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ON THE GENERALIZATION OF WIARDA'S METHOD OF SOLUTION OF NON-LINEAR FUNCTIONAL EQUATIONS

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Consider the functional equation $F(y) = f$. The following theorem is proved: Let $F(y)$ be a nonlinear operator in Hilbert space H such that the Gateaux differential $F'(y)$ exists and is a symmetric operator on a closed set $E \subset H$.

Under some assumptions (see (8) and (9)), an iterative process (10) is proposed, whose convergence is of order $|y_n - y^*| \leqq k\alpha^n$ with $\alpha < 1$.

Let us denote by B a Banach space, by H a Hilbert (separable and complete) space. Let an equation

$$Ay = f \tag{1}$$

be given, with $A = I - \lambda K$, where K is a linear bounded operator and λ a real parameter. The iterative method of Wiarda is based on the following theorem.

Theorem 1. *Let K be a symmetric operator and $-(\lambda Ky, y) \geqq 0$ for every $y \in H$. Let the inequality $0 < \vartheta < 1/(1 + \|\lambda K\|)$ hold. Then the equation (1) has a unique solution y^*. The iterative process defined by*

$$y_{n+1} = (1 - \vartheta)\, y_n + \lambda\vartheta K y_n + \vartheta f$$

is covergent in the norm of H to the solution y^* and its error satisfies

$$\|y_n - y^*\| \leqq \frac{q^n}{1 - q} \|y_1 - y_0\|\,,$$

where $q = \|(1 - \vartheta) I - \vartheta\lambda K\|$ and y_0 is an arbitrary element from H.

The iterative method of Wiarda is a modification of the method of successive approximations. Its advantage is that it can be used for solving of linear equations with operators with norm not necessarily less than one. We shall generalize the theorem 1 for the solution of non-linear equations in general.

Let an equation

(2) $$F(y) = f$$

be given, where $F(y)$ is an arbitrary operator which maps B into B.

Lemma 1. *Let $F(y)$ be an arbitrary operator which maps B into B and let P be a linear bounded operator in B such that P^{-1} exists. Let the following conditions be fulfilled*:

1. *There exists a closed set $E \subset B$ and a real number $\alpha (0 < \alpha < 1)$ such that for every $u, v \in E$*

(3) $$\|R(u) - R(v)\| \leqq \alpha \|u - v\| ,$$

where $R = I - PF$.

2. *The closed sphere $\Omega(y_1, r)$, where*

$$y_1 = y_0 - PF(y_0) + Pf , \quad r = \frac{\alpha}{1 - \alpha} \|y_1 - y_0\|$$

and y_0 is an arbitrary element from E, lies in E. Then the equation (2) has a unique solution y^* in the sphere $\Omega(y_1, r)$. The sequence $\{y_n\}$ defined by

(4) $$y_{n+1} = y_n - PF(y_n) + Pf , \quad n = 0, 1, 2, \ldots$$

is convergent in the norm of B to the solution y^* of (2) and the error of the approximation y_n satisfies the inequality

$$\|y^* - y_n\| \leqq \frac{\alpha^n}{1 - \alpha} \|y_1 - y_0\| .$$

The proof of this theorem follows from Banach's theorem [3], where we put $T(y) = y - PF(y) + Pf$.

Remark 1. If $\alpha = M(R) < 1$, where

(5) $$M(R) = \sup_{\substack{u,v \in B \\ u \neq v}} \frac{\|R(u) - R(v)\|}{\|u - v\|} ,$$

then the condition (3) is fulfilled.

The equation (4) can be used to solve non-linear problems, if a real number $\alpha(0 < \alpha < 1)$ can be found such that the inequality (3) holds on the closed set $E \subset B$. One method is described in the following lemma:

Lemma 2. *Let $T(y)$ be an operator which maps H into H, which has the continuous Gateaux derivative $T'(y)$ on the closed set $E \subset H$, and let $\sup_{y \in E} \|T'(y)\| < 1$. For every $y \in E$ let $T'(y)$ be a symmetric operator in such that the inequality*

(6) $$(T'(y)\, h, h) \geqq m\|h\|^2 \; (m > 0)$$

holds for every $y \in E$ and $h \in H$. If ,

$$\widetilde{M}(I - T) = \sup_{\substack{u,v \in E \\ u \neq v}} \frac{\|T(v) - T(u) - (v - u)\|}{\|u - v\|} ,$$

then

$$\alpha = \tilde{M}(I - T) = \sup_{y \in E} \|I - T'(y)\| < 1 .$$

Proof. We have

$$\|T'(y)\| = \sup_{\|h\|=1} |(T'(y) h, h)|$$

for every $y \in E$ and $h \in H$. Now

$$(7) \qquad \|I - T'(y)\| = \sup_{\|h\|=1} |(h - T'(y) h, h)| = \sup_{\|h\|=1} |1 - (T'(y) h, h)| = \\ = \sup_{\|h\|=1} \{1 - (T'(y) h, h)\} ,$$

because $0 < (T'(y) h, h) \leqq \|T'(y)\| < 1$ for every $y \in E$ and $h \in H$ with $\|h\| = 1$. From (6) and (7), $\|I - T'(y)\| \leqq 1 - m < 1$. Therefore

$$\alpha = \sup_{y \in E} \|I - T'(y)\| \leqq 1 - m < 1 .$$

Then

$$\|T(u) - T(v) - (u - v)\| = \left\| \int_v^u (T'(y) - I) \, \mathrm{d}y \right\| \leqq \\ \leqq \int_0^{\|u-v\|} \|T'(t) - I\| \, \mathrm{d}t \leqq \alpha \|u - v\| .$$

Hence $\tilde{M}(I - T) < 1$ and this concludes the proof.

Theorem 2. *Let P be an operator with properties as in Lemma 1, and let $F(y)$ be an operator which maps H into H and has the continuous Gateaux derivative $F'(y)$ on the closed set $E \subset H$. For every $y \in H$ let $PF'(y)$ be a symmetric operator in H such that the inequalities*

$$(8) \qquad \sup_{y \in E} \|PF'(y)\| < 1$$

and

$$(9) \qquad (PF'(y) h, h) \geqq m \|h\|^2 \, (m > 0)$$

hold for every $y \in E$, $h \in H$. Put

$$(10) \qquad y_{n+1} = y_n - PF(y_n) + Pf , \quad n = 0, 1, 2, \ldots ,$$

$$\alpha = \sup_{y \in E} \|I - PF'(y)\| , \quad r = \frac{\alpha}{1 - \alpha} \|y_1 - y_0\| ,$$

where y_0 is an arbitrary element from E. Let $\Omega(y_1, r)$ be a closed sphere contained in E. Then the equation (2) has a unique solution y^* in the sphere $\Omega(y, r)$. The sequence $\{y_n\}$ defined by (10) converges in the norm of H to the solution y^* of (2) and its error satisfies

$$\|y^* - y_n\| \leqq \frac{\alpha^n}{1 - \alpha} \|y_1 - y_0\| .$$

Proof. From (8), $\sup_{y \in E} \|PF'(y)\| < 1$. Further from (9),

$$\alpha = \sup_{y \in E} \|I - PF'(y)\| = \sup_{y \in E} [\sup_{\|h\|=1} |(h - PF'(y)\, h, h)|] =$$
$$= \sup_{y \in E} [\sup_{\|h\|=1} \{1 - (PF'(y)\, h, h)\}] \leqq 1 - m < 1\,.$$

From lemmas 2, 1 we obtain our theorem.

Remark 2. From the proof it also follows that α may be replaced by α', where $\alpha < \alpha' = 1 - m < 1$.

Let an equation

$$y - \lambda\Phi(y) = f \tag{11}$$

be given, where λ is a real parameter, $\Phi(y)$ is an operator which maps B into B.

Remark 3. If we put $P = I$, then $R = \lambda\Phi$. To solve equation (11) we use the iterative formulae

$$y_{n+1} = \lambda\Phi(y_n) + f\,, \quad n = 0, 1, 2, \ldots$$

If $\alpha = M(\lambda\Phi) < 1$ (where $M(\lambda\Phi)$ is defined by equation 5), then the equation (11) has a unique solution in the sphere $\Omega(y_1, r)$, where

$$r = \frac{\alpha}{1 - \alpha} \|y_1 - y_0\|\,.$$

Theorem 3. *Let $\Phi(y)$ be an operator which maps H into H and has continuous Gateaux derivative $\Phi'(y)$ on the closed set $E \subset H$ for every $y \in H$; assume it is a symmetrical operator in H and such that the inequality $-(\lambda\Phi'(y)\, h, h) \geqq 0$ holds for every $y \in H$ and $h \in H$. Let ϑ satisfy*

$$0 < \vartheta < \frac{1}{1 + \sup_{y \in E} \|\lambda\Phi'(y)\|}\,. \tag{12}$$

Put

$$y_{n+1} = (1 - \vartheta)\, y_n + \vartheta\lambda\Phi(y_n) + \vartheta f\,, \quad n = 0, 1, 2, \ldots, \tag{13}$$

$$\alpha = \sup_{y \in E} \|I - \vartheta(I - \lambda\Phi'(y))\|\,, \quad r = \frac{\alpha}{1 - \alpha} \|y_1 - y_0\|\,.$$

Let $\Omega(y_1, r)$ be a sphere which lies in E. Then the equation (11) has a unique solution y^* in the sphere $\Omega(y_1, r)$. The sequence $\{y_n\}$ defined by (13) converges in the norm of H to the solution y^* of (11) and the error $\|y^* - y_n\|$ of the approximation y_n satisfies

$$\|y^* - y_n\| \leqq \frac{\alpha^n}{1 - \alpha} \|y_1 - y_0\|\,.$$

Proof. According to (12),

$$\sup_{y \in E} \|\vartheta(I - \lambda\Phi'(y))\| \leqq \vartheta(1 + \sup_{y \in E} \|\lambda\Phi'(y)\|) < 1$$

and

$$\alpha = \sup_{y \in E} \Big[\sup_{\|h\|=1} |1 - \vartheta(\{I - \lambda\Phi'(y)\}\, h, h)| \Big] =$$
$$= \sup_{y \in E} \Big[\sup_{\|h\|=1} \{1 - \vartheta + \vartheta(\lambda\Phi'(y)\, h, h)\} \Big] \leqq 1 - \vartheta < 1 .$$

Our theorem follows from lemmas 2, 1.

Remark 4. The number α may be replaced by α', where $\alpha < \alpha' = 1 - \vartheta < 1$.

Let an equation

$$F(y) = 0 \tag{14}$$

be given, where $F(y)$ is an operator which maps H into H. If we put $P = [F'(y_0)]^{-1}$ in (10), we obtain the Newton-Kantorowitch iterative process. We prove the following theorems.

Theorem 4. *Let $F(y)$ be an operator which maps H into H and has continuous Gateaux derivative $F'(y)$ on the closed set $E \subset H$, and let $[F'(y_0)]^{-1} = \Gamma_0$, where y_0 is an arbitrary element from E. Let $\Gamma_0 F'(y)$ be a symmetric operator in H for every $y \in H$, and such that the inequality*

$$(\Gamma_0 F'(y)\, h, h) \geqq m\|h\|^2 \; ; \quad (m > 0) \tag{15}$$

holds for every $y \in E$ and $h \in H$. Let the inequality

$$0 < \vartheta < \frac{1}{\sup_{y \in E} \|\Gamma_0 F'(y)\|} \tag{16}$$

hold for the number ϑ. Put

$$y_{n+1} = y_n - \vartheta \Gamma_0 F(y_n) , \quad n = 0, 1, 2, \ldots \tag{17}$$

$$\alpha = \sup_{y \in E} \|I - \vartheta \Gamma_0 F'(y)\| , \quad r = \frac{\alpha}{1 - \alpha} \|y_0 - y_1\| .$$

Let $\Omega(y_1, r)$ be a sphere which is contained in E. Then the equation (14) has a unique solution y^* in the sphere $\Omega(y_1, r)$. The sequence $\{y_n\}$ defined by (17) converges in the norm of H to y^* and its error satisfies

$$\|y^* - y_n\| \leqq \frac{\alpha^n}{1 - \alpha} \|y_1 - y_0\| .$$

Proof. According to (16) and (17),

$$\sup_{y \in E} \vartheta \|\Gamma_0 F'(y)\| < 1 , \quad \alpha = \sup_{y \in E} \|I - \vartheta \Gamma_0 F'(y)\| =$$
$$= \sup_{y \in E} \Big[\sup_{\|h\|=1} |1 - \vartheta(\Gamma_0 F'(y)\, h, h)| \Big] \leqq 1 - m\vartheta < 1 .$$

Remark 5. The number α in theorem 4 may be replaced by α', where $\alpha < \alpha' = 1 - m\vartheta < 1$.

Theorem 5. *Let $F(y)$ be an operator which maps B into B and such that it has Gateaux derivatives $F'(y)$, $F''(y)$ and $\Gamma_0 = [F'(y_0)]^{-1}$ on a convex closed set $E \subset B$. Let the following inequalities be fulfilled. $\|\Gamma_0 F''(y)\| \leqq K$ for every $y \in E$ and $K . d(\omega) < 1$, where $d(\omega) = \sup_{x,y \in E} \|x - y\|$.*

Define the sequence $\{y_n\}$ by

$$y_{n+1} = y_n - \Gamma_0 F(y_n)\,, \quad n = 0, 1, 2, \dots\,; \quad y_0 \in E\,. \tag{18}$$

Let $\Omega(y_1, r)$ be a sphere, where

$$r = \frac{\alpha}{1 - \alpha} \|y_1 - y_0\|\,, \quad \alpha = \sup_{y \in E} \|I - \Gamma_0 F'(y)\|\,,$$

which lies in E.

Then the equation (14) has a unique solution y^* in the sphere $\Omega(y_1, r)$. The sequence $\{y_n\}$ defined by (18) converges in the norm of B to y^* and its error satisfies

$$\|y^* - y_n\| \leqq \frac{\alpha^n}{1 - \alpha} \|y_1 - y_0\|\,.$$

Proof. Denote $S(y) = y - \Gamma_0 F(y)$. Then $S'(y) = I - \Gamma_0 F'(y)$, $S''(y) = -\Gamma_0 F''(y)$ and $S'(y_0) = 0$. We have

$$\|I - \Gamma_0 F'(y)\| = \|S'(y) - S'(y_0)\| =$$

$$= \left\|\int_{y_0}^{y} S''(t)\,\mathrm{d}t\right\| \leqq \int_0^{\|y - y_0\|} \|S''(t)\|\,\mathrm{d}t \leqq K\,d(\omega) < 1\,.$$

Hence $\sup_{y \in E} \|I - \Gamma_0 F'(y)\| < 1$. This and Lemmas 2, 1 conclude the proof.

Remark 6. The assumption of symmetricity of the operator $F'(y)$ in Theorems 2, 3, 4 may be replaced by an assumption of potentiality of the operator $F(y)$ (cf. [2] § 5, theorem 5.1).

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Резюме

ОБ ОБОБЩЕНИИ МЕТОДА ВИАРДА ДЛЯ РЕШЕНИЯ НЕЛИНЕЙНЫХ ФУНКЦИОНАЛЬНЫХ УРАВНЕНИЙ

ЙОСЕФ КОЛОМЫ (Josef Kolomý), Прага

Обобщение метода Виарда основано на следующей теореме:

Теорема 3. *Пусть оператор* $\Phi(y) \in (H \to H)$ (H — *гильбертово пространство*) *имеет на замкнутом множестве* $E \subset H$ *непрерывную производную Гато* $\Phi'(y)$. *Пусть* $\Phi'(y)$ *является симметричным оператором для всех* $y \in E$ *в* H. *Пусть выполнены следующие неравенства:*

$$0 < \vartheta < \frac{1}{1 + \sup\limits_{y \in E} \|\lambda \Phi'(y)\|}$$

и $-(\lambda \Phi'(y)\, h, h) \geqq 0$ *для всех* $y \in E$ *и* $h \in H$. *Обозначим*

$$y_{n+1} = (1 - \vartheta)\, y_n + \vartheta \lambda \Phi(y_n) + \vartheta f\,, \quad y_0 \in E\,, \quad n = 0, 1, 2, \ldots, \tag{1}$$

$$\alpha = \sup_{y \in E} \|I - \vartheta(I - \lambda \Phi'(y))\|\,, \quad r = \frac{\alpha}{1 - \alpha} \|y_1 - y_0\|\,.$$

Пусть замкнутый шар $\Omega(y_1, r) \subset E$. *Тогда уравнение* (11) *имеет единственное решение* y^* *в шаре* $\Omega(y_1, r)$. *Последовательность* $\{y_n\}$, *построеная по соотношению* (1), *сходится к* y^* *по норме* H, *и имеет место оценка:*

$$\|y^* - y_n\| \leqq \frac{\alpha^n}{1 - \alpha} \|y_1 - y_0\|\,.$$

Приведенные результаты применяются к решению вообще нелинейных функциональных уравнений модифицированным методом Ньютона-Канторовича.