemigroup of S then T need not be simple even in the case when T has an idempotent. Let f.i. S be the multiplicative group of real numbers >0. S is then a (trivial) completely simple semigroup. Let T be the subsemigroup of real numbers $\geqq 1$. Then T contains the idempotent 1 and is not simple, since f.i. the subsemigroup of all real numbers $\geqq 2$ is a proper ideal of T.

We first prove

Lemma 1,1. *Let S be a completely simple semigroup without zero and T a simple subsemigroup of S containing an idempotent. Then*

1. *T is completely simple. If T contains more then one element, T is completely simple without zero.*

2. *If L_α is a minimal left ideal of S, and $L'_\alpha = T \cap L_\alpha \neq \emptyset$, then L'_α is a minimal left ideal of T.*

3. *Conversely, if L'_α is a minimal left ideal of T, then there exists a uniquely determined minimal left ideal L_α of S such that $L'_\alpha = T \cap L_\alpha$.*

Proof. 1. An idempotent $e \neq 0$ of any semigroup is called primitive if there does not exist an idempotent $x \neq e$ and $x \neq 0$ such that $xe = ex = x$.

It is known that all non-zero idempotents of a completely simple semigroup (with zero or without zero) are primitive. Further it is known (see D. Rees [7]): If a simple semigroup T contains a non-zero primitive idempotent, then T is completely simple. In our case: Since T is simple and it contains an idempotent $e \in S$, T is completely simple if e is not a zero element of T or if T reduces to e.

Suppose that T contains more then one element and $e = z$ is a zero of T. We show that this case is impossible. Let be $a \in T$, $a \neq z$. By definition of a zero element we have $az = za = z$. The idempotent $z \in T$ is contained in a group-component $G_{\alpha\beta}$ of S. The element a cannot be contained in $G_{\alpha\beta}$ since otherwise $az = z^2$ (in $G_{\alpha\beta}$) would imply $a = z$. Hence $a \in G_{\gamma\delta}$ for some γ, δ and $G_{\alpha\beta} \cap G_{\gamma\delta} = \emptyset$. Denote by e' the unit element of $G_{\gamma\delta}$, hence $e' \neq z$. The relation $az = z$ implies $e'az = e'z$, $az = e'z$, $z = e'z$; analogously $za = z$ implies $zae' = ze'$, $za = ze'$, $z = ze'$. Hence $z = e'z = ze'$. But this contradicts to the fact that e' is a primitive idempotent of S.

2. Let L_α be a minimal left ideal of S and suppose $L_\alpha \cap T = L'_\alpha \neq \emptyset$. Then for $a \in T$ and $x \in L'_\alpha$ we have $ax \in aL'_\alpha \subset aL_\alpha \subset L_\alpha$, further $ax \in T \, . \, T \subset T$, hence $ax \in T \cap L_\alpha = L'_\alpha$. Therefore L'_α is a left ideal of T.

We next prove that L'_α is a minimal left ideal of T. Since T is completely simple, it is the union of its minimal left ideals. Hence L'_α is either a minimal left ideal of T or there is a minimal left ideal L of T such that $L \subsetneqq L'_\alpha$. Suppose this second case. Let e be the idempotent contained in L. We have $L'_\alpha e \subset L'_\alpha L \subset L$. On the other hand every idempotent $\in L_\alpha$ is a right unit of the semigroup L_α, hence $L'_\alpha e = L'_\alpha$. Therefore $L'_\alpha \subset L$, i.e. $L'_\alpha = L$, contrary to the supposition.

3. Write $S = \bigcup_{\beta \in \Lambda_2} L_\beta$ as the decomposition of S into the union of its minimal left ideals. Let L'_α be a minimal left ideal of T. We have

$$(1) \qquad L'_\alpha = L'_\alpha \cap S = L'_\alpha \cap \{ \bigcup_{\beta \in \Lambda_2} L_\beta \} = \bigcup_{\beta \in \Lambda_2} \{ L'_\alpha \cap L_\beta \} .$$

If $L'_\alpha \cap L_\beta \neq \emptyset$, then $T(L'_\alpha \cap L_\beta) \subset L'_\alpha \cap L_\beta$, hence $L'_\alpha \cap L_\beta$ is a left ideal of T. Since L'_α is minimal, there is exactly one non-empty summand on the right hand side of (1). Hence there is a unique minimal left ideal of S — say L_α, $\alpha \in \Lambda_2$, — such that $L'_\alpha = L'_\alpha \cap L_\alpha$. We have $L'_\alpha \subset T$, $L'_\alpha \subset L_\alpha$, hence $L'_\alpha \subset T \cap L_\alpha$. On the other hand it has been proved above that $T \cap L_\alpha$ is a minimal left ideal of T, therefore $L'_\alpha = T \cap L_\alpha$, q.e.d.

Remark. The following problem arises. Does there exist a completely simple semigroup S containing a simple subsemigroup T without idempotents (hence T simple but not completely simple)?

Write $S = \bigcup_{\gamma \in \Lambda_1} \bigcup_{\delta \in \Lambda_2} G_{\gamma\delta}$ and suppose that $T \subset S$ is simple. Then there is at least one couple $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$ such that $T \cap G_{\alpha\beta} = P_{\alpha\beta} \neq \emptyset$. We show that the semigroup $P_{\alpha\beta}$ itself must be a simple semigroup. Choose $a, b \in P_{\alpha\beta}$. We then also have $a^3 \in P_{\alpha\beta} \subset T$. Since T is simple, there exist $x, y \in T$ such that $xa^3y = b$. If $x \in R_\gamma$ for some γ, we have $b = xa^3y \in R_\gamma a^3 y \subset R_\gamma$, therefore $\gamma = \alpha$; hence $x \in R_\alpha$ and $x \in T \cap R_\alpha$. Analogously $y \in T \cap L_\beta$. Further we have $\xi = xa \in R_\alpha G_{\alpha\beta} = R_\alpha R_\alpha L_\beta = R_\alpha L_\beta = G_{\alpha\beta}$, and $\xi = xa \in T \cdot T = T$, hence $\xi \in T \cap G_{\alpha\beta} = P_{\alpha\beta}$. Analogously $\eta = ay \in P_{\alpha\beta}$. Since $(xa)\, a(ay) = \xi a \eta = b$, this implies: To every couple $a, b \in P_{\alpha\beta}$ there exist elements $\xi, \eta \in P_{\alpha\beta}$ such that $\xi\, a\, \eta = b$. Hence $P_{\alpha\beta}$ is simple. (Note also that $P_{\alpha\beta}$ being a subset of $G_{\alpha\beta}$ satisfies the left and right cancellation law.)

We have proved: *If there is a completely simple semigroup containing a simple subsemigroup without idempotents, there must exist a group G such that G contains a simple subsemigroup H without idempotents.* Conversely, the existence of such a group G would give a positive answer to our question, since G is itself a (trivial) completely simple semigroup.

Now such groups really exist. Consider f.i. the set G of all couples (a, b) of real numbers with $a \neq 0$ and introduce in G a multiplication by $(a, b) \odot (c, d) = (ac, bc + d)$. Then G is a group with $(1, 0)$ as unit element. Next let S be the subset of all couples (a, b) with $a > 0$, $b > 0$. Then S is easily seen to be a simple semigroup (without idempotents). This shows that the assumption in Lemma 1,1 that T contains an idempotent is an essential one. {Analogous examples due to O. Andersen can be found in the recent book [3], p. 51.}

If $G_{\alpha\beta}$ is commutative (and hence all $G_{\gamma\delta}$ commutative), then $P_{\alpha\beta}$ is commutative and since a simple commutative semigroup without zero is a group, $P_{\alpha\beta}$ is a group. This implies:

Corollary 1,1. *Let S be completely simple without zero and let the group-components of S be commutative. Then every simple subsemigroup of S is completely simple.*

The following lemma is well known. We prove it only for the sake of completeness (since we shall use it several times).

Lemma 1,2. *Let S be a semigroup without zero containing a minimal left ideal and suppose that S is the union of its minimal left ideals. Then S is a simple semigroup.*

Proof. Write $S = \bigcup_{\alpha \in \Lambda_2} L_\alpha$, where L_α runs through all minimal left ideals of S. Suppose that N is a two-sided ideal of S, $N \subset S$. Then $NS \subset N$. On the other hand $NS = N\{\bigcup_\alpha L_\alpha\} = \bigcup_\alpha NL_\alpha$, and since (with respect to the minimality of L_α) $NL_\alpha = L_\alpha$, we have $NS = \bigcup_\alpha L_\alpha = S$, i.e. $S \subset N$, therefore $S = N$. This proves our lemma.

Now we introduce a topological restriction. If S is a semigroup and at the same time a topological space, and the multiplication is continuous, S is called a topological semigroup. Supposing that S is a Hausdorff compact space the corresponding semigroup will be called a compact semigroup. It is known that any compact semigroup contains always at least one idempotent. Moreover if S has not a zero, there exists at least one (non-zero) minimal left ideal and at least one (non-zero) minimal right ideal, and the minimal ideals are closed. This implies (see K. NUMAKURA [5], p. 103): A compact simple semigroup without zero is completely simple.

We now prove:

Theorem 1.1. *Let S be a compact simple semigroup without zero. Then every closed subsemigroup T of S is a completely simple semigroup.*

Remark. In the finite case this was proved by A. K. Сушкевич ([10], p. 59).

Proof. Let $S = \bigcup_{\alpha \in \Lambda_2} L_\alpha$ be the decomposition of S into the union of its minimal left ideals. Consider the set of all left ideals L_α for which $L_\alpha \cap T \neq \emptyset$. Let this set be $\{L_\gamma, \gamma \in \Lambda_2'\}$. For every $\gamma \in \Lambda_2'$ denote $L_\gamma \cap T = L_\gamma'$.

The closed subset $L_\gamma' \neq \emptyset$ is a left ideal of T. In fact, for every $a \in T$ and every $x \in L_\gamma'$ we have $ax \in aL_\gamma' \subset SL_\gamma = L_\gamma$, further $ax \in T \cdot T \subset T$, hence $ax \in T \cap L_\gamma = L_\gamma'$.

We next show that L_γ' is a minimal left ideal of T. Let L' be a left ideal of T and suppose that $L' \subset L_\gamma'$. Since T is closed and hence compact (in the relative topology), T is a compact semigroup and there is necessarily a minimal left ideal L^* of T with $L^* \subset L'$. Again L^* is closed (hence compact), therefore it contains an idempotent e. We have $e \in L^* \subset L' \subset L_\gamma' \subset L_\gamma$. In particular $L_\gamma' e \subset L_\gamma' L^* \subset TL^* \subset L^*$. On the other hand every idempotent $e \in L_\gamma$ is a right unit of the semigroup L_γ, hence $L_\gamma' e = L_\gamma'$. Therefore $L_\gamma' \subset L^*$ and $L^* = L' = L_\gamma'$. This proves that L_γ' is a minimal left ideal of T.

It follows from $T = T \cap S = \bigcup_{\gamma \in \Lambda'_2} L_\gamma'$ that T is the union of its minimal left ideals. By Lemma 1,2 we conclude that T is a simple semigroup. Since we can prove by the same argument that T contains also a minimal right ideal of T, T is completely simple.

Remark. The following (trivial) example shows that the supposition that T is closed cannot be – in general – dropped. Let G be the group of complex numbers

$\{z \mid |z| = 1\}$ in the obvious topology, $z_0 = e^{2\pi i \vartheta}$ with an irrational ϑ and $T = \{z_0^n \mid n = 1, 2, \ldots\}$. The sub-semigroup T is clearly not simple, since f.i. $T^2 \neq T$. Also the subsemigroup $T_0 = T \cup \{1\}$ (containing an idempotent) is not simple since it contains an infinity of ideals of T_0.

Lemma 1,3. *Let S be a completely simple semigroup without zero and T a simple subsemigroup of S containing an idempotent. Then there exists a unique greatest simple subsemigroup $T_1 \supset T$ of S having the same idempotents as T. The semigroup T_1 can be written in the form $T_1 = \{\bigcup_{\alpha \in \Lambda_1'} R_\alpha\} \cap \{\bigcup_{\beta \in \Lambda_2'} L_\beta\}$ with suitably chosen minimal right and left ideals R_α, L_β of S respectively.*

Proof. By Lemma 1,1 T is completely simple. Therefore we may write $T = \bigcup_{\alpha \in \Lambda'_1} R'_\alpha = \bigcup_{\beta \in \Lambda'_2} L'_\beta$, where R'_α, L'_β run through all minimal left and right ideals of T respectively.

Choose – in the sense of Lemma 1,1 – R_α, L_β such that $R'_\alpha = R_\alpha \cap T$, $L'_\beta = L_\beta \cap T$. We have

$$T = T^2 = \{\bigcup_{\alpha \in \Lambda'_1} R'_\alpha\} \cdot \{\bigcup_{\beta \in \Lambda'_2} L'_\beta\} = \bigcup_\alpha \bigcup_\beta \{R'_\alpha L'_\beta\} \subset \bigcup_{\alpha \in \Lambda'_1} \bigcup_{\beta \in \Lambda'_2} \{R_\alpha L_\beta\} =$$
$$= \{\bigcup_{\alpha \in \Lambda'_1} R_\alpha\} \cap \{\bigcup_{\beta \in \Lambda'_2} L_\beta\} = \bigcup_{\alpha \in \Lambda'_1} \bigcup_{\beta \in \Lambda'_2} G_{\alpha\beta} = T_1 \,,$$

and T_1 is a semigroup which contains exactly the same idempotents as T.

For $\beta \in \Lambda'_2$ denote $L''_\beta = L_\beta \cap T_1$. We prove that L''_β is a minimal left ideal of T_1. If $x \in L''_\beta$, $a \in T_1$, we have $ax \in aL''_\beta \subset aL_\beta \subset L_\beta$, further $ax \in T_1 . T_1 \subset T_1$, hence $ax \in T_1 \cap L_\beta = L''_\beta$. This proves that L''_β is a left ideal of T_1. Now

$$L''_\beta = L_\beta \cap T_1 = L_\beta \cap \{(\bigcup_{\alpha \in \Lambda'_1} R_\alpha) \cap (\bigcup_{\beta \in \Lambda'_2} L_\beta)\} = L_\beta \cap \{\bigcup_{\alpha \in \Lambda'_1} R_\alpha\} =$$
$$= \{\bigcup_{\alpha \in \Lambda'_1} \{L_\beta \cap R_\alpha\} = \bigcup_{\alpha \in \Lambda'_1} G_{\alpha\beta} \,.$$

Suppose that L is a left ideal of T_1 and $L \subset L''_\beta$. Then there is at least one summand $G_{\alpha_0\beta}$, $\alpha_0 \in \Lambda'_1$ with $G_{\alpha_0\beta} \cap L \neq \emptyset$. Now a left ideal of any semigroup which has a non-empty intersection with a subgroup contains the whole subgroup. Therefore $G_{\alpha_0\beta} \subset L$. In particular L contains the idempotent $e_{\alpha_0\beta}$. We have $e_{\alpha_0\beta} \in L \subset L''_\beta \subset L_\beta$ and $L''_\beta e_{\alpha_0\beta} \subset L''_\beta L \subset L$. On the other hand every idempotent $\in L_\beta$ is a right unit of L_β, in particular $L''_\beta e_{\alpha_0\beta} = L''_\beta$. Therefore $L''_\beta \subset L$, i.e. $L = L''_\beta$. This proves that L''_β is a minimal left ideal of T_1. Analogously we prove that for $\alpha \in \Lambda'_1$ $R_\alpha \cap T_1 = R''_\alpha$ is a minimal right ideal of T_1.

Since $T_1 = \{\bigcup_{\alpha \in \Lambda'_1} R_\alpha\} \cap \{\bigcup_{\beta \in \Lambda'_2} L_\beta\} = T_1 \cap \{\bigcup_{\beta \in \Lambda'_2} L_\beta\} = \bigcup_{\beta \in \Lambda'_2} \{L_\beta \cap T_1\} = \bigcup_{\beta \in \Lambda'_2} L''_\beta$ and T_1 contains also a minimal right ideal, it follows by Lemma 1,2 that T_1 is a completely simple semigroup.

Let now finally T_2 be a simple subsemigroup of S with $T_2 \supset T$. T_1 is a union of disjoint maximal groups of S. Since also T_2 is completely simple, it is also a union of groups. The maximal groups belonging to different idempotents are disjoints. If there were $T_2 - T_1 \neq \emptyset$, T_2 would contain at least one group-component, and hence at least one idempotent, not contained in T_1. This contradiction proves the maximality of T_1 and completes the proof of our theorem.

The suppositions of Lemma 1,3 hold if, in particular, S is compact and simple, and T is any closed subsemigroup of S. In this case T_1 is closed. To prove this denote $A = \bigcup_{\alpha \in A'_1} R_\alpha$, $B = \bigcup_{\beta \in A'_2} L_\beta$. Then $T_1 = A \cap B$ and is it sufficient to prove that A and B are closed. Since S is the union of all minimal right ideals of S, $S - A$ is clearly the largest right ideal of S that does not meet T, hence the largest right ideal contained in the open set $S - T$. Therefore it is sufficient to prove that the largest right ideal R^* of S contained in $S - T$ is open. Let be $x \in R^*$. Since R^* is a right ideal, we have $x \cup xS \subset R^* \subset S - T$. Since $S - T$ is open, we may apply a lemma of A. D. WALLACE ([11], Lemma 1) which says that there is an open set V about x such that $V \cup VS \subset S - T$. Now since $V \cup VS$ is a right ideal, we have $V \cup VS \subset R^*$, hence $V \subset R^*$. This proves the following

Corollary 1,3. *If S is compact simple, $T \subset S$ closed, then the maximal subsemigroup T_1 of S having the same idempotents as T is closed.*

If in Lemma 1,3 the subsemigroup T contains a maximal group of S, then all group-components of T are maximal groups of S and $T = T_1$.

This combined with Theorem 1,1, Lemma 1,3 and Corollary 1,3 gives

Theorem 1,2. *If S is a compact simple semigroup without zero and T a closed subsemigroup of S, then there exists a unique greatest subsemigroup $T_1 \supset T$ having the same idempotents as T. The semigroup T_1 is closed and completely simple and it can be written in the form $T_1 = \{\bigcup_\alpha R_\alpha\} \cap \{\bigcup_\beta L_\beta\}$ with suitably chosen minimal right and left ideals R_α and L_β of S respectively. If, moreover, T contains a maximal group of S, then $T = T_1$.*

2

In this section we shall try to find analogous results for completely simple semigroups with zero.

If S is compact simple with zero, and T a closed subsemigroup of S, T need not be simple even in the case when T contains an idempotent $\neq 0$. This can be shown on the example $S = \{0, a_1, a_2, a_3, a_4\}$ with the multiplication table

	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	a_2	0	0
a_2	0	0	0	a_1	a_2
a_3	0	a_3	a_4	0	0
a_4	0	0	0	a_3	a_4

The subsemigroup $T = \{0, a_1, a_2\}$ contains the idempotent $a_1 \neq 0$ but T is not a simple semigroup since $\{0, a_2\}$ is clearly a proper two-sided ideal of T different from (0) and T.

Lemma 2,1. *Let S be a completely simple semigroup with zero 0 for which $S^2 \neq 0$. Let T be a simple subsemigroup of S containing an idempotent but not containing the zero element 0. Then*

1. *T is a completely simple semigroup. If* card $T > 1$, *T is a completely simple semigroup without zero.*

2. *If L_α is a minimal left ideal of S and $T \cap L_\alpha = L'_\alpha \neq \emptyset$, then L'_α is a minimal left ideal of T.*

3. *Conversely, if L'_α is a minimal left ideal of T, then there exists a uniquely determined minimal left ideal L_α of S such that $L'_\alpha = T \cap L_\alpha$.*

Proof. 1. If T reduces to a single element, T is completely simple. If T has more than one element, then since T is simple and contains an idempotent e, T is completely simple if e is not a zero element of T. For in this case e (being a primitive idempotent of S) is also a primitive idempotent of T and we may use the known result mentioned at the beginning of the proof of Lemma 1,1.

We show that the case that T contains at least two elements and $e = z$ is the zero element of T is impossible. Let be $a \in T$, $a \neq z$. Write S as the union of minimal left ideals $S = \bigcup_\beta L_\beta$ and suppose $a \in L_\beta$. As remarked above, L_β can be written in the form $L_\beta = \{\bigcup_\alpha G_{\alpha\beta}\} \cup P_\beta$, where $G_{\alpha\beta}$ are maximal groups of S and P_β is a semigroup with $P_\beta^2 = 0$. The element a is not contained in P_β since otherwise we would have $a^2 = 0$, $z = az = a^2z = 0$, which contradicts to $z \neq 0$. Therefore there is a group $G_{\alpha_0\beta}$ with $a \in G_{\alpha_0\beta}$. Analogously z is contained in a minimal left ideal $L_\delta = \{\bigcup_\gamma G_{\gamma\delta}\} \cup$ $\cup P_\delta$ and z cannot be contained in P_δ since otherwise we would have $z = z^2 = 0$, contrary to the assumption. Hence there is a group $G_{\gamma_0\delta}$ with $z \in G_{\gamma_0\delta}$. We have $G_{\gamma_0\delta} \cap G_{\alpha_0\beta} = \emptyset$ since $G_{\gamma_0\delta} = G_{\alpha_0\beta}$ and $z^2 = az(=z)$ would imply $a = z$. If e' is the unit element of the group $G_{\alpha_0\beta}$, $az = z$ implies $e'az = e'z$, $az = e'z$, $z = e'z$ and $za = z$ implies $zae' = ze'$, $za = ze'$, $z = ze'$; hence $e'z = ze' = z$. Since $z \neq 0$ and $e' \neq z$, this contradicts to the fact that e' is a primitive idempotent of S.

2. The proofs of the second and third assertions follow in the same lines as in Lemma 1,1.

Theorem 2,1. *Let S be a compact simple semigroup with zero 0 satisfying $S^2 \neq 0$. Let T be a closed subsemigroup of S which contains more then one element and which does not contain 0. Then T is a completely simple semigroup without zero.*

Remark. The case T has one element is trivial.

Proof. Write $S = \bigcup_{\alpha \in \Lambda_2} L_\alpha$, L_α running through all minimal left ideals of S.[1]) Let $\{L_\gamma \mid \gamma \in \Lambda'_2 \subset \Lambda_2\}$ be the set of all minimal left ideals of S for which $L'_\gamma = L_\gamma \cap T \neq \emptyset$. Since $0 \text{ non} \in T$, $L'^2_\gamma \neq (0)$ for every $\gamma \in \Lambda'_2$ and $T = S \cap T = \{\bigcup_{\alpha \in \Lambda_2} L_\alpha\} \cap T = \bigcup_{\gamma \in \Lambda'_2} L'_\gamma$. Clearly the closed subset L'_γ is again a left ideal of T.

We prove that L'_γ is a minimal left ideal of T. Suppose that there is a left ideal L of T with $L \subset L'_\gamma$. Since T is closed (hence compact), there is a minimal left ideal L^* of T with $L^* \subset L \subset L'_\gamma$ and L^* is closed. L^* contains an idempotent $e \neq 0$. The relation $e \in L^* \subset L \subset L'_\gamma \subset L_\gamma$ implies $L_\gamma e \subset L_\gamma L_\gamma = L^2_\gamma$. Since $L'^2_\gamma \neq (0)$ and $L'^2_\gamma \subset L^2_\gamma$, we have $L^2_\gamma \neq (0)$, and with respect to the minimality of L_γ we have $L^2_\gamma = L_\gamma$ and $L_\gamma e = L_\gamma$. Hence e is a right unit of L_γ, therefore $L'_\gamma e = L'_\gamma$. Since L^* is a left ideal of T, we have $L'_\gamma = L'_\gamma e \subset L'_\gamma L^* \subset L^*$, i.e. $L'_\gamma \subset L^*$, whence $L'_\gamma = L^*$. This proves that L'_γ is a minimal left ideal of T.

Now Lemma 1,2 implies that $T = \bigcup_{\gamma \in \Lambda'_2} L'_\gamma$ is a simple semigroup and hence – by Lemma 2,1 – T is a completely simple semigroup without zero.

Lemma 2,2. *Let S be a completely simple semigroup with zero 0 satisfying $S^2 \neq (0)$ and T a simple subsemigroup of S containing an idempotent but not containing the zero element 0. Then there exists a unique greatest simple subsemigroup $T_1 \supset T$ of S having (exactly) the same idempotents as T. The semigroup T_1 can be written in the form*

$$T_1 = [\{\bigcup_{\alpha \in \Lambda'_1} R_\alpha\} \cap \{\bigcap_{\beta \in \Lambda'_2} L_\beta\}] - \{0\}$$

with suitably chosen minimal right and left ideals R_α, L_β of S respectively.

Proof. By Lemma 2,1 T is completely simple (without zero if card $T > 1$) and we may write $T = \bigcup_{\alpha \in \Lambda'_1} R'_\alpha = \bigcup_{\beta \in \Lambda'_2} L'_\beta$, where R'_α, L'_β run through all minimal left and right ideals of T respectively. Choose – in the sense of Lemma 2,1 – R_α and L_β such that $R'_\alpha = R_\alpha \cap T$, $L'_\beta = L_\beta \cap T$. We have

$$T = T^2 = \{\bigcup_{\alpha \in \Lambda'_1} R'_\alpha\} \cdot \{\bigcup_{\beta \in \Lambda'_2} L'_\beta\} = \bigcup_{\alpha \in \Lambda'_1} \bigcup_{\beta \in \Lambda'_2} \{R'_\alpha L'_\beta\}.$$

Each of the $R'_\alpha . L'_\beta$'s is a group and $L'_\beta R'_\alpha = T$. Denote $R'_\alpha L'_\beta = G'_{\alpha\beta}$ and let be $e_{\alpha\beta}$ the unit element of $G'_{\alpha\beta}$.

We have $0 \text{ non} \in L'_\beta R'_\alpha$ and $L'_\beta R'_\alpha \subset L_\beta R_\alpha$, and since $L_\beta R_\alpha$ is a two-sided ideal of S, we have $L_\beta R_\alpha = S$. Further $0 \text{ non} \in R'_\alpha L'_\beta$ and $R'_\alpha L'_\beta \subset R_\alpha L_\beta$, hence $R_\alpha L_\beta \neq \{0\}$. It is known (A. H. Clifford [2], R. P. Rich [8]) that if R, L are two minimal right and left ideals of S respectively and $LR = S$, $RL \neq \{0\}$, then $RL = R \cap L$ is a group

[1]) We use the following result: A compact simple semigroup with zero is completely simple (see f.i. [12]).

with zero. Hence for $\alpha \in \Lambda'_1$, $\beta \in \Lambda'_2$, $R_\alpha L_\beta = R_\alpha \cap L_\beta$ is a group with zero (having $e_{\alpha\beta}$ as unity element). Write $R_\alpha L_\beta = G_{\alpha\beta} \cup \{0\}$. We then have

$$T = \bigcup_{\alpha \in \Lambda'_1} \bigcup_{\beta \in \Lambda'_2} \{R'_\alpha L'_\beta\} \subset \bigcup_{\alpha \in \Lambda'_1} \bigcup_{\beta \in \Lambda'_2} R_\alpha L_\beta = \bigcup_{\alpha \in \Lambda'_1} \bigcup_{\beta \in \Lambda'_2} \{R_\alpha \cap L_\beta\} = \{\bigcup_{\alpha \in \Lambda'_1} R_\alpha\} \cap \{\bigcup_{\beta \in \Lambda'_2} L_\beta\}.$$

Since T does not contain 0 we may write

$$T \subset T_1 = [\{\bigcup_{\alpha \in \Lambda'_1} R_\alpha\} \cap \{\bigcup_{\alpha \in \Lambda'_2} L_\beta\}] - \{0\},$$

and T_1 contains clearly the same idempotents as T. Moreover, T_1 is a union of maximal groups of S.

To prove that T_1 is a semigroup it is sufficient to show that $a, b \in T_1$ imply $ab \neq 0$. Let be $a \in R_\alpha \cap L_\beta$, $b \in R_\gamma \cap L_\delta$. Denote by $\bar{a}$ the element of $[R_\alpha \cap L_\beta] - \{0\} = G_{\alpha\beta}$ for which $\bar{a}a = e_{\alpha\beta}$, by $\bar{b}$ the element of $[R_\gamma \cap L_\delta] - \{0\} = G_{\gamma\delta}$ for which $b\bar{b} = e_{\gamma\delta}$. If there were $ab = 0$, we would have $\bar{a}ab\bar{b} = 0$, i.e. $e_{\alpha\beta}e_{\gamma\delta} = 0$. But $e_{\alpha\beta}$, $e_{\gamma\delta}$ are contained in the semigroup T which does not contain 0. Hence $ab \neq 0$.

The proof that T_1 is simple follows by the same argument as in Lemma 1,3.

It rests to prove the maximality property of T_1. Suppose that T_2 is a simple subsemigroup of S with $T_2 \supseteq T_1$. If T_2 contains the zero element, T_2 has an idempotent not contained in T_1. We may suppose therefore that T_2 does not contain 0. The proof then follows in the same way as in the proof of Lemma 1,3.

Lemma 2,2 together with Theorem 2,1 implies

Theorem 2,2. *If S is a compact simple semigroup with zero satisfying $S^2 \neq 0$ and T a closed subsemigroup of S which does not contain 0, then there exists a unique greatest simple subsemigroup $T_1 \supset T$ of S having exactly the same idempotents as T. The semigroup T_1 can be written in the form $T_1 = [\{\bigcup_\alpha R_\alpha\} \cap \{\bigcup_\beta L_\beta\}] - \{0\}$ with suitably chosen minimal right and left ideals R_α, L_β of S. If, moreover, T contains a maximal group of S, then $T = T_1$.*

Remark. The largest right and left ideals contained in $S - T$ are open and $A = \{\bigcup_{\alpha \in \Lambda'_1} R_\alpha\} - \{0\}$, $B = \{\bigcup_{\beta \in \Lambda'_2} L_\beta\} - \{0\}$ respectively are their complements in S. Hence $T_1 = A \cap B$ is again closed.

3

Let S be a completely simple semigroup without zero and H a simple subsemigroup of S containing all idempotents $\in S$. By Lemma 1,1 H is then completely simple.

We shall study coset decompositions of S modulo H. The possibillity of such decompositions is a priori not evident.

We shall write $S = \bigcup_{\alpha \in \Lambda_2} L_\alpha$, where L_α runs through all minimal left ideals of S and $H = \bigcup_{\alpha \in \Lambda_2} L'_\alpha$, where L'_α runs through all minimal left ideals of H. By Lemma 1,1 there is a one-to-one correspondence between L_α and L'_α such that $L'_\alpha = L_\alpha \cap H$. The decompositions $S = \bigcup_{\beta \in \Lambda_1} R_\beta$, $H = \bigcup_{\beta \in \Lambda_1} R'_\beta$ have an analogous meaning.

Denote further $R_\alpha L_\beta = R_\alpha \cap L_\beta = G_{\alpha\beta}$, $R'_\alpha L'_\beta = R'_\alpha \cap L'_\beta = G'_{\alpha\beta}$. We then have $S = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$, $H = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G'_{\alpha\beta}$; the groups $G_{\alpha\beta}$ are pairwise disjoints; further $G'_{\alpha\beta} \subset G_{\alpha\beta}$. The unit element of $G_{\alpha\beta}$ will be denoted by $e_{\alpha\beta}$.

Lemma 3,1. *If $a \in S$, then $a \in Ha$.*

Proof. Since $a \in S$, there is a group $G_{\mu\nu}$ with $a \in G_{\mu\nu}$. Hence $a = e_{\mu\nu} a \in Ha$.

Lemma 3,2. *If $L_\alpha \neq L_\beta$ and $a \in L_\alpha$, $b \in L_\beta$, then $Ha \cap Hb = \emptyset$.*

Proof. $Ha \subset SL_\alpha \subset L_\alpha$, $Hb \subset SL_\beta \subset L_\beta$ and since $L_\alpha \cap L_\beta = \emptyset$, we have $Ha \cap \cap Hb = \emptyset$.

Lemma 3,3. *If $a \in L_\alpha$, then for any $\lambda \in \Lambda_2$ we have $L'_\lambda a = L'_\alpha a$ and $L'_\alpha a = Ha$.*

Proof. The element a is contained in a minimal right ideal of S, say R_β. Then $a \in R_\beta \cap L_\alpha = G_{\beta\alpha}$ and $a = e_{\beta\alpha} a$. We have $L'_\lambda a = L'_\lambda e_{\beta\alpha} a$. The set $L'_\lambda e_{\beta\alpha}$ is contained in H and by a known theorem it is also a minimal left ideal of H. But $L'_\lambda e_{\beta\alpha} \subset L'_\lambda L_\alpha \subset \subset L_\alpha$. Hence $L'_\lambda e_{\beta\alpha} \subset L_\alpha \cap H = L'_\alpha$, and with respect to the minimality of L'_α we have $L'_\lambda e_{\beta\alpha} = L'_\alpha$. Hence $L'_\lambda a = (L'_\alpha e_{\beta\alpha})\, a = L'_\alpha a$. Moreover, $Ha = (\bigcup_{\lambda \in \Lambda_2} L'_\lambda)\, a = \bigcup_{\lambda \in \Lambda_2} (L'_\lambda a) = L'_\alpha a$. This proves our assertion.

Lemma 3,4. *If $a, b \in L_\alpha$, then either $Ha \cap Hb = \emptyset$ or $Ha = Hb$.*

Proof. Since $Ha = L'_\alpha a$, $Hb = L'_\alpha b$, it is sufficient to show that $L'_\alpha a \cap L'_\alpha b \neq \emptyset$ implies $L'_\alpha a = L'_\alpha b$.

Suppose $L'_\alpha a \cap L'_\alpha b \neq \emptyset$. Then there exist $u, v \in L'_\alpha$ such that $ua = vb$. The relation $u \in L'_\alpha = L'_\alpha \cap H = L'_\alpha \cap \{\bigcup_{\beta \in \Lambda_1} R'_\beta\} = \bigcup_{\beta \in \Lambda_1} (L'_\alpha \cap R'_\beta) = \bigcup_{\beta \in \Lambda_1} G'_{\beta\alpha}$ implies that there is a $\gamma \in \Lambda_1$ with $u \in G'_{\gamma\alpha}$. Find the element $u' \in G'_{\gamma\alpha}$ with $u'u = e_{\gamma\alpha} \in L'_\alpha$. We then have $u'ua = u'vb$, i.e. $e_{\gamma\alpha} a = u'vb$. Hence $L'_\alpha a = (L'_\alpha e_{\gamma\alpha})\, a = (L'_\alpha u'v)\, b \subset L'_\alpha L'_\alpha L'_\alpha b = L'_\alpha b$, i.e. $L'_\alpha a \subset L'_\alpha b$. By the same argument we prove $L'_\alpha b \subset L'_\alpha a$, hence $L'_\alpha a = L'_\alpha b$, q.e.d.

Since (by Lemma 3,1) $a \in Ha$, we have $S = \bigcup_{\eta \in S} H\eta$ and omitting equal summands we get a decomposition (2) $S = \bigcup_{\xi \in A} H\xi$ with pairwise disjoint summands, where ξ runs through a subset $A \subset S$.

If L_α is fixed chosen, then for any $\xi \in L_\alpha$ we have $L'_\alpha \xi \subset L'_\alpha L_\alpha \subset L_\alpha$, and since $L'_\alpha \xi = H\xi$, we conclude that L_α can be covered by cosets of the form $L'_\alpha \xi$, $\xi \in L_\alpha$. Again omitting equal terms we may write $L_\alpha = \bigcup_{\xi \in A_\alpha} L'_\alpha \xi$, where A_α is a suitably chosen subset of L_α and the summands are disjoints.

It is easy to find a complete system of the ξ's which are sufficient for the construction of such a decomposition. Write $L_\alpha = \bigcup_{\beta \in \Lambda_1} G_{\beta\alpha}$ and choose a fixed summand, say $G_{\delta\alpha}$,

$\delta \in \Lambda_1$. Construct the right coset decomposition of the group $G_{\delta\alpha}$ modulo the subgroup $G'_{\delta\alpha}$ of the form $G_{\delta\alpha} = \bigcup_{\nu \in I} G'_{\delta\alpha} \cdot \eta_{\delta\alpha}^{(\nu)}$. We then have

$$L_\alpha = \bigcup_{\beta \in \Lambda_1} G_{\beta\alpha} = \bigcup_{\beta \in \Lambda_1} e_{\beta\alpha} G_{\delta\alpha} = \bigcup_{\beta \in \Lambda_1} e_{\beta\alpha} \{ \bigcup_{\nu \in I} G'_{\delta\alpha} \eta_{\delta\alpha}^{(\nu)} \} = \bigcup_{\nu \in I} \{ \bigcup_{\beta \in \Lambda_1} e_{\beta\alpha} G'_{\delta\alpha} \eta_{\delta\alpha}^{(\nu)} \} =$$

$$= \bigcup_{\nu \in I} \{ \bigcup_{\beta \in \Lambda_1} G'_{\beta\alpha} \eta_{\delta\alpha}^{(\nu)} \} = \bigcup_{\nu \in I} L'_\alpha \eta_{\delta\alpha}^{(\nu)} .$$

Note further that since $G'_{\delta\alpha}\eta_{\delta\alpha}^{(\nu)} \neq G'_{\delta\alpha}\eta_{\delta\alpha}^{(\mu)}$ for $\mu \neq \nu$, we have $L'_\alpha \eta_{\delta\alpha}^{(\nu)} \cap L'_\alpha \eta_{\delta\alpha}^{(\mu)} = \emptyset$. Finally, since $S = \bigcup_{\alpha \in \Lambda_2} L_\alpha = \bigcup_{\alpha \in \Lambda_2} \bigcup_{\nu \in I} L'_\alpha \eta_{\delta\alpha}^{(\nu)} = \bigcup_{\alpha \in \Lambda_2} \bigcup_{\nu \in I} H \eta_{\delta\alpha}^{(\nu)}$, we see that the ξ's in (2) can be chosen in an arbitrary – but fixed – minimal right ideal R_δ of S.

Summarily we proved:

Theorem 3,1. *If* $S = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$ *is a completely simple semigroup without zero and* $H = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G'_{\alpha\beta}$ *a simple subsemigroup of* S *containing all idempotents* $\in S$, *then*:

1. *There exists a coset decomposition* $S = \bigcup_{\xi^{(\nu)} \in A} H\xi^{(\nu)}$ *into pairwise disjoin classes,* A *being a suitably chosen subset of* S.

2. A *can be chosen as a subset of an arbitrary fixed chosen minimal right ideal of* S.

3. *If* L_α *is a minimal left ideal of* S *and* $L'_\alpha = L_\alpha \cap H$, *then there exists a coset decomposition of* L_α *of the form* $L_\alpha = \bigcup_{\xi^{(\mu)} \subset A_\alpha} L'_\alpha \xi^{(\mu)}$ *with pairwise disjoint summands.*

4. *If* $G_{\delta\alpha} = \bigcup_{\nu \in I} G'_{\delta\alpha}\eta_{\delta\alpha}^{(\nu)}$ $(\delta \in \Lambda_1$ *fixed*$)$ *is the coset decomposition of the group* $G_{\delta\alpha}$ *with respect to the subgroup* $G'_{\delta\alpha}$, *we may choose* $A_\alpha = \{\eta_{\delta\alpha}^{(\nu)}, \nu \in I\}$.

In some applications (see the forthcomming paper [9]) double coset decompositions mod (H, K) are needed. We end therefore our investigations with the proof of the following theorem:

Theorem 3,2. *Let* $S = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$ *be a completely simple semigroup without zero and* H, K *two simple subsemigroups of* S *both containing all idempotents* $\in S$. *Then there exists a double coset decomposition of* S *into pairwise disjoint classes of the form* $S = \bigcup_{\xi^{(\nu)} \in B} H\xi^{(\nu)}K$, $B \subset S$. *Denoting* $G'_{\alpha\beta} = H \cap G_{\alpha\beta}$, $G''_{\alpha\beta} = K \cap G_{\alpha\beta}$, *we have* $HaK \cap G_{\alpha\beta} = G'_{\alpha\beta} a G''_{\alpha\beta}$ *for any* $a \in S$. *Further* $G'_{\alpha\beta} a G''_{\alpha\beta}$ *is exactly one class of the double coset decomposition of the group* $G_{\alpha\beta}$ *modulo* $(G'_{\alpha\beta}, G''_{\alpha\beta})$.

Proof. 1. If $a \in S$, there is a group $G_{\alpha\beta}$ with $a \in G_{\alpha\beta}$. If $e_{\alpha\beta}$ is the unit element of $G_{\alpha\beta}$, we have $a = e_{\alpha\beta} a e_{\alpha\beta} \subset HaK$. Hence $S = \bigcup_{x \in S} HxK$.

We show that if $a, b \in S$, then either $HaK = HbK$ or $HaK \cap HbK = \emptyset$. Suppose $HaK \cap HbK \neq \emptyset$. Then there exist elements $u, w \in H$, $v, z \in K$ such that

$$uav = wbz \,. \tag{3}$$

Suppose $a \in R_\alpha$, $u \in L'_\beta$ for some α and β (L'_β being a minimal left ideal of H). $e_{\alpha\beta} \in R_\alpha \cap L'_\beta$ is a right unit for every element $\in L'_\beta$ and a left unit for every element $\in R_\alpha$. Since L'_β is a minimal left ideal of H, we have $L'_\beta u = L'_\beta$, hence there exists an $u^* \in L'_\beta \subset H$ with $u^*u = e_{\alpha\beta}$. Let analogously be $a \in L_\gamma$, $v \in R''_\delta$ for some γ and δ (R''_δ being a minimal right ideal of K). The idempotent $e_{\delta\gamma} \in R''_\delta \cap L_\gamma$ is a left unit for every element $\in R''_\delta$ and a right unit for every element $\in L_\gamma$. Since R''_δ is a minimal right ideal of K, we have $vR''_\delta = R''_\delta$, hence there is an element $v^* \in R''_\delta \subset K$ such that $vv^* = e_{\delta\gamma}$. The relation (3) implies

$$u^*uavv^* = u^*wbzv^* \,,$$

$$e_{\alpha\beta}ae_{\delta\gamma} = (u^*w)\, b(zv^*) \,,$$

$$a = (u^*w)\, b(zv^*) \,.$$

We have therefore $HaK = H(u^*w)\, b(zv^*)\, K \subset HbK$. Analogously we prove $HbK \subset HaK$, hence $HaK = HbK$.

Omitting in $S = \bigcup_{x \in S} HxK$ the classes which occur more times, we get a decomposition of the required form. This proves the first statement of our theorem.

2. Consider next the set $G'_{\varrho\sigma}aG''_{\tau\omega}$. We prove that this product is independent of σ and τ (i.e. we have the same set for every couple σ, τ). Let f.i. be $a \in G_{\mu\nu}$; then $e_{\mu\nu}ae_{\mu\nu} = a$ and

$$G'_{\varrho\sigma}aG''_{\tau\omega} = (G'_{\varrho\sigma}e_{\mu\nu})\, a(e_{\mu\nu}G''_{\tau\omega}) = G_{\varrho\nu}aG''_{\mu\omega}$$

and this is clearly independent of σ and τ. Further

$$HaK = [\bigcup_{\varrho\in\Lambda_1}\bigcup_{\sigma\in\Lambda_2} G'_{\varrho\sigma}]\, a[\bigcup_{\tau\in\Lambda_1}\bigcup_{\omega\in\Lambda_2} G''_{\tau\omega}] = \bigcup_{\varrho}\bigcup_{\sigma}\bigcup_{\tau}\bigcup_{\omega} G'_{\varrho\sigma}aG''_{\tau\omega} \tag{4}$$

and

$$G'_{\varrho\sigma}aG''_{\tau\omega} \subset G_{\varrho\sigma}aG_{\tau\omega} = R_\varrho L_\sigma a R_\tau L_\omega \subset R_\varrho \cap L_\omega = G_{\varrho\omega} \,.$$

Now α, β being fixed $G_{\alpha\beta}$ contains those and only those summands of (4) which are of the form $G'_{\alpha\sigma}aG''_{\tau\beta}$. Since all these summands are identical we may choose $\sigma = \beta$, $\tau = \alpha$. Thus we have $HaK \cap G_{\alpha\beta} = G'_{\alpha\beta}aG''_{\alpha\beta}$. This proves the second assertion.

3. It is possible to caracterise the set $T_{\alpha\beta} = G'_{\alpha\beta}aG''_{\alpha\beta}$ in terms of the elements of the group $G_{\alpha\beta}$. Clearly $T_{\alpha\beta} = (G'_{\alpha\beta}e_{\alpha\beta})\, a(e_{\alpha\beta}G''_{\alpha\beta})$. Further $e_{\alpha\beta}ae_{\alpha\beta} \subset R_\alpha L_\beta a R_\alpha L_\beta \subset R_\alpha L_\beta = G_{\alpha\beta}$. Hence, denoting $a_{\alpha\beta} = e_{\alpha\beta}ae_{\alpha\beta} \in G_{\alpha\beta}$, we have $T_{\alpha\beta} = G'_{\alpha\beta}a_{\alpha\beta}G''_{\alpha\beta}$ and $a_{\alpha\beta} \in T_{\alpha\beta}$ since $e_{\alpha\beta}a_{\alpha\beta}e_{\alpha\beta} = a_{\alpha\beta}$. This says that $T_{\alpha\beta}$ is a class of the double coset decomposition of $G_{\alpha\beta}$ modulo the subgroups $(G'_{\alpha\beta}, G''_{\alpha\beta})$. Our theorem is completely proved.

References

[1] *R. H. Bruck:* A survey of binary systems. Ergebnisse der Math., Neue Folge, Vol. *20*, Springer, 1958.
[2] *A. H. Clifford:* Semigroups without nilpotent ideals. Amer. J. Math. *71* (1949), 834—844.
[3] *A. H. Clifford-G. B. Preston:* The algebraic theory of semigroups. Vol. *I.* Math. Surveys, No. 7, Amer. Math. Society, Providence, R. I., 1961.
[4] *Е. С. Ляпин:* Полугруппы. Гос. изд. физ.-мат. лит., Москва, 1960, 1—592.
[5] *K. Numakura:* On bicompact semigroups. Math. J. Okayama Univ. *1* (1952), 99—108.
[6] *G. B. Preston:* Embedding any semigroup in a *D*-simple semigroup. Trans. Amer. Math. Soc. *93* (1959), 351—355.
[7] *D. Rees:* Note on semigroups. Proc. Cambridge Philos. Soc. *37* (1941), 434—435.
[8] *R. P. Rich:* Completely simple ideals of a semigroup. Amer. J. Math. *71* (1949), 883—885.
[9] *Št. Schwarz:* Probabilities on non-commutative semigroups. Czechoslovak Math. J. (to appear).
[10] *А. К. Сушкевич:* Теория обобщенных групп. Гос. научно-тех. изд. Украины, Харьков-Киев, 1937, 1—176.
[11] *A. D. Wallace:* A note on mobs. Ann. Acad. Brasil. Ci. *24* (1952), 329—336.
[12] *W. M. Faucett, R. J. Koch, K. Numakura:* Complements of maximal ideals in compact semigroups. Duke Math. J. *22* (1955), 655—662.

Резюме

ПОДПОЛУГРУППЫ ПРОСТЫХ ПОЛУГРУПП

ШТЕФАН ШВАРЦ (Štefan Schwarz), Братислава

Целью настоящей работы является исследование строения подполугрупп простых (в особенности вполне простых) полугрупп. Приведем некоторые результаты:

Если S — вполне простая полугруппа без нуля и H — простая подполугруппа, обладающая идемпотентом, то H вполне проста. Если S — простая быкомпактная полугруппа, то всякая замкнутая подполугруппа вполне проста.

Аналогичные результаты имеют место и для вполне простых полугрупп с нулем, если в качестве H рассмотриваются подполугруппы, несодержащие нуля полугруппы S.

Существует тесная связь между односторонними идеалами полугрупп S и H.

Исследуется строение максимальных подполугрупп из S, имеющие в точности все идемпотенты данной подполугруппы H.

Пусть S — вполне проста и H, K — две подполугруппы из S, содержащие все идемпотенты $\in S$. В одделе 3 исследуется разложение полугруппы S по двойному модулю (H, K).