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NOTE ON STRONG GENERALIZED JACOBIANS

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There is a question in the Banach theory of continuous plane transformations whether the equality

$$\iint_{E_2} N(z, T)\, dz = \iint_{\mathscr{D}} |\mathscr{J}_s(w, T)|\, dw$$

holds whenever T is a sAC transformation defined in a domain $\mathscr{D} \subset E_2$ (cf. [4], p. 419, (i)); here N and $\mathscr{J}_s$ are the corresponding Banach multiplicity function and strong generalized Jacobian respectively. It is the purpose of this paper to show that the answer is negative.

I

We shall first briefly review certain aspects of so-called Banach theory of continuous plane transformations, namely the notions of (strong) bounded variation, absolute continuity and generalized Jacobian. The definition of strong bounded variation and absolute continuity was introduced by S. BANACH [1], the notion of strong generalized Jacobian is due to J. SCHAUDER. For an excellent presentation of the Banach theory and its far-going generalisation the reader should consult T. RADO's monograph [4], which will be frequently used as a reference in the sequel.

Throughout this paper E_2 is the Euclidean plane which is identified with the set of all finite complex numbers. In studying plane transformations it is convenient to imagine the plane is given in duplicate; the variable in the first and the second plane will be denoted by w and z respectively. In this sense we shall speak of the w- and z-plane.

If $\mathscr{M}$ is a Lebesgue measurable set in E_2, then $|\mathscr{M}|$ denotes its measure. For $w \in E_2$ the same symbol $|w|$ is used to denote the absolute value of the complex number w; any misunderstanding is, clearly, impossible.

The term *Jordan region* is taken to mean the closure of the interior of a simple closed curve. (A more explicit term "simply connected Jordan region", as used in [4], appears to be unnecessary for our purposes, since there are no

Jordan regions of higher connectivity met in the sequel.) If the boundary curve of a Jordan region $\mathfrak{R}$ is a simple closed polygon then $\mathfrak{R}$ is termed a *polygonal region*.

Let $\mathfrak{R}$ be a Jordan region in the w-plane and let $\mathscr{C}$ be its boundary curve which is assumed to be oriented in the counterclockwise sense. Further, let T be a continuous mapping from $\mathfrak{R}$ into the z-plane. If $z \in T(\mathscr{C})$, we put $\mu(z, T, \mathfrak{R}) = 0$. In the contrary case $\mu(z, T, \mathfrak{R})$ is set to be equal to the topological index of the point z with respect to the closed oriented curve (= path) $T(\mathscr{C})$ — the image of $\mathscr{C}$ under T. (For a precise definition see [4], IV. 1.24, II.4.34.)

To the end of this chapter $\mathscr{D}$ will stand for a bounded domain in the w-plane, T will be a continuous mapping from $\mathscr{D}$ into the z-plane. In accordance with [4], the set $T(\mathscr{D})$ is assumed to be bounded.

Let $\mathscr{I}(T, \mathscr{D})$ be the set of all $w \in \mathscr{D}$ which are isolated points of the set $T^{-1}(T(w))$ and let us define in $\mathscr{D}$ the *strong local index function* $i_s(w, T)$ as follows.

For $w \in \mathscr{D} - \mathscr{I}(T, \mathscr{D})$ put $i_s(w, T) = 0$. If $w \in \mathscr{I}(T, \mathscr{D})$, choose a Jordan region $\mathfrak{R} \subset \mathscr{D}$ such that

$$w \in \mathfrak{R}^0 , \quad \mathfrak{R} \cap T^{-1}(T(w)) = \{w\} ,^{1}) \tag{1}$$

and put $i_s(w, T) = \mu(T(w), T, \mathfrak{R})$. (It is proved in [4], IV.1.75, that $\mu(T(w),T, \mathfrak{R})$ does not depend upon the choice of $\mathfrak{R}$ fulfilling (1), so that our definition is consistent.)

The values taken by i_s in $\mathscr{D}$ are integers and, according to the Radó countability theorem, we have

$$|i_s(w, T)| \leqq 1$$

with exception of a countable set of w's in $\mathscr{D}$.

Given a point z in the z-plane and a subset $\mathscr{D}_1$ in $\mathscr{D}$, $N(z, T, \mathscr{D}_1)$ will denote the number (possibly zero or infinite) of points of the set $T^{-1}(z) \cap \mathscr{D}_1$. The function $N(z) = N(z, T, \mathscr{D})$ is non-negative and measurable (cf. [4], IV. 2.6), so that the integral

$$\iint N \tag{2}$$

(the Lebesgue integral of N over the whole z-plane) is available. In the case the integral (2) is finite the transformation T is said to be *strongly of bounded variation* (briefly: T is sBV) in $\mathscr{D}$ (cf. [4], IV. 4.1, IV. 2.13, IV. 2.11).

Let us assume to the end of chap. I that T is sBV in $\mathscr{D}$. If $S(w)$ denotes a square (i. e. a Cartesian product of two closed one-dimensional intervals of equal positive length) of centre w, then there exists a finite derivative

$$D(w, T) = \lim \frac{|T(S^0(w))|}{|S(w)|} , \quad |S(w)| \to 0 \quad (^1)$$

[1]) $\mathscr{M}^0$ is the interior of the set $\mathscr{M} \subset E_2$.
([1]) We write $S^0(w)$ instead of $(S(w))^0$.

for almost every $w \in \mathscr{D}$ (cf. [4], IV. 2.32). The function D (which is defined almost everywhere in $\mathscr{D}$) is integrable over $\mathscr{D}$ and, as proved in [4], IV. 2.33, IV. 2.34, the inequality

$$\iint_{\mathscr{D}} D \leqq \iint N \tag{3}$$

is valid.

Let us define for almost every $w \in \mathscr{D}$ the *strong generalized Jacobian* $\mathscr{J}_s(w, T)$ as follows:

$$\mathscr{J}_s(w, T) = i_s(w, T) \,.\, D(w, T)$$

(cf. [4], IV. 3.21). The function $\mathscr{J}_s$ is measurable and the inequality $|\mathscr{J}_s| \leqq D$ is satisfied almost everywhere in $\mathscr{D}$.

It is proved in [4], IV. 2.45, that in (3) the sign of equality holds if and only if the implication

$$(\mathscr{M} \subset \mathscr{D}, |\mathscr{M}| = 0) \Rightarrow |T(\mathscr{M})| = 0 \tag{4}$$

is valid (cf. also IV. 2.42). The transformation T is said to be *strongly absolutely continuous* (briefly: T is sAC) in $\mathscr{D}$, if it is sBV in $\mathscr{D}$ and if the implication (4) is fulfilled (cf. [4], IV. 4.1, IV. 2.42, IV. 2.39).

In [4], p. 419, section (i), T. Radó raised a problem whether the equality

$$\iint_{\mathscr{D}} |\mathscr{J}_s| = \iint N \tag{5}$$

is valid whenever T is sAC in $\mathscr{D}$. It will be shown the answer is negative.

Denoting by $\mathscr{I}_0(T, \mathscr{D})$ the set of all $w \in \mathscr{I}(T, \mathscr{D})$ with $i_s(w, T) = 0$ we have the following assertion (contained implicitly in [5], IV. 5. 3):

If T is sAC *in $\mathscr{D}$, the equality* (5) *appears to be equivalent to any of the following conditions* (6), (7):

$$|T(\mathscr{I}_0(T, \mathscr{D}))| = 0\,, \tag{6}$$

$$|\mathscr{J}_s(w, T)| = D(w, T) \textit{ for almost every } w \in \mathscr{D}. \tag{7}$$

Proof. The implication (7) $\Rightarrow$ (5) is obvious. (Note that the sign of equality holds in (3).) Suppose now the equality (5) is true. Then $\iint_{\mathscr{D}} |\mathscr{J}_s| = \iint_{\mathscr{D}} D$ and, since $|\mathscr{J}_s| \leqq D$ almost everywhere in $\mathscr{D}$, we have (7). From (7) we obtain $D(w, T) = 0$ for almost every $w \in \mathscr{I}_0 = \mathscr{I}_0(T, \mathscr{D})$. Hence it follows in view of the inequality

$$\iint_{\mathscr{I}_0} D \geqq |T(\mathscr{I}_0)|$$

(which is an easy consequence of the theorem IV. 2.50 in [4], where Φ is taken to mean the characteristic function of the set $\mathscr{I}_0$, so that $\iint \sigma(z, \Phi) \geqq |T(\mathscr{I}_0)|$) the relation (6).

We have so (7) $\Leftrightarrow$ (5) $\Rightarrow$ (6). The remaining implication (6) $\Rightarrow$ (7) being a consequence of IV. 3.23 in [4], our assertion is proved.

However, it is not a priori clear that the conditions (6), (7) need not be fulfilled automatically whenever T is sAC in $\mathscr{D}$. In chapter II an example (example 1) will be given of a sAC transformation T not fulfilling (5) (and, consequently, neither (6) nor (7)).

As proved in [4], IV. 3.23, the implication (6) $\Rightarrow$ (7) is valid provided T is sBV in $\mathscr{D}$. It will be shown by an example (example 2) that (7) is possible with $|T(\mathscr{I}_0(T, \mathscr{D}))| > 0$ for a sBV transformation T. (In the second case T is not sAC in $\mathscr{D}$ as it follows from the assertion stated above.)

II

In this chapter the examples mentioned in chap. I are given. The example 1 is constructed in sections 2,1—2,3, the example 2 is given in 2,4.

2,1. Put $z_1 = -1 = w_1$, $z_2 = -i = w_2$ (i is the imaginary unity), $z_3 = 1 = w_3$, $z_4 = i = w_4$ and denote by $\mathfrak{T}$ and $\mathfrak{R}$ the polygonal region in the z- and the w-plane bounded by the polygon $z_1z_2z_3z_4z_1$ and $w_1w_2w_3w_4w_1$ respectively. Let $\mathscr{S}$ be the segment z_2z_4 and choose a number ε with

$$0 < \varepsilon < |\mathfrak{R}| \,. \tag{8}$$

Then there exists a one-one continuous mapping V of $\mathscr{S}$ onto a set $V(\mathscr{S}) = \mathscr{C} \subset \mathfrak{R}$ [2]) such that

$$V(z_2) = w_2\,, \quad V(z_4) = w_4\,, \quad \mathscr{C} - \{w_2, w_4\} \subset \mathfrak{R}^0\,, \quad |\mathscr{C}| > |\mathfrak{R}| - \varepsilon\,.$$

Proof. See [3] and Fig. 1.

2,2. Let us keep the notation introduced in previous section. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the polygonal region (= triangle) of boundary $z_1z_2z_4z_1$ and $z_2z_4z_3z_2$ resp. Let $\mathfrak{R}_1$ be the Jordan region bounded by the simple closed curve which is union of w_1w_2, $\mathscr{C}$, w_4w_1. Further, let $\mathfrak{R}_2$ be the Jordan region bounded by the simple closed curve which is the union of $\mathscr{C}$, w_4w_3, w_3w_2. Put $\mathscr{S}_0 = \mathscr{S} - \{z_2, z_4\}$, $\mathscr{C}_0 = \mathscr{C} - \{w_2, w_4\}$. We have then

$$\mathfrak{T}_1 \cup \mathfrak{T}_2 = \mathfrak{T}\,, \quad \mathfrak{T}_1 \cap \mathfrak{T}_2 = \mathscr{S}\,, \quad \mathfrak{T}^0 = \mathfrak{T}_1^0 \cup \mathfrak{T}_2^0 \cup \mathscr{S}_0\,, \tag{9}$$

$$\mathfrak{R}_1 \cup \mathfrak{R}_2 = \mathfrak{R}\,, \quad \mathfrak{R}_1 \cap \mathfrak{R}_2 = \mathscr{C}\,, \quad \mathfrak{R}^0 = \mathfrak{R}_1^0 \cup \mathfrak{R}_2^0 \cup \mathscr{C}_0\,. \tag{10}$$

The mapping V (defined in $\mathscr{S}$ as yet) possesses an extension to a homeomorphism between $\mathfrak{T}$ and $\mathfrak{R}$ such that $V(z) = z$ for every z of the boundary of $\mathfrak{T}$.[3]) Moreover, the extended homeomorphism can be so chosen that the following relations (11)—(13) be fulfilled.

$$V(\mathfrak{T}_k) = \mathfrak{R}_k \quad (k = 1, 2)\,, \tag{11}$$

$$(\mathscr{N} \subset \mathfrak{T}_1^0\,, \quad |\mathscr{N}| = 0) \Rightarrow |V(\mathscr{N})| = 0\,, \tag{12}$$

$$(\mathscr{M} \subset \mathfrak{R}_2^0\,, \quad |\mathscr{M}| = 0) \Rightarrow |V^{-1}(\mathscr{M})| = 0\,. \tag{13}$$

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[2]) V is a mapping from the z-plane into the w-plane.

[3]) Here, for simplicity, no distinction is made between the points in the z- and the w-plane having the same coordinates.

Proof. The relations (9) are obvious. The proof of the relations (10) may be left to the reader (cf. [2], theorems 11.7 and 11.8, pp. 118, 119).

Put $V(z) = z$ for every point z lying on segments z_1z_2, z_4z_1. In this way we obtain a homeomorphism between the boundaries of $\mathfrak{T}_1$ and $\mathfrak{R}_1$. According to the assertion 3,6 in chapter III, this homeomorphism can be extended to such a homeomorphism V between $\mathfrak{T}_1$ and $\mathfrak{R}_1$ that the implication (12) be satisfied. Similarly defining V as the identity mapping on z_2z_3, z_3z_4, we see that the homeomorphism between the boundaries of $\mathfrak{T}_2$, $\mathfrak{R}_2$ obtained in this way possesses an extension to a homeomorphism between $\mathfrak{T}_2$, $\mathfrak{R}_2$ satisfying the implication (13). In view of (9), (10), this procedure yields a one-one continuous mapping V of $\mathfrak{T}$ onto $\mathfrak{R}$ having all the required properties.

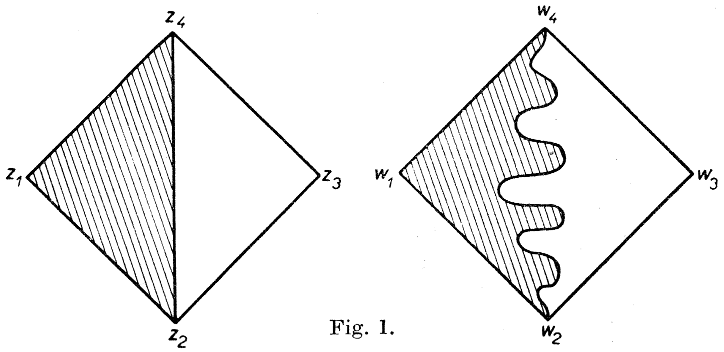

Fig. 1.

2,3. Let Z be the mapping of $\mathfrak{T}$ onto itself defined as follows:

$$Z(x + iy) = - x + iy .$$

Clearly, Z is a homeomorphism of $\mathfrak{T}$ onto itself leaving all the points of $\mathscr{S}$ unchanged. Further, $Z(\mathfrak{T}_1) = \mathfrak{T}_2$, $Z(\mathfrak{T}_2) = \mathfrak{T}_1$, $Z^{-1} = Z$.

Let us now define the mapping T of the Jordan region $\mathfrak{R}$ into itself as follows:

$$w \in \mathfrak{R} - \mathfrak{R}_2 \Rightarrow T(w) = w ,$$

$$w \in \mathfrak{R}_2 \Rightarrow T(w) = V(Z(V^{-1}(w))) .$$

We are now going to prove the following assertions:

(i) *T is a continuous mapping of $\mathfrak{R}$ onto $\mathfrak{R}_1$, leaving all the points of $\mathfrak{R}_1$ unchanged. If restricted to $\mathfrak{R}_2$, T yields a homeomorphism of $\mathfrak{R}_2$ onto $\mathfrak{R}_1$. Further we have the implications*

$$z \in \mathfrak{R}_1^0 \Rightarrow N(z, T, \mathfrak{R}^0) = 2 , \qquad N(z, T, \mathfrak{R}_1^0) = N(z, T, \mathfrak{R}_2^0) = 1 , \text{ [2])}$$

$$z \in \mathscr{C}_0 \Rightarrow N(z, T, \mathfrak{R}^0) = 1 , \quad N(z, T, \mathfrak{R}_1^0) = N(z, T, \mathfrak{R}_2^0) = 0 ,$$

$$z \in E_2 - (\mathfrak{R}_1^0 \cup \mathscr{C}_0) \Rightarrow N(z, T, \mathfrak{R}^0) = N(z, T, \mathfrak{R}_1^0) = N(z, T, \mathfrak{R}_2^0) = 0 .$$

Proof. For $w \in \mathscr{C}$ we have $V^{-1}(w) \in \mathscr{S}$ so that $V(Z(V^{-1}(w))) = V(V^{-1}(w)) = w$. We see that all the points of $\mathscr{C} \cup (\mathfrak{R} - \mathfrak{R}_2) = \mathfrak{R}_1$ (see (10)) remain unchanged under T. Further it is easy to see that T carries $\mathfrak{R}_2$ homeomorphically onto $\mathfrak{R}_1$. Since $\mathfrak{R}_1$, $\mathfrak{R}_2$ are closed sets and the mapping T is continuous in every of them, it is continuous in $\mathfrak{R}_1 \cup \mathfrak{R}_2 = \mathfrak{R}$. For $z \in \mathfrak{R}_1^0$ we have $T^{-1}(z) \cap \mathfrak{R}^0 = \{z, V(Z(V^{-1}(z)))\}$, $V(Z(V^{-1}(z))) \in \mathfrak{R}_2^0$; $z \in \mathscr{C}_0$ implies $T^{-1}(z) \cap \mathfrak{R}^0 = \{z\}$ and for $z \in E_2 - (\mathfrak{R}_1^0 \cup \mathscr{C}_0)$ we have $T^{-1}(z) \cap \mathfrak{R}^0 = \emptyset$.

(ii) $i_s(w, T) = 0$ *for every* $w \in \mathscr{C}_0$, $|i_s(w, T)| = 1$ *for every* $w \in \mathfrak{R}^0 - \mathscr{C}_0$.

Proof. It is immediately seen from (i) that $\mathscr{I}(T, \mathfrak{R}^0) = \mathfrak{R}^0$. Let w_0 be any point of $\mathscr{C}_0$ and put $z_0 = T(w_0)$. We have then $z_0 = w_0$, $T^{-1}(z_0) \cap \mathfrak{R}^0 = \{w_0\}$. Let $\mathfrak{K}$ be a closed circular disc of center w_0, $\mathfrak{K} \subset \mathfrak{R}^0$. According to the definition of the function i_s, we have the equality $i_s(w_0, T) = \mu(z_0, T, \mathfrak{K})$. Suppose, if possible, that

$$\mu(z_0, T, \mathfrak{K}) \neq 0 . \tag{14}$$

Then there exists a circular disc $\mathfrak{K}_1$ of center z_0 such that

$$z_1 \in \mathfrak{K}_1 \Rightarrow \mu(z_1, T, \mathfrak{K}) = \mu(z_0, T, \mathfrak{K}) \tag{15}$$

(cf. [4], IV. 1.25, (a), (c)). By (i) we have $T(\mathfrak{K}) \subset \mathfrak{R}_1$, so that $T(\mathfrak{K}) \cap \mathfrak{R}_2^0 = \emptyset$. Hence it follows by [4], IV. 1.25 (e), the implication

$$z_2 \in \mathfrak{R}_2^0 \Rightarrow \mu(z_2, T, \mathfrak{K}) = 0 . \tag{16}$$

Since $\mathscr{C}$ is contained in the boundary of $\mathfrak{R}_2$, we have $\mathfrak{K}_1 \cap \mathfrak{R}_2^0 \neq \emptyset$. Choose an arbitrary point $z_{12} \in \mathfrak{K}_1 \cap \mathfrak{R}_2^0$. In view of (16) we obtain the equality $\mu(z_{12}, T, \mathfrak{K}) = 0$, which contradicts (15). Therefore (14) is impossible and $i_s(w_0, T) = 0$.

Noting that T is biunique in every domain $\mathfrak{R}_1^0$, $\mathfrak{R}_2^0$, we see that $|i_s(w, T)| = 1$ for any $w \in \mathfrak{R}_1^0 \cup \mathfrak{R}_2^0 = \mathfrak{R}^0 - \mathscr{C}_0$ (cf. [4], IV. 4.52).

(iii) $D(w, T) \geqq 1$ *for almost every* $w \in \mathscr{C}_0$.

Proof. Denote by $S(w)$ the square of center w. According to the theorem (10.2) in [6], p. 129, we have a set $\widehat{\mathscr{C}} \subset \mathscr{C}_0$ with $|\mathscr{C}_0 - \widehat{\mathscr{C}}| = 0$ such that

$$(w \in \widehat{\mathscr{C}}, \quad |S(w)| \to 0) \Rightarrow \frac{|S^0(w) \cap \mathscr{C}_0|}{|S(w)|} \to 1 . \ (^1) \tag{17}$$

Since T appears to be the identity mapping in $\mathscr{C}_0 \subset \mathfrak{R}_1$, we have $T(S^0(w)) \supset T(S^0(w) \cap \mathscr{C}_0) = S^0(w) \cap \mathscr{C}_0$, $\dfrac{|T(S^0(w))|}{|S(w)|} \geqq \dfrac{|S^0(w) \cap \mathscr{C}_0|}{|S(w)|}$ for any $w \in \mathscr{C}_0$. Hence it follows by (17) the inequality $D(w, T) \geqq 1$ for almost every $w \in \widehat{\mathscr{C}}$ (and, consequently, for almost every $w \in \mathscr{C}_0$).

(iiii) *The transformation* T *is* sAC *in* $\mathfrak{R}^0$.

Proof. As it follows from (i), T is sBV in $\mathfrak{R}^0$. Therefore it is sufficient to verify the implication

$$(\mathscr{M} \subset \mathfrak{R}^0, |\mathscr{M}| = 0) \Rightarrow |T(\mathscr{M})| = 0 .$$

Let $\mathscr{M} \subset \mathfrak{R}^0$ be any set with $|\mathscr{M}| = 0$ and put $\mathscr{M}^* = \mathscr{M} - \mathfrak{R}_1$, so that $\mathscr{M}^* \subset \mathfrak{R}_2^0$, $|\mathscr{M}^*| = 0$. According to (13) we obtain $|V^{-1}(\mathscr{M}^*)| = 0$ and the set $\mathscr{N} = Z(V^{-1}(\mathscr{M}^*))$ is immediately seen to satisfy the relations

$$\mathscr{N} \subset \mathfrak{T}_1^0 , \quad |\mathscr{N}| = 0 .$$

Hence it follows by (12) $|V(\mathscr{N})| = |T(\mathscr{M}^*)| = 0$. Since $\mathfrak{R}_1$ remains unchanged under T, we have $|T(\mathscr{M} \cap \mathfrak{R}_1)| = |\mathscr{M} \cap \mathfrak{R}_1| = 0$ and the set $T(\mathscr{M}) = T(\mathscr{M} \cap \mathfrak{R}_1) \cup T(\mathscr{M}^*)$ is of measure zero.

Conclusion of example 1. As proved in (iiii), T is sAC in $\mathscr{D} = \mathfrak{R}^0$. By (ii), (iii) we obtain $\mathscr{I}_0(T, \mathscr{D}) = T(\mathscr{I}_0(T, \mathscr{D})) = \mathscr{C}_0$,[3]) $D(w, T) \neq \mathscr{J}_s(w, T)$ for almost every $w \in \mathscr{C}_0$. Since $|\mathscr{C}_0| > |\mathfrak{R}| - \varepsilon > 0$ (see 2,1), we see that neither (6) nor (7) are valid. Moreover, given $\varepsilon > 0$, the transformation T can be so chosen that the measure of the set of all $w \in \mathscr{D}$ with $D(w, T) = |\mathscr{J}_s(w, T)|$ does not exceed ε.

Clearly, T is also sAC in $\mathfrak{R}_1^0$ and in $\mathfrak{R}_2^0$, so that we have by (i)

$$\iint_{\mathfrak{R}_k^0} D = \iint N(z, T, \mathfrak{R}_k^0) = |\mathfrak{R}_1^0| , \quad k = 1, 2 .$$

Hence it follows in view of (ii)

$$\iint_{\mathscr{D}} |\mathscr{J}_s| = \iint_{\mathfrak{R}_1^0} D + \iint_{\mathfrak{R}_2^0} D = 2|\mathfrak{R}_1^0| .$$

Since $\mathfrak{R}_1^0 \subset \mathfrak{R} - \mathscr{C}$, we have $|\mathfrak{R}_1^0| \leqq |\mathfrak{R}| - |\mathscr{C}| < \varepsilon$, so that $\iint_{\mathscr{D}} |\mathscr{J}_s| < 2\varepsilon$. On the other hand, (i) implies $\iint N(z, T, \mathscr{D}) = 2|\mathfrak{R}_1^0| + |\mathscr{C}_0| > |\mathfrak{R}| - \varepsilon$. Since ε was an arbitrary number with (8), we can conclude:

To any $\varkappa > 0$ it is possible to find a sAC transformation T in $\mathscr{D}$ that the inequality

$$\iint_{\mathscr{D}} |\mathscr{J}_s(w, T)| < \varkappa \iint N(z, T, \mathscr{D})$$

be satisfied.

2,4. We shall now give the example 2 (cf. chap. I.). Let us keep the notation introduced in 2,1—2,2 and let V be the homeomorphism of $\mathfrak{T}$ onto $\mathfrak{R}$ as described in 2,2. Further, let Z be the mapping of $\mathfrak{T}$ onto itself defined in 2,3.

Let us define the mapping T^* of $\mathfrak{T}$ into $\mathfrak{R}$ as follows: $T^*(z) = V(z)$ if $z \in \mathfrak{T}_1$, $T^*(z) = V(Z(z))$ if $z \in \mathfrak{T} - \mathfrak{T}_1$. We have then the following assertions (i*), (ii*).

(i*) *T^* is a continuous mapping of the Jordan region $\mathfrak{T}$ onto the Jordan region $\mathfrak{R}_1$ which is biunique in every set $\mathfrak{T}_1$, $\mathfrak{T}_2$ itself. Further we have the implications*

$$w \in E_2 - (\mathfrak{R}_1^0 \cup \mathscr{C}_0) \Rightarrow N(w, T^*, \mathfrak{T}^0) = 0 ,^{4)}$$

[4]) T^* is a mapping from the z-plane into the w-plane. Therefore the meaning of symbols w, z is interchanged if compared with chap. I.

$$w \in \mathscr{C}_0 \Rightarrow N(w, T^*, \mathfrak{T}^0) = 1 ,$$
$$w \in \mathfrak{R}_1^0 \Rightarrow N(w, T^*, \mathfrak{T}^0) = 2 .$$

(ii*) $i_s(z, T^*) = 0$ *for* $z \in \mathscr{S}_0$, $|i_s(z, T^*)| = 1$ *for* $z \in \mathfrak{T}^0 - \mathscr{S}_0$.

The proofs are similar to those given in 2,3 and are omitted.

Conclusion of example 2. It follows directly from (i*) that T^* is s BV in $\mathfrak{T}^0 = \mathscr{D}_*$. By (ii*) we obtain $\mathscr{I}_0(T^*, \mathscr{D}_*) = \mathscr{S}_0$, $|\mathscr{J}_s(z, T^*)| = |i_s(z, T^*)| \,.\, D(z, T^*) = D(z, T^*)$ for almost every $z \in \mathscr{D}_* - \mathscr{S}_0$ (and, consequently, for almost every $z \in \mathscr{D}_*$); on the other hand, we have $|T^*(\mathscr{I}_0(T^*, \mathscr{D}_*))| = |\mathscr{C}_0| > 0$.

III

It is the purpose of this chapter to prove the assertion 3,6 which has been used in chap. II, section 2,2.

In fact, the procedure used by M. H. A. Newman [2] for the proof of the theorem 17.1, p. 169, applies also in our case and therefore we shall restrict ourselves only to several remarks.

3,1. Let $\mathfrak{P}_1$, $\mathfrak{P}_2$ be polygonal regions of boundaries $\mathscr{C}_1$, $\mathscr{C}_2$ resp. The mapping T of the polygon $\mathscr{C}_1$ onto the polygon $\mathscr{C}_2$ is said to be *quasilinear* (= q. l.), if there exists a subdivision of $\mathscr{C}_1$ into a finite number of non-overlapping[5]) segments such that T is linear[6]) in every of them. Similarly, the mapping T of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$ is termed q. l. if it is possible to subdivide $\mathfrak{P}_1$ into a finite number of non-overlapping[7]) triangles, T being linear in every of them. A q. l. mapping which appears simultaneously to be a homeomorphism is termed a *quasilinear homeomorphism* (briefly: q. l. homeomorphism). If T is a q. l. homeomorphism of $\mathscr{C}_1$ onto $\mathscr{C}_2$ (of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$), then T^{-1} is a q. l. homeomorphism of $\mathscr{C}_2$ onto $\mathscr{C}_1$ (of $\mathfrak{P}_2$ onto $\mathfrak{P}_1$). Further, let $\mathfrak{P}_3$ be a polygonal region of boundary $\mathscr{C}_3$ and let V be a q. l. homeomorphism of $\mathscr{C}_2$ onto $\mathscr{C}_3$ (of $\mathfrak{P}_2$ onto $\mathfrak{P}_3$). Then $V(T)$ is a q. l. homeomorphism of $\mathscr{C}_1$ onto $\mathscr{C}_3$ (of $\mathfrak{P}_1$ onto $\mathfrak{P}_3$).

3,2. Let $\mathfrak{Q}$ be a rectangle (i. e. a Cartesian product of two closed one-dimensional intervals of positive length) and let T be a q. l. homeomorphism of the boundary of $\mathfrak{Q}$ onto itself. Then T possesses an extension to a q. l. homeomorphism of $\mathfrak{Q}$ onto itself.

Proof. (Cf. [2], theorem 6.1, p. 146.) Let $\mathscr{S}_1, \ldots, \mathscr{S}_m$ be the subdivision of the boundary of $\mathfrak{Q}$ into non-overlapping segments, T being linear in every

[5]) The segments $\mathscr{S}_1, \ldots, \mathscr{S}_m$ are said to be non-overlapping, if $\mathscr{S}_i$, $\mathscr{S}_k$ have at most end-points in common whenever $i \neq k$.

[6]) The mapping T is termed linear in $\mathscr{M} \subset E_2$, if there exist (finite complex) constants a, b, c such that $T(u + iv) = au + bv + c$ for any $u + iv \in \mathscr{M}$.

[7]) The triangles $\mathfrak{T}_1, \ldots, \mathfrak{T}_m$ are said to be non-overlapping if $\mathfrak{T}_i{}^0 \cap \mathfrak{T}_k{}^0 = \emptyset$ whenever $i \neq k$.

$\mathscr{S}_i$. Let w_0 be the center of $\mathfrak{Q}$ and denote by $\mathfrak{T}_k$ ($k = 1, \ldots, m$) the triangle obtained as a convex envelope of w_0 and $\mathscr{S}_k$. For every k extend T to a linear mapping of $\mathfrak{T}_k$ into $\mathfrak{Q}$ such that $T(w_0) = w_0$. It is immediately seen that this procedure yields a q. l. homeomorphism of $\mathfrak{Q}$ onto itself.

3,3. Let $\mathfrak{P}$ be a polygonal region of boundary $\mathscr{C}$ and suppose that there exists a q. l. homeomorphism of $\mathfrak{P}$ onto a rectangle $\mathfrak{Q}$.[8]) Then every q. l. homeomorphism of $\mathscr{C}$ onto the boundary of $\mathfrak{Q}$ possesses an extension to a q. l. homeomorphism of $\mathfrak{P}$ onto $\mathfrak{Q}$.

Proof. (Cf. [2], p. 146, corollary.) Let T_1 be a q. l. homeomorphism of $\mathfrak{P}$ onto $\mathfrak{Q}$ and let T_2 be any q. l. homeomorphism of $\mathscr{C}$ onto the boundary of $\mathfrak{Q}$. Then $T_2(T_1^{-1})$ appears to be a q. l. homeomorphism of the boundary of $\mathfrak{Q}$ onto itself. According to 3.2, the mapping $T_2(T_1^{-1})$ can be extended to a q. l. homeomorphism T of $\mathfrak{Q}$ onto itself. The transformation $T(T_1)$ is then the required extension of T_2.

3,4. Let us keep the notation introduced in 3,3. Suppose that all the segments of the polygon $\mathscr{C}$ are parallel to the coordinate axes. Then there exists a q. l. homeomorphism of $\mathfrak{P}$ onto $\mathfrak{Q}$.

Proof. The procedure used in [2] for the proof of the theorem 6.2, pp. 146 to 148, applies in our case; every homeomorphism which is met in the course of the proof is only to be chosen as a q. l. homeomorphism.

3,5. Let $\mathfrak{S}$, $\mathfrak{Q}$ be a Jordan region and a square resp. and let T be homeomorphism of the boundary of $\mathfrak{Q}$ onto the boundary of $\mathfrak{S}$. Then T can be extended to a homeomorphism of $\mathfrak{Q}$ onto $\mathfrak{S}$ that the following implications be satisfied:

$$(\mathscr{N} \subset \mathfrak{Q}^0\,, \quad |\mathscr{N}| = 0) \Rightarrow |T(\mathscr{N})| = 0\,,$$
$$(\mathscr{M} \subset \mathfrak{S}^0\,, \quad |\mathscr{M}| = 0) \Rightarrow |T^{-1}(\mathscr{M})| = 0\,.$$

Proof. It is sufficient to modify a little the proof of the theorem 17.1 in [2], pp. 169—173. (Let us point out that our terminology and notation does not quite agree with that used in [2]; especially, the meaning of the term "Jordan region" is different.) Let us subdivide $\mathfrak{Q}^0$ into a countable system of squares $\mathfrak{Q}_1, \mathfrak{Q}_2, \ldots$ as described in the proof quoted above, p. 169 (cf. fig. 64, p. 170). The squares are numbered in conformity with fig. 64 l. c. The subdivision of $\mathfrak{Q}^0$ is imitated in $\mathfrak{S}^0$ so as to obtain a countable system of polygonal regions $\mathfrak{P}_1, \mathfrak{P}_2, \ldots$ as described in the proof just quoted, where f is changed for T. The polygonal regions are numbered so that $\mathfrak{P}_k$ corresponds to $\mathfrak{Q}_k$ (cf. p. 172, l. c.). For every k ($k = 1, 2, \ldots$) let us choose a homeomorphism $T_k = T$ of $\mathfrak{Q}_k$ onto $\mathfrak{P}_k$ that all the conditions in l. c., p. 172, be satisfied; moreover, let T be *quasilinear* in $\mathfrak{Q}_k$. Such a q. l. homeomorphism exists in view of 3,4, 3,3,

[8]) It can be proved that such a q. l. homeomorphism always exists. However, the following assertion 3.4 will be sufficient for our purposes.

since the boundary polygon of $\mathfrak{P}_k$ consists of segments parallel to the coordinate axes. It is proved in l. c. that the mapping T of $\mathfrak{Q}^0$ onto $\mathfrak{S}^0$ obtained in this way yields an extension of the prescribed homeomorphism between the boundaries of $\mathfrak{Q}$, $\mathfrak{S}$ to a homeomorphism of $\mathfrak{Q}$ onto $\mathfrak{S}$.

Let now $\mathscr{N}$ be any subset in $\mathfrak{Q}^0$, $|\mathscr{N}| = 0$. Since T is quasilinear in every square $\mathfrak{Q}_k$, we have

$$|T(\mathscr{N})| \leqq \sum_{k=1}^{\infty} |T(\mathscr{N} \cap \mathfrak{Q}_k)| = 0 .$$

As the inverse mapping T^{-1} appears to be q. l. in every $\mathfrak{P}_k$, we have similarly for any $\mathscr{M} \subset \mathfrak{S}^0$

$$|\mathscr{M}| = 0 \Rightarrow |T^{-1}(\mathscr{M})| \leqq \sum_{k=1}^{\infty} |T^{-1}(\mathscr{M} \cap \mathfrak{P}_k)| = 0 .$$

3,6. *Let $\mathfrak{S}_1$, $\mathfrak{S}_2$ be Jordan regions and let V be a homeomorphism of the boundary of $\mathfrak{S}_1$ onto the boundary of $\mathfrak{S}_2$. Then V possesses an extension to a homeomorphism between $\mathfrak{S}_1$, $\mathfrak{S}_2$ satisfying the implications*

$$(\mathscr{N} \subset \mathfrak{S}_1^0, |\mathscr{N}| = 0) \Rightarrow |V(\mathscr{N})| = 0 ,$$
$$(\mathscr{M} \subset \mathfrak{S}_2^0, |\mathscr{M}| = 0) \Rightarrow |V^{-1}(\mathscr{M})| = 0 .$$

Proof. By 3,5 there exists such a homeomorphism T_i of $\mathfrak{S}_i$ onto a square $\mathfrak{Q}_i$ ($i = 1, 2$) that the implications

$$(\mathscr{N} \subset \mathfrak{S}_i^0, |\mathscr{N}|) \Rightarrow |T_i(\mathscr{N})| = 0 ,$$
$$(\mathscr{M} \subset \mathfrak{Q}_i^0, |\mathscr{M}| = 0) \Rightarrow |T_i^{-1}(\mathscr{M})| = 0$$

are valid. The mapping $T_2(V(T_1^{-1}))$ carries the boundary of $\mathfrak{Q}_1$ homeomorphically onto the boundary of $\mathfrak{Q}_2$. In view of 3,5, the mapping $T_2(V(T_1^{-1}))$ (considered in the boundary of $\mathfrak{Q}_1$) can be extended to a homeomorphism U of $\mathfrak{Q}_1$ onto $\mathfrak{Q}_2$ such that

$$(\mathscr{M} \subset \mathfrak{Q}_1^0, |\mathscr{M}| = 0) \Rightarrow |U(\mathscr{M})| = 0 ,$$
$$(\mathscr{M} \subset \mathfrak{Q}_2^0, |\mathscr{M}| = 0) \Rightarrow |U^{-1}(\mathscr{M})| = 0 .$$

The homeomorphism $V = T_2^{-1}(U(T_1))$ is easily seen to possess all the required properties.

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Резюме

ЗАМЕТКА К СИЛЬНЫМ ОБОБЩЕННЫМ ЯКОБИАНАМ

ИОСЕФ КРАЛ (Josef Král), Прага
(Поступило в редакцию 27/VI 1958)

Пусть $\mathscr{D}$-ограниченная область на плоскости и T — ограниченное сильно абсолютно непрерывное отображение области $\mathscr{D}$ в плоскость. Обозначим символом $N(z, T)$ индикатрису Банаха и символом $\mathscr{J}_s(w, T)$ сильный обобщенный якобиан, соответствующие отображению T. (Определение этих понятий имеется в [4].) В [4], стр. 419, арт. (i) поставлен вопрос о справедливости соотношения

$$\iint_{\mathscr{D}} |\mathscr{J}_s(w, T)| \, \mathrm{d}w = \iint_{E_2} N(z, T) \, \mathrm{d}z \,. \tag{*}$$

Доказывается, что необходимым и достаточным условием для справедливости равенства (*) является требование, чтобы образ множества всех точек, в которых локальный индекс отображения T равен нулю, был множеством меры нуль (достаточность этого условия доказана в [4]).

Приводится пример, показывающий, что для каждого $\varkappa > 0$ можно построить такое сильно абсолютно непрерывное отображение открытого квадрата $\mathscr{D}$ в себя, чтобы имело место неравенство

$$\iint_{\mathscr{D}} |\mathscr{J}_s(w, T)| \, \mathrm{d}w < \varkappa \iint_{E_2} N(z, T) \, \mathrm{d}z \,.$$