Czechoslovak Mathematical Journal

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Addition to my paper "Generalized ordinary differential equations and continuous dependence on a parameter"

Czechoslovak Mathematical Journal, Vol. 9 (1959), No. 4, 564–573

Persistent URL: http://dml.cz/dmlcz/100382

ADDITION TO MY PAPER "GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND CONTINUOUS DEPENDENCE ON A PARAMETER"

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(Received December 8, 1958)

This paper contains an improved treatment of section 3 of [1]. It follows that the results of section 4 of [1] are valid in a more general form. Further the results of section 5,1 [1] are formulated and proved in a correct way.

We shall use definitions and notations introduced in [1].

1. We shall give new proofs of Theorems 3,1 and 3,2 [1]. In these proofs weaker assumptions concerning the monotonity properties of $\psi(\eta)$ are needed.

Let $\psi(\eta)$ be defined for $0 \leqq \eta \leqq \sigma$, $(\sigma > 0)$, $\psi(0) = 0$, $\psi(\eta) \geqq 0$. Let

$$\sum_{j=1}^{\infty} 2^j \psi\left(\frac{\eta}{2^j}\right) \tag{1}$$

converge uniformly. (This assumption is specially fulfilled if $\psi(\eta)$ is nondecreasing and $\sum_{j=1}^{\infty} 2^j \psi\left(\frac{\sigma}{2^j}\right) < \infty$.) Let us put

$$\Psi(\eta) = \sum_{j=1}^{\infty} \frac{2^j}{\eta}\, \psi\left(\frac{\eta}{2^j}\right) \quad \text{for} \quad 0 < \eta \leqq \sigma\,, \quad \Psi(0) = 0\,.$$

If $\frac{\sigma}{2^{k+1}} \leqq \eta \leqq \frac{\sigma}{2^k}$ $(k = 0, 1, 2, \ldots)$, $\zeta = 2^k\eta$, then $\Psi(\eta) = \sum_{j=k+1}^{\infty} \frac{2^j}{\zeta}\, \psi\left(\frac{\zeta}{2^j}\right)$.

As (1) convergences uniformly, it follows, that $\Psi(\eta) \to 0$ for $\eta \to 0+$.

Let $\tau_* < \tau^* \leqq \tau_* + \sigma$. Let Q be the square $\tau_* \leqq t \leqq \tau^*$, $\tau_* \leqq \tau \leqq \tau^*$.

Let $V(\tau, t)$ be defined on Q, $\tau_* \leqq \lambda_1 < \lambda_2 \leqq \tau^*$. Let us denote

$$S_i(V; \lambda_1, \lambda_2) = S_i = \sum_{j=0}^{2i-1} [V(\zeta_j, \zeta_{j+1}) - V(\zeta_j, \zeta_j)]\,, \tag{1a}$$

$$Z_i(V; \lambda_1, \lambda_2) = Z_i = \sum_{j=0}^{2i-1} [V(\zeta_{j+1}, \zeta_{j+1}) - V(\zeta_{j+1}, \zeta_j)] \tag{1b}$$

where

$$\zeta_j = \lambda_1 + \frac{j}{2^i}(\lambda_2 - \lambda_1)\,, \quad j = 0, 1, 2, \ldots, 2^i\,.$$

Lemma 1. *If* $\frac{\partial V}{\partial t} = v(\tau, t)$ *is continuous and if* $|v(\tau, t) - v(\tau', t)| \leqq C|\tau - \tau'|$ *for* $\tau, \tau', t \in \langle \tau_*, \tau^* \rangle$, *then* $\int_{\lambda_1}^{\lambda_2} \mathrm{D}V$ *exists and*

$$\int_{\lambda_1}^{\lambda_2} \mathrm{D}V = \int_{\lambda_1}^{\lambda_2} v(\tau, \tau)\, \mathrm{d}\tau\,, \tag{2}$$

$$\int_{\lambda_1}^{\lambda_2} \mathrm{D}V = \lim_{i\to\infty} S_i = \lim_{i\to\infty} Z_i\,. \tag{3}$$

Proof. Let $h(t) = \int_{\lambda_1}^{t} v(\tau, \tau)\, \mathrm{d}\tau$. As

$$|h(t) - h(\tau) - V(\tau, t) + V(\tau, \tau)| = \left| \int_\tau^t v(\xi, \xi)\, \mathrm{d}\xi - \int_\tau^t v(\tau, \xi)\, \mathrm{d}\xi \right| \leqq$$

$$\leqq \frac{C}{2}|t - \tau|^2\,,$$

it follows, that $h(t) + \varepsilon t$ (resp. $h(t) - \varepsilon t$), $\varepsilon > 0$ is an upper (lower) function of V, the integral $\int_{\lambda_1}^{\lambda_2} \mathrm{D}V$ exists and (2) holds (cf. [1], section 1,1). Further

$$\left| \int_{\lambda_1}^{\lambda_2} \mathrm{D}V - S_i \right| = \left| \sum_{j=0}^{2i-1} \left[\int_{\zeta_j}^{\zeta_{j+1}} v(\tau, \tau)\, \mathrm{d}\tau - \int_{\zeta_j}^{\zeta_{j+1}} v(\zeta_j, \tau)\, \mathrm{d}\tau \right] \right| \leqq \sum_{j=0}^{2i=1} \int_{\zeta_j}^{\zeta_{j+1}} C(\tau - \zeta_j)\, \mathrm{d}\tau =$$

$$= \frac{\mathrm{C}}{2} \frac{(\lambda_2 - \lambda_1)^2}{2^i}\,.$$

Similarly

$$\left| \int_{\lambda_1}^{\lambda_2} \mathrm{D}V - Z_i \right| \leqq \frac{C}{2} \frac{(\lambda_2 - \lambda_1)^2}{2^i}$$

and (3) holds.

Theorem 1. *Let* $U(\tau, t)$ *be defined and continuous on* Q *and let*

$$|U(\tau + \eta, t + \eta) - U(\tau + \eta, t) - U(\tau, t + \eta) + U(\tau, t)| \leqq \psi(\eta)$$

if $0 < \eta \leqq \sigma$ *and if* $(\tau + \eta, t + \eta)$, $(\tau + \eta, \tau)$, $(\tau, t + \eta)$, $(\tau, t) \in Q$. *Then* $\int_{\tau_*}^{\tau^*} \mathrm{D}U$

exists and

$$|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U - U(\lambda_1, \lambda_2) + U(\lambda_1, \lambda_1)| \leqq \tfrac{1}{2}(\lambda_2 - \lambda_1)\, \Psi(\lambda_2 - \lambda_1)\,, \tag{4}$$

$$|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U - U(\lambda_2, \lambda_2) + U(\lambda_2, \lambda_1)| \leqq \tfrac{1}{2}(\lambda_2 - \lambda_1)\, \Psi(\lambda_2 - \lambda_1) \tag{5}$$

for $\tau^* \leqq \lambda_1 < \lambda_2 \leqq \tau^*$.

Proof. By a usual approximating process we find such a sequence of functions $U_k(\tau, t)$ on Q that

i) $U_k(\tau, t)$ has continuous derivatives of the second order,

ii) $U_k(\tau, t) \to U(\tau, t)$ uniformly,

iii) for every $\vartheta > 0$ there is such a $K(\vartheta)$, that

$$|U_k(\tau + \eta, t + \eta) - U_k(\tau + \eta, t) - U_k(\tau, t + \eta) + U_k(\tau, t)| \leqq \psi(\eta)$$

if

$$k > K(\vartheta)\,,\ \tau_* + \vartheta \leqq \tau < \tau + \eta \leqq \tau^* - \vartheta\,,$$
$$\tau_* + \vartheta \leqq t < t + \eta \leqq \tau^* - \vartheta\,.$$

Let

$$\vartheta > 0\,,\quad \tau_* + \vartheta \leqq \lambda_1 < \lambda_2 \leqq \tau^* - \vartheta\,.$$

According to Lemma 1 $\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k$ exists and $\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k = \lim_{i\to\infty} S_i(U_k; \lambda_1, \lambda_2)$.

$$S_{i+1}(U_k; \lambda_1, \lambda_2) - S_i(U_k; \lambda_1, \lambda_2) =$$

$$= \sum_{j=0}^{2^i-1} [U_k(\xi_j, \xi_j + \eta) - U_k(\xi_j, \xi_j) - U_k(\xi_j - \eta, \xi_j + \eta) + U_k(\xi_j - \eta, \xi_j)]\,,$$

$$\xi_j = \lambda_1 + \frac{j}{2^i}(\lambda_2 - \lambda_1) + \frac{\lambda_2 - \lambda_1}{2^{i+1}}\,,\quad \eta = \frac{\lambda_2 - \lambda_1}{2^{i+1}}\,.$$

If $k > K(\vartheta)$, then

$$|S_{i+1}(U_k; \lambda_1, \lambda_2) - S_i(U_k; \lambda_1, \lambda_2)| \leqq 2^i \psi\left(\frac{\lambda_2 - \lambda_1}{2^{i+1}}\right)$$

and

$$\left|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k - U_k(\lambda_1, \lambda_2) + U_k(\lambda_1, \lambda_1)\right| = |\lim_{i\to\infty} S_i - S_0| \leqq$$

$$\leqq \sum_{i=0}^{\infty} |S_{i+1} - S_i| \leqq \sum_{i=0}^{\infty} 2^i \psi\left(\frac{\lambda_2 - \lambda_1}{2^{i+1}}\right) = \frac{1}{2}(\lambda_2 - \lambda_1)\, \Psi(\lambda_2 - \lambda_1)\,. \tag{6}$$

Similarly

$$|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k - U_k(\lambda_2, \lambda_2) + U_k(\lambda_2, \lambda_1)| \leqq \tfrac{1}{2}(\lambda_2 - \lambda_1)\, \Psi(\lambda_2 - \lambda_1)\,. \tag{7}$$

Let $0 < \delta \leqq \sigma$ and let $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ be such a subdivision of $\langle\lambda_1, \lambda_2\rangle$ (i. e. $\lambda_1 = \alpha_0 < \alpha_1 < \cdots < \alpha_s = \lambda_2$, $\alpha_0 \leqq \tau_1 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{s-1} \leqq \leqq \tau_s \leqq \alpha_s$) that

$$\tau_j - \alpha_{j-1} < \delta\,, \quad \alpha_j - \tau_j < \delta\,. \tag{8}$$

Let us put

$$R(V, A) = \sum_{j=1}^{s} [V(\tau_j, \alpha_j) - V(\tau_j, \alpha_{j-1})]\,.$$

Then

$$\left|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k - R(U_k, A)\right| =$$

$$= \left|\sum_{j=1}^{s}\left[\int_{\alpha_{j-1}}^{\tau_j} \mathrm{D}U_k - U_k(\tau_j, \tau_j) + U_k(\tau_j, \alpha_{j-1}) + \int_{\tau_j}^{\alpha_j} \mathrm{D}U_k - U_k(\tau_j, \alpha_j) + \right.\right.$$

$$\left.\left. + U_k(\tau_j, \tau_j)\right]\right| \leqq \sum_{j=1}^{s} \frac{1}{2}\,[(\tau_j - \alpha_{j-1})\,\Psi(\tau_j - \alpha_{j-1}) + (\alpha_j - \tau_j)\,\Psi(\alpha_j - \tau_j)] \leqq$$

$$\leqq \frac{\lambda_2 - \lambda_1}{2} \sup_{0<t\leqq\delta} \Psi(t)\,.$$

If A_1, A_2 are subdivisions of $\langle\lambda_1, \lambda_2\rangle$ which fulfil (8), then

$$|R(U_k, A_2) - R(U_k, A_1)| \leqq (\lambda_2 - \lambda_1) \sup_{0<t\leqq\delta} \Psi(t)\,. \tag{9}$$

As $U_k \to U$ it follows that $R(U_k, A_1) \to R(U, A_1)$, $R(U_k, A_2) \to R(U, A_2)$ and

$$|R(U, A_2) - R(U, A_1)| \leqq (\lambda_2 - \lambda_1) \sup_{0<t\leqq\delta} \Psi(t)\,. \tag{10}$$

Consequently $\int_{\lambda_1}^{\lambda_2} \mathrm{D}U$ exists as $\Psi(\delta) \to 0$ with $\delta \to 0$.[1])

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[1]) Summarizing the results of [1], section 1 we obtain, that $\int_{\lambda_1}^{\lambda_2} \mathrm{D}V$ exists if and only if for every $\varepsilon > 0$ there exists such a positive function $\delta(\tau)$ that

$$|R(V, A_2) - R(V, A_1)| < \varepsilon \tag{*}$$

if the subdivisions $A_1 = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$, $A_2 = \{\alpha_0', \tau_1', \alpha_1', \ldots, \tau_r', \alpha_r'\}$ of $\langle\lambda_1, \lambda_2\rangle$ fulfil the conditions

$$\begin{aligned} \tau_j - \alpha_{j-1} &< \delta(\tau_j)\,, \quad \alpha_j - \tau_j < \delta(\tau_j)\,, \quad j = 1, 2, \ldots, s\,, \\ \tau_i' - \alpha_{j-1}' &< \delta(\tau_j')\,, \quad \alpha_j' - \tau_j' < \delta(\tau_j')\,, \quad j = 1, 2, \ldots, r\,. \end{aligned} \tag{**}$$

In this case $|\int_{\lambda_1}^{\lambda_2} \mathrm{D}V - R(V, A_1)| \leqq \varepsilon$.

Let δ_1 be such a positive constant, that $(\lambda_2 - \lambda_1)\,\Psi(\delta) < \varepsilon$ for $0 < \delta < \delta_1$. We proved that (*) is fulfilled for $V = U$ (cf. (10)) if $\delta(\tau) = \delta_1$. As in this case ($\delta(\tau) = \delta_1 = \text{const}$) (**) is equivalent to the usual conditions (that the respective subdivisions A_1, A_2 are fine enough) of the Riemann theory of integration, we may say, that $\int_{\lambda_1}^{\lambda_2} \mathrm{D}U$ exists in the sense of Riemann.

It follows from (9) and (10) that

$$|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k - R(U_k, A_1)| \leqq (\lambda_2 - \lambda_1) \sup_{0<t\leqq\delta} \Psi(t) ,$$

$$|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U - R(U, A_1)| \leqq (\lambda_2 - \lambda_1) \sup_{0<t\leqq\delta} \Psi(t) .$$

As δ is arbitrary and $R(U_k, A_1) \to R(U, A_1)$, $\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k \to \int_{\lambda_1}^{\lambda_2} \mathrm{D}U$.

Passing to the limit for $k \to \infty$ in (6) and (7) we obtain (4) and (5) with the additional assumption $\tau_* < \lambda_1 < \lambda_2 < \tau^*$.

Let $\zeta \in (\tau_*, \tau^*)$. As $\int_{\zeta}^{\lambda} \mathrm{D}U$ is uniformly continuous in λ on (τ_*, τ^*) and $U(\tau, t)$ is continuous on Q, $\int_{\lambda_1}^{\lambda_2} \mathrm{D}U$ exists if $\tau_* \leqq \lambda_1 < \lambda_2 \leqq \tau^*$ (cf. Theorem 1,3,5, [1]) and (4) and (5) hold for $\tau_* \leqq \lambda_1 < \lambda_2 \leqq \tau^*$. Theorem 1 is proved.

Let S be the set of such (τ, t) that $\tau_* \leqq \tau \leqq \tau^*$, $\tau_* \leqq t \leqq \tau^*$, $|\tau - t| \leqq \sigma$.

Theorem 2. *Let the functions $U_k(\tau, t)$, $k = 0, 1, 2, \ldots$ be defined and continuous on S and let*

$$|U_k(\tau + \eta, t + \eta) - U_k(\tau + \eta, t) - U_k(\tau, t + \eta) + U_k(\tau, t)| \leqq \psi(\eta)$$

if

$$0 < \eta \leqq \sigma\,, \quad (\tau + \eta, t + \eta)\,, \ (\tau + \eta, t)\,, \ (\tau, t + \eta)\,, \ (\tau, t) \in S\,.$$

Let $U_k(\tau, t) \to U(\tau, t)$ uniformly on S with $k \to \infty$. (11)

Then

$$\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k \to \int_{\lambda_1}^{\lambda_2} \mathrm{D}U \text{ with } k \to \infty \text{ uniformly for } \tau_* \leqq \lambda_1 \leqq \lambda_2 \leqq \tau^*\,. \tag{12}$$

Proof. From (2) and (3) we obtain in a similar manner as in the proof of the preceding theorem that

$$|\int_{\lambda_1}^{\lambda_2} \mathrm{D}U_k - B(U_k, A)| \leqq (\lambda_2 - \lambda_1) \sup_{0<t\leqq\delta} \Psi(t)$$

in the subdivision A of $\langle\lambda_1, \lambda_2\rangle$ fulfils (8), $0 < \delta \leqq \sigma$ and (12) follows from (11), as $B(U_k, A) \to B(U, A)$ with $k \to \infty$ (cf. Theorem 1,3,4, [1]).

Note 1. The results of section 4, [1] (specially Theorems 4,1,1, 4,1,2, 4,2,1, Lemma 4,1,1) are valid, if we omit the assumption that $\eta^{-1}\psi(\eta)$ is nondecreasing (we assume of course, that $\sum_{j=1}^{\infty} 2^j \psi\left(\frac{\sigma}{2^j}\right) < \infty$; $\psi(\eta) = \omega_1(\eta)\,\omega_2(\eta)$ is nondecreasing, as $\omega_1(\eta)$ and $\omega_2(\eta)$ are nondecreasing).

Note 2. Suppose that the values of the function $U(\tau, t)$ $(\tau, t \in \langle\tau_*, \tau^*\rangle)$ belong tc a Banach space B. For $X \in B$ let $|X|$ be the norm of X. Let $\langle\lambda_1, \lambda_2\rangle \subset$

$\subset \langle \tau_*, \tau^* \rangle$. If $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ is a subdivision of $\langle \lambda_1, \lambda_2 \rangle$, put

$$R(U, A) = \sum_{j-1}^{s} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})] .$$

Suppose that for every $\varepsilon > 0$ there exists such a function $\delta(\tau) > 0$ that

$$|R(U, A) - R(U, A')| < \varepsilon$$

if the subdivisions $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$, $A' = \{\alpha'_0, \tau'_1\, \alpha'_1, \ldots, \tau'_r, \alpha'_r\}$ of $\langle \lambda_1, \lambda_2 \rangle$ fulfil the conditions

$$\begin{aligned} \tau_i - \alpha_{i-1} &< \delta(\tau_i) , \quad \alpha_i - \tau_i < \delta(\tau_i) , \quad i = 1, 2, \ldots, s , \\ \tau'_i - \alpha'_{i-1} &< \delta(\tau'_i) , \quad \alpha'_i - \tau'_i < \delta(\tau'_i) , \quad i = 1, 2, \ldots, r . \end{aligned} \tag{1*}$$

Then there exists such a $W \in B$ that

$$|W - R(U, A)| \leqq \varepsilon$$

if the subdivision A of $\langle \lambda_1, \lambda_2 \rangle$ fulfils (1*). In this case we define

$$\int_{\lambda_1}^{\lambda_2} \mathrm{D}U(\tau, t) = W .$$

We shall show that Theorem 1 remains valid if the values of U belong to B. Let us put

$$\overline{S}(U; \lambda_1, \lambda_2) = \lim_{i\to\infty} S_i(U; \lambda_1, \lambda_2) , \tag{2*}$$

$$\overline{Z}(U; \lambda_1, \lambda_2) = \lim_{i\to\infty} Z_i(U; \lambda_1, \lambda_2) , \tag{3*}$$

where S_i, Z_i are defined by the formulas (1a), (1b). In the same manner as we deduced (6) we obtain that the limits in (2*) and (3*) exist and that

$$|\overline{S}(U; \lambda_1, \lambda_2) - U(\lambda_1, \lambda_2) + U(\lambda_1, \lambda_1)| \leqq \tfrac{1}{2}(\lambda_2 - \lambda_1)\, \Psi(\lambda_2 - \lambda_1) , \tag{4*}$$

$$|\overline{Z}(U; \lambda_1, \lambda_2) - U(\lambda_2, \lambda_2) + U(\lambda_2, \lambda_1)| \leqq \tfrac{1}{2}(\lambda_2 - \lambda_1)\, \Psi(\lambda_2 - \lambda_1) . \tag{5*}$$

Obviously

$$\varphi S_i(U; \lambda_1, \lambda_2) = S_i(\varphi U; \lambda_1, \lambda_2) , \quad \varphi Z_i(U; \lambda_1, \lambda_2) = Z_i(\varphi U; \lambda_1, \lambda_2)$$

where φ is a linear functional on B. It follows that

$$\varphi \overline{S}(U; \lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} \mathrm{D}\varphi U(\tau, t) = \varphi \overline{Z}(U; \lambda_1, \lambda_2) .$$

Hence

$$\overline{S}(U; \lambda_1, \lambda_2) = \overline{Z}(U; \lambda_1, \lambda_2) \tag{6*}$$

and

$$\overline{S}(U; \lambda_1, \lambda_2) + \overline{S}(U; \lambda_2, \lambda_3) = \overline{S}(U; \lambda_1, \lambda_3) . \tag{7*}$$

Let $\delta > 0$ and let A be a subdivision of $\langle \lambda_1, \lambda_2 \rangle$, $\alpha_i - \tau_i < \delta$, $\tau_i - \alpha_{i-1} < \delta$, $i = 1, 2, \ldots, s$.

It follows from (4*)—(7*) that

$$|\overline{S}(U; \lambda_1, \lambda_2) - R(U, A)| \leqq (\lambda_2 - \lambda_1) \sup_{0<t\leqq\delta} \Psi(t) .$$

Consequently $\int\limits_{\lambda_1}^{\lambda_2} \mathrm{D}U(\tau, t)$ exists and

$$\int\limits_{\lambda_1}^{\lambda_2} \mathrm{D}U(\tau, t) = \overline{S}(U; \lambda_1, \lambda_2) .$$

From (4*), (5*) and (6*) we obtain (4) and (5). Theorem 1 holds.

Theorem 2 follows from Theorem 1 in the same manner as in the scalar case.

2. Some considerations in section 5,1, [1] are not correct (especially p. 444, formulas in the 3rd and 5th line from below and p. 445 formula in the 6th line from above). The aim of section 5,1, [1] was to prove that the solutions of generalized linear differential equations are unique and to establish the variation-of-constants formula. Here we shall give the correct proofs. The author intends to use the variation-of-constants formula for an investigation of linear differential equations. Therefore we shall work with general moduli of continuity ω_3, ω_4 while in section 5,1 [1] it was assumed that the respective moduli of continuity are powers of η.

Let $\omega_3(\eta)$ $\omega_4(\eta)$ be nondecreasing on $\langle 0, \sigma \rangle$, $\omega_3(\eta) \geqq c\eta$, $\omega_4(\eta) \geqq c\eta$ $(c > 0)$, $\omega_5(\eta) = \max(\omega_3(\eta), \omega_4(\eta))$, $\psi_4(\eta) = \omega_4^2(\eta)$, $\psi_5(\eta) = \omega_5(\eta)\, \omega_4(\eta)$.

Suppose that $\sum\limits_{j=1}^{\infty} 2^j \psi_5 \left(\dfrac{\sigma}{2^j}\right) < \infty$ and put $\Psi_i(\eta) = \sum\limits_{j=1}^{\infty} \dfrac{2^j}{\eta} \psi_i \left(\dfrac{\eta}{2^j}\right)$, $i = 4,5$.

Let $A(t)$, $t \in (-\infty, \infty)$ be an $n \times n$-matrix and let $B(t)$ be an n-vector and suppose that

$$\|A(t_2) - A(t_1)\| \leqq \omega_4(|t_2 - t_1|) \quad \text{for} \quad |t_2 - t_1| \leqq \sigma , \tag{13}$$

$$\|B(t_2) - B(t_1)\| \leqq \omega_3(|t_2 - t_1|) \quad \text{for} \quad |t_2 - t_1| \leqq \sigma . \tag{14}$$

Lemma 2. *Let $c \in E_n$. There exists at most one regular*[2]) *solution $x(\tau)$ of*

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[2]) Let $F(x, t) \in F(G, \omega_1, \omega_2, \sigma)$ (cf. [1], section 4,1). Let $x(\tau)$, $\tau \in \langle \tau_1, \tau_2 \rangle$ be a solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) .$$

$x(\tau)$ is a regular solution, if there is such a $\sigma' > 0$, that $\|x(\tau_3) - x(\tau_4)\| \leqq 2\omega_1(|\tau_3 - \tau_4|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle$, $|\tau_3 - \tau_4| \leqq \sigma'$. This definition is equivalent to Definition 4,2,1 [1], as the interval $\langle \tau_1, \tau_2 \rangle$ is compact.

Let $y(\tau)$ be a solution of

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \mathrm{D}[A(t)\, y + B(t)] \tag{16}$$

on an interval I [I may be closed, open or $I = (-\infty, \infty)$]. Let $\langle \tau_1, \tau_2 \rangle$ be a compact subinterval of I . $y(\tau)$ is continuous on I (cf. Theorem 1,3,6 and Definition 2,1,1 [1] and therefore there exists such a bounded open subset G of E_{n+1} [$(n + 1)$-dimensional Euclidean space], that contains all the points $(y(\tau), \tau)$ for $\tau \in \langle \tau_1, \tau_2 \rangle$. Obviously $A(t)\, y + B(t) \in F(G, K_3\omega_5(\eta), K_3\omega_4(\eta), \sigma)$ if K_3 is great enough. $y(\tau)$, $\tau \in \langle \tau_1, \tau_2 \rangle$ is regular [with respect to $F(G, K_3\omega_5(\eta), K_3\omega_4(\eta), (\sigma)]$, if there is such a $\sigma' > 0$, that $\|y(\tau_4) - y(\tau_3)\| \leqq 2K_3\omega_5(|\tau_4 - \tau_3|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle$, $|\tau_4 - \tau_3| \leqq \sigma'$. We shall say that $y(\tau)$ is a regular solution of (16), if for every compact subinterval $\langle \tau_1, \tau_2 \rangle$ of I there exist such positives K_4 and σ' that $\|y(\tau_4) - (\tau_3)\| \leqq K_4\omega_5(|\tau_4 - \tau_3|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle$, $|\tau_4 - \tau_3| \leqq \sigma'$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}A(t)\,x\,, \tag{15}$$

which fulfils $x(t_0) = c$.

In order to prove Lemma 2 we may use the proof of Lemma 5,1 [1]. At the same time Lemma 2 is a consequence of Theorem 1 [2].

Lemma 3. *The regular solutions of* (15) *are defined for* $\tau \in (-\infty, \infty)$.

Lemma 3 is a consequence of Lemma 2 and Theorem 4,2,1 [1], where we put $F_1(x, t) = F_0(x, t) = A(t)\,x$, $x_0(\tau) = 0$, $\varepsilon = 1$, $G = E(x, t;\, \|x\| < 1)$. It follows that for every $T > 0$ there is such a $\delta > 0$ that the regular solutions of (15) which fulfil $\|x(0)\| < \delta$ are defined on $\langle 0, T\rangle$ and fulfil $\|x(\tau)\| < 1$ on $\langle 0, T\rangle$. By the substitution $t' = -t$ we obtain that the solutions of (15) which fulfil $\|x(0)\| < \delta'$ are defined on $\langle -T, 0\rangle$ and fulfil $\|x(\tau)\| < 1$ on $\langle -T, 0\rangle$.

The fundamental matrix of (15) is a $n \times n$-matrix $\Phi(\tau)$, $\Phi(0) = E$[3]) every column of which is a regular solution of (15). It follows from Lemmas 2 and 3 that $\Phi(\tau)$ is defined uniquely for $\tau \in (-\infty, \infty)$.

Our aim is to establish the variation-of-constants formula for the solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A(t)\,x + B(t)]\,. \tag{16}$$

Let $A_k(t)$, $B_k(t)$ be such matrices and vectors, that $A_k(t) \to A(t)$, $B_k(t) \to B(t)$ uniformly on every bounded interval, $\frac{\mathrm{d}}{\mathrm{d}t} A_k(t) = a_k(t)$, $\frac{\mathrm{d}}{\mathrm{d}t} B_k(t) = b_k(t)$ are continuous, $A_k(t)$ fulfil (13), $B_k(t)$ fulfil (14).

As the generalized equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A_k(t)\,x + B_k(t)]$$

is equivalent to the classical equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_k(t)\,x + b_k(t)\,,$$

we have the variation-of-constants formula

$$x_k(s) = \Phi_k(s)[z + \int_0^s \Xi_k(t)\,b_k(t)\,\mathrm{d}t] =$$
$$= \Phi_k(s)[z + \int_0^s \mathrm{D}\Xi_k(\tau)\,B_k(t)]\,, \quad x_k(0) = z\,, \tag{17}$$

where $\Phi_k(\tau)$ is the fundamental matrix of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A_k(t)\,x] \tag{18}$$

and $\Xi_k(\tau) = \Phi_k^{-1}(\tau)$.

[3]) E is the unit matrix.

According to Lemma 2 and Theorem 4,2,1, [1] $\Phi_k(\tau) \to \Phi(\tau)$ uniformly on every bounded interval. As $\Xi_k^*(\tau)$[4] is the fundamental matrix of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -\mathrm{D}A_k^*(t)\,x\,, \tag{19}$$

it follows similarly, that $\Xi_k^*(\tau) \to \Xi^*(\tau)$ uniformly on every bounded interval, where $\Xi^*(\tau)$ is the fundamental matrix of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -\mathrm{D}A^*(t)\,x\,.$$

Passing to the limit for $k \to \infty$ in $\Phi_k(\tau)\Xi_k(\tau) = E$ we obtain that $\Phi(\tau)\,\Xi(\tau) = E\ (\Xi = \Xi^{**})$.

Let $T > 0$. As the columns of $\Xi_k^*(\tau)$ are regular solutions of (19), there exist such K_{5k} and σ_k' that

$$\|\Xi_k(t_2) - \Xi_k(t_1)\| \leqq K_{5k}\omega_4(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in \langle -T, T\rangle\,,\ |t_2 - t_1| \leqq \sigma_k'\,.$$

It follows that there exist such constants K_{6k}, that

$$\|\Xi_k(t_2) - \Xi_k(t_1)\| \leqq K_{6k}\omega_4(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in \langle -T, T\rangle\,,\ |t_2 - t_1| \leqq \sigma\,.$$

According to Lemma 4,1,1 [1] there exist such $\sigma^* > 0$ $(\sigma^* \leqq \sigma)$ and $L > 0$ (independent on k) that

$$\|\Xi_k(t_2) - \Xi_k(t_1)\| \leqq L\omega_4(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in \langle -T, T\rangle\,,\ |t_2 - t_1| \leqq \sigma^*\,. \tag{20}$$

[5]

Similarly

$$\|\Phi_k(t_2) - \Phi_k(t_1)\| \leqq L'\omega_4(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in \langle -T, T\rangle\,,\ |t_2 - t_1| \leqq \sigma^*\,. \tag{21}$$

Theorem 2 together with (14) and (17) implies that

$$\int_0^s \mathrm{D}\Xi_k(\tau)\,B_k(t) \to \int_0^s \mathrm{D}\Xi(\tau)\,B(t) \quad \text{uniformly with} \quad k \to \infty \quad \text{for} \quad s \in \langle -T, T\rangle\,. \tag{22}$$

Consequently the uniform limit $\lim_{k\to\infty} x_k(t) = x(t)$ exists. From (13), (14), (17), (20), (21) and Theorem 1 we obtain that

$$\Big\|\int_{t_1}^{t_2} \mathrm{D}\Xi_k(\tau)\,B_k(t)\Big\| \leqq K_7\omega_3(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in \langle -T, T\rangle\,, \quad |t_2 - t_1| \leqq \sigma^*\,,$$

$$\|x_k(t_2) - x_k(t_1)\| \leqq K_8\omega_5(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in \langle -T, T\rangle\,, \quad |t_2 - t_1| \leqq \sigma^*\,.$$

[4] Ξ_k^* is the conjugate transpose to Ξ_k.

[5] The number σ^* which occurs in Lemma 4,1,1 [1] depends only on K, G_1, ω_1, ω_2, not on the right-hand side of equation (4,1,04). Let us denote by $\xi_{kj}^*(\tau)$ $(\xi_j^*(\tau))$ the j-th column of the matrix $\Xi_k^*(\tau)$ $(\Xi^*(\tau))$. In the present case we choose such an open and bounded set G_1 that contains all the points $(\xi_{k,j}^*(\tau), \tau)$, $(\xi_j^*(\tau), \tau)$, $k = 1, 2, 3, \ldots$ $j = 1, 2, \ldots, n$, $\tau \in \langle -T, T\rangle$.

According to Theorem 4,1,1 [1] $x(\tau)$ is a solution of (16).[6]) Passing to the limit for $k \to \infty$ in (17) we obtain the required variation-of-constants formula

$$x(s) = \Phi(s)[z + \int_0^s \mathrm{D}\Xi(\tau)\, B(t)]\,, \quad x(0) = z\,. \tag{23}$$

As the difference of two regular solutions of (16) is a regular[7]) solution of (15), Lemma 2 implies that $x(\tau)$ is the only regular solution of (16), which fulfils $x(0) = z$.

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Резюме

ДОБАВЛЕНИЕ К МОЕЙ СТАТЬЕ „ОБОБЩЕННЫЕ ОБЫКНОВЕННЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ И НЕПРЕРЫВНАЯ ЗАВИСИМОСТЬ ОТ ПАРАМЕТРА“

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(Поступило в редакцию 8/XII 1959 г.)

Дается улучшенное изложение третьего параграфа статьи [1]. В новых доказательствах теорем этого параграфа употребляются более слабые предположения, касающиеся монотонности функции $\psi(\eta)$. Следовательно, в этих более общих предположениях верны и основные результаты статьи [1], содержащиеся в четвертом параграде. Далее, для обобщенных линейных уравнений выводится формула вариации постоянных (что представляет корректное изложение параграфа 5,1 статьи [1]).

[6]) $A_k(t)\, x + B_k(t) \in F(G, K_9\omega_5(\eta), \omega_4(\eta), \sigma)$ where G is a suitable bounded set and K_9 is great enough.

[7]) Cf. footnote [2]).