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# DIRECT DECOMPOSITIONS OF LATTICES, II

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The main result of this paper is that the completion by cuts of partially ordered sets with $O$, $I$ is multiplicative; i. e. that

$$\widetilde{\mathsf{P}_A P_a} = \mathsf{P}_A \tilde{P}_a$$

where $\mathsf{P}$ denotes direct product and $\sim$ cut-completion. This is then applied to an analysis of the Glivenko-Stone theorem.

We shall, in general, use the notation of LT[1]) with some exceptions. $P$ will mean a p. o. (partially ordered) set. In $P$, $\bar{a}$ is the set of $x \leqq a$ (M-closure); $\cup$, $\cap$ and $\subset$ are set-joins, meets and inclusions, reserving $\vee$, $\wedge$, $\leqq$ for the lattice operations; $\delta$ is the Kronecker delta,

$$\delta_b^a = \begin{cases} 0 & \text{if} \quad a \neq b\,, \\ I & \text{if} \quad a = b\,; \end{cases}$$

$\tilde{P}$ is the completion by cuts of a p. o. set $P$. The direct ("cardinal" in LT) product of p. o. sets $P_a$ ($a \in A \neq \emptyset$) will be denoted by $\mathsf{P}_A P_a$; and in $P = \mathsf{P}_A P_a$ the equality sign means "is isomorphic to"; if then $x \in P$ and $[x_a]_A$ correspond, we shall write $x = [x_a]_A$ (and also use $[x_a]_{a \in A}$ or $[x_a]$ merely).

## 1. CUT-COMPLETION OF DIRECT PRODUCTS

The following lemma is easily verified:

**Lemma 1.** *Let* $x_b \equiv [x_a^b]_{a \in A} \in \mathsf{P}_A P_a$. *Then* $\bigvee_b x_b$ *exists if and only if* $\bigvee_b x_a^b$ *exists for each* $a \in A$, *where upon*

$$\bigvee_b x_b \equiv \bigvee_b [x_a^b]_a = [\bigvee_b x_a^b]_a\,;$$

*also dually.*

Let a p. o. set $P$ have extremal elements, and $P = \mathsf{P}_A P_a$; then every $P_a$ has extremal elements, so that every

$$e_a = [\delta_i^a]_{i \in A}$$

[1]) G. BIRKHOFF, Lattice Theory, 2nd. ed., New York 1948.

is in $P$ (the *central* elements — see LT, II, § 9). Then the set of all these $e_a$ generates a complete atomic Boolean subalgebra of $P$. Also, using the isomorphism of $P = \mathsf{P}_A P_a$ and lemma 1 repeatedly, we see that for any $x \in P$ there exist $x \wedge e_a$, $x \vee e'_a$, etc., in $P$, and that quite generally

**Lemma 2.** $x = \bigvee(x \wedge e_a) = \bigwedge(x \vee e'_a)$ *for all* $x \in P$.

**Lemma 3.** *If $e$ is central in $P$ and $\bigvee x_a$ exists, then*

$$e \wedge \bigvee x_a = \bigvee(e \wedge x_a)\,;$$

*also dually.*

Proof. There is a direct decomposition $P = P_1 P_2$ in which $e = [I, 0]$; let then $x_a = [x_1^a, x_2^a]$. Using lemma 1 twice,

$$e \wedge \bigvee x_a = [I, 0] \wedge \bigvee[x_1^a, x_2^a] = [I, 0] \wedge [\bigvee x_1^a, \bigvee x_2^a] =$$
$$= [\bigvee x_1^a, 0] = \bigvee[x_1^a, 0] = \bigvee([I, 0] \wedge [a_1^a, x_1^a]) = \bigvee(e \wedge x_a)\,.$$

We recall that (cf. LT, IV, §§ 5–7) $X \in \tilde{P}$ if and only if $X = X^{*+} \subset P$ ("closed" subset); also that

$$\text{l. u. b. of } X_a \text{ in } \tilde{P} = (\bigcup X_a)^{*+}\,,$$
$$\text{g. l. b. of } X_a \text{ in } \tilde{P} = \bigcap X_a$$

(all $X_a \in \tilde{P}$); finally that the injection $P \to \tilde{P}$ is $x \to \bar{x} = x^{*+}$. In a series of italicised statements we will prove our main result:

**Theorem 1.** *Let $P$ be a p. o. set with extremal elements, and $P = \mathsf{P}_A P_a$. Then $\tilde{P} = \mathsf{P}_A \tilde{P}_a$ under an extended map.*

(More explicitly, if $f$ is the isomorphism $P \to \mathsf{P}_A P_a$, and $g$ the isomorphism $\tilde{P} \to \mathsf{P}_A \tilde{P}_a$ to be constructed, then $g$ is an extension of $f$, *i. e.* $f \subset g$.)

(1) As before, form central elements $e_a = [\delta_i^a]_{i \in A}$. Using the lemma of LT, II, § 8, we may and shall identify $P_a$ with $\bar{e}_a$; and then, in $x = [x_a]_A$, the $x_a$ *is* $x \wedge e_a$.

(2) *If $X \in \tilde{P}$, then* $(\bigcup_{a \in A} (X \cap \bar{e}_a))^* \subset X^*$. For let $y \in (\bigcup_a (X \cap \bar{e}_a))^*$. Let $x \in X$, $a \in A$. Then $y \geqq x_a$, for all $a$; thus $y = [y_a] \geqq [x_a] = x$, for all $x \in X$; thus finally $y \in X^*$.

(3) If $X \in \tilde{P}$, then $(\bigcup_a (X \cap \bar{e}_a))^{*+} \supset X^{*+} \supset (\bigcup_a (X \cap \bar{e}_a))^{*+}$ — the latter inclusion is trivial. Re-phrasing, *for every* $x \in \tilde{P}$,

$$x = \bigvee_a (x \wedge e_a)\,.$$

(4) If $X \in \tilde{P}$, then $\bigcap_a (X \cup \bar{e}'_a)^{*+} \supset X^{*+} \supset \bigcap_a (X \cup \bar{e}'_a)^{*+}$ (the former inclusion is trivial). Indeed, let $y \in \bigcap_a (X \cup \bar{e}'_a)^{*+}$; *i. e.*, for every $a \in A$: $y \leqq t$ whenever $t \geqq$ all $x \in X$ and $t \geqq e'_a$. Take any $t \geqq$ all $x \in X$. Then $t \vee e'_a \geqq$ all $x \in X$ again, and $\geqq e'_a$, implying $y \leqq t \vee e'_a$, for every $a \in A$; from lemma 2 we conclude $y \leqq \bigwedge_a (t \vee e'_a) = t$, for all our $t \in X^*$, i. e. $y \in X^{*+}$. Re-phrasing, *for every* $x \in \tilde{P}$,

$$x = \bigwedge_a (x \vee e'_a)\,.$$

(5) *Each $e_a$ is central in $\tilde{P}$.* For it is complemented in $P$, in $\tilde{P}$; and applying the results of (3), (4) to the direct decomposition $P = \bar{e}_a\bar{e}'_a$ which takes $e_a$ into $[I, O]$, we see that

$$x = (x \wedge e_a) \vee (x \wedge e'_a) = (x \vee e_a) \wedge (x \vee e'_a)$$

for all $x \in \tilde{P}$, and conclude that $e_a$ is central in $\tilde{P}$.[2])

(6) Set $Q_a = \{x \wedge e_a : x \in \tilde{P}\}$, the M-closure of $e_a$ in $\tilde{P}$. Then

$$x \to [x \wedge e_a]_{a \in A}$$

is a meet-homomorphism taking $\tilde{P}$ into $\mathsf{P}_A Q_a$; this meet-homomorphism is obviously an extension of the isomorphic map $P = \mathsf{P}_A P_a$ – see (1). Choosing any $x_a \in Q_a$, we have $\bigvee x_a \to [x_a]$, since $e_a \wedge \bigvee_{b \in A} x_b = \bigvee_b (e_a \wedge x_b) = x_a$ ($e_a$ central in $\tilde{P}$, lemma 3; $e_a \wedge x_b \leqq e_b \wedge e_b = 0$ for $a \neq b$); thus the mapping is *onto* $\mathsf{P}_A Q_a$. Finally, $x \wedge e_a = y \vee e_a$ for all $a \in A$ implies $x = \bigvee(x \wedge e_a) = \bigvee(y \wedge e_a) = y$, so that the map is $1-1$. Now, a $1-1$ meet-homomorphism onto is an isomorphism (LT, II, § 5, ex. 7a), and we obtain $\tilde{P} = \mathsf{P}_A Q_a$.

(7) *If $e$ is central in $P$, $X \subset \bar{e}$* (M-closure in $P$), *then $X$ is closed in $P$ if and only if it is closed in $\bar{e}$*; i. e. $X \in \tilde{P}$ precisely when $X \in \tilde{\bar{e}}$. For let $X \subset \bar{e}$. If $y$ is such that $y \leqq t$ whenever $t \geqq$ all $x \in X$ and $t \leqq e$ (i. e. $y \in (^{*+})$-closure of $X$ in $\bar{e}$), and if $z \geqq$ all $x \in X$, then $z \wedge e \geqq$ all $x \in X$ again, so that $y \leqq z \wedge e$ by assumption, $y \leqq z$; thus $y$ is in the $(^{*+})$-closure of $X$ in $P$; the converse being obvious, we see that $(^{*+})$-closures in $\bar{e}$ and in $P$ coincide.

(8) From this we conclude $Q_a = \tilde{P}_a$. For $Q_a$ consists of $X \subset \bar{e}_a$ closed in $P$, thus in $\bar{e}_a = P_a$ also; conversely $\tilde{P}_a$ consists of $X = \bar{e}_a$ closed in $\bar{e}_a$, therefore in $P$ also. Thus finally $\tilde{P} = \mathsf{P}_A \tilde{P}_a$, q. e. d.

Thus presence of the extremal elements is a sufficient condition for $\widetilde{\mathsf{P}_A P_a} = \mathsf{P}_A \tilde{P}_a$. The converse theorem also holds, in non-trivial decompositions.[3])

**Theorem 2.** *Let $P$, $P_a$ ($a \in A$) be p. o. sets, with $A$ and all $P_a$ containing more than one element. If*

$$P = \mathsf{P}_A P_a \quad \textit{and} \quad \tilde{P} = \mathsf{P}_A \tilde{P}_a$$

*then $P$, and consequently all $P_a$ also, contains both $O$, $I$.*

Proof. Assume that $I$ non $\in P$, say. Then some $P_0$ will also have $I$ non $\in P_0$. Take any element $x \in \tilde{P} = \mathsf{P}_A \tilde{P}_a$ whose $o$-th coordinate is $I$ and other coordinates are arbitrarily fixed $x_a \in P_a$. By definition of completion by cuts, $x$ is the $(^{*+})$-closure of the set of elements $y \in P$ with $y \leqq x$ in $\tilde{P}$, *i. e.*

$$x = (\bar{x} \cap P)^{*+}.$$

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[2]) LT, II, exercise a) in § 8; $\tilde{P}$ is a lattice. Incidentally, the result of this exercise can be easily extended to the case when $L$ is merely p. o.

[3]) The motivation of Theorem 2 is LT, IV, § 7, exercise 4.

Now consider the set $\bar{x} \cap P$. It contains all elements $[y_a] \in P = \mathsf{P}_A P_a$ with $y_a \leqq x_a$ for $a \neq o$, but with quite general $y_0 \in P_0$. Then $(\bar{x} \cap P)^*$ is void, for no element of $P = \mathsf{P}_A P_a$ can have $o$-th coordinate $\geqq$ all $y_0 \in P_0$ (recall $I$ non $\in P_0$). Thus $(\bar{x} \cap P)^{*+} = P$, *i. e.* $x = I$ in $\tilde{P}$. But this cannot hold for all $x$'s of the type described, for there is more than one such; a contradiction.

## 2. AN ANALYSIS OF THE GLIVENKO-STONE THEOREM

A consequence of theorem 1 is the

**Lemma 4.** *If $P$ is a p. o. set, then every central element of $P$ remains central in $\tilde{P}$.*

For if $e$ goes into $[I, 0]$ under a decomposition $P = P_1 P_2$, then it must go into $[I, 0]$ again in the extended map taking $\tilde{P} = \tilde{P}_1 \tilde{P}_2$ (this indeed is our statement (5)).

Conversely, of course, an element of a lattice $P$ which is central in $\tilde{P}$ is only neutral in $P$; and it is not difficult to construct an example to show that it need not be central in $P$ (*i. e.*, not complemented).

**Lemma 5.** *Let $P$ be a p. o. set. If*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad in \ \tilde{P}$$

*whenever $x \in P$ but $y, z \in \tilde{P}$,*[4]*) then $\tilde{P}$ is distributive.*

Proof. Take $X$, $Y$, $Z$ in $\tilde{P}$; in any case

$$X \wedge (Y \vee Z) \geqq (X \wedge Y) \vee (X \wedge Z) \text{ in } \tilde{P}$$

($\wedge$, $\vee$ are bounds in $\tilde{P}$; however, $\wedge$ is also set-meet). Take any $u \in P$, $u \in X \wedge \wedge (Y \vee Z)$; thus $u \in X$, $u \in Y \vee Z$, and therefore $u \in \bar{u} \wedge (Y \vee Z)$. By assumption, $u \in \bar{u} \wedge (Y \vee Z) = (\bar{u} \wedge Y) \vee (\bar{u} \wedge Z) \subset (X \wedge Y) \vee (X \wedge Z)$; we conclude that also $X \wedge (Y \vee Z) \leqq (X \wedge Y) \vee (X \wedge Z)$. Thus L6′ holds in $\tilde{P}$ (LT, IX, § 1).

As a special case, we obtain the

**Lemma 6.** *If all elements of a distributive lattice $D$ are neutral in $\tilde{D}$, then $\tilde{D}$ is also distributive.*

Now take for $P$ a Boolean algebra $B$. The famous Glivenko-Stone theorem states that $\tilde{B}$ is then also Boolean. Using only the results of this paper, we have, first, that every element of $B$ is central in $\tilde{B}$ (lemma 4); therefore the condition of lemma 6 is satisfied, so that, secondly, $\tilde{B}$ is distributive. Having got thus far, one is tempted to seek conditions for complementation of $\tilde{B}$; thus showing that every element of $\tilde{B}$ is neutral and complemented, i. e. central. Surprisingly enough, this direction leads to a theorem which by itself is a new proof of the Glivenko-Stone theorem. Namely, we will show that this last is a consequence of Birkhoff's theorem 17 in LT, X, § 13.

Let $P$ be a p. o. set with $0$, $I$. We generalise trivially a definition of LT (VIII, § 8) by

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[4]) If $P$ is also a lattice, then this condition implies, and is stronger than, distributivity of $P$.

calling $P$ *orthocomplemented* if there exists a map $x \to x'$ taking $P$ into itself and such that, for all $x$, $y$ in $P$,

$$x \wedge x' = 0\,, \quad x \vee x' = I\,, \quad x = x''\,, \quad x \leqq y \quad \text{implies} \quad x' \geqq y'\,.$$

Note that $x = x''$ implies $x \to x'$ is $1-1$ onto, *i. e.* a dual automorphism, so that conversely $x' \geqq y'$ implies $x \leqq y$. In lattices we can conclude $x' \wedge y' = (x \vee y)'$ and dually; and then we may dispense with the condition $x \vee x' = I$. An orthocomplemented lattice with unique complements is a Boolean algebra (LT, X, theorem 17). But of course there are non-Boolean orthocomplemented modular lattices – see LT, VIII; possibly the simplest is in the fig. 1.

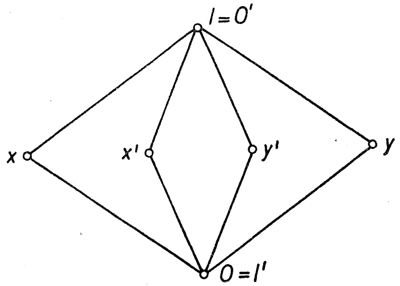


Fig. 1.

**Theorem 3.** *If $P$ is orthocomplemented, then $\tilde{P}$ is such also (under an extended dual automorphism).*

Proof. Let capitals denote elements of $\tilde{P}$, *i. e.* closed subsets of $P$; let $X'$ be the set of all $x'$ with $x \in X$, so that $X'^{*} = X^{+\prime}$, etc. (recall that $x \to x'$ is onto). We proceed to show that the map $X \to X'^{+}$ has the desired properties.

First, $X'^{+}$ is closed, since $(X'^{+})^{*+} = X^{*\prime*+} = (X^{*+})'^{+} = X'^{+}$. Similarly, the map is an extension of $x \to x'$ (interpreted in $\tilde{P}$, of course): $\bar{x}'^{+} = x^{*+\prime+} = (x')^{+*+}$, and this is readily shown to be $\overline{x'}$. Again, the map has period two, since $X'^{+\prime+} = X^{*\prime\prime+} = X^{*+} = X$. Also $X \subset Y$ implies $X' \subset Y'$, $X'^{+} \supset Y'^{+}$. Since $\tilde{P}$ is a lattice and $X \wedge X'^{+} = X \cap X'^{+} = 0$ is obvious, we conclude that $\tilde{P}$ is orthocomplemented.

**Theorem 4.** *If $B$ is a Boolean algebra, then so is $\tilde{B}$.*

For proof it suffices to show that $\tilde{B}$ has unique complements and then to apply our theorem 3 and the theorem 17 of LT, X already mentioned.

Now, if $X \wedge Y = 0$, then $x \wedge y = 0$, $y \leqq x'$, for all $x \in X$, $y \in Y$; *i. e.*, $Y \subset X'^{+}$. Conversely, $B = (X \cup Y)^{*+}$ implies $(X \cup Y)^{*} = I$; then $t \in X'^{+} \vee Y'^{+}$ implies $t \leqq$ all $x'$, all $y'$, $t' \geqq$ all $x$, all $y$, $t' \in (X \cup Y)^{*} = I$, $t = 0$. Thus we have $X'^{+} \wedge Y'^{+} = 0$; as before, this has as consequence $X'^{+} \subset Y'^{+\prime+} = Y$. We conclude that the only complement $Y$ of $X$ in $\tilde{B}$ is $X'^{+}$.

## Резюме

# ПРЯМЫЕ РАЗЛОЖЕНИЯ В СТРУКТУРАХ, II

ОТОМАР ГАЕК (Otomar Hájek), Прага

Пусть $\mathbf{P}_A P_a$ – прямое произведение системы частично упорядоченных (част. уп.) множеств $P_a$, и пусть $\tilde{P}$ обозначает пополнение част. уп. множества $P$ с помощью сечений (т. е. метод Дедекинда в част. уп. множествах). Доказываются следующие теоремы:

*Если в част. уп. множествах $P_a$ существуют экстремальные элементы $O$, $I$, то $\widetilde{\mathbf{P}_A P_a} = \mathbf{P}_A \tilde{P}_a$ при гомоморфизме, являющимся естественным продолжением разлагающево гомоморфизма $\mathbf{P}_A P_a \to P_a$.*

*Обратно, в нетривиальных разложениях, из $\widetilde{\mathbf{P}_A P_a} = \mathbf{P}_A \tilde{P}_a$ следует наличие экстремальнных элементов у всех $P_a$.*

Этот результат применяется к анализу отдельных предложений теоремы Гливенко-Стоне (пополнение булевой алгебры есть булева алгебра). Наконец, теорема Гливенко-Стоне выводится как следствие из одной теоремы Г. Биркгофа, которая является таким образом более основной.