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# ON THE EXISTENCE AND STABILITY OF THE PERIODIC SOLUTION OF THE SECOND KIND OF A CERTAIN MECHANICAL SYSTEM

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In this paper the existence and stability of the periodic solution of the second kind of the system

$$\ddot{x} + x = \varepsilon\, f(x, \dot{x}, \varphi, \dot{\varphi}, \varepsilon)\ ,$$
$$\ddot{\varphi} = \varepsilon\, M(\dot{\varphi}) + \varepsilon^2\, g(x, \dot{x}, \varphi, \dot{\varphi}, \varepsilon)$$

is examined, $f$ and $g$ being $2\pi$-periodic in $\varphi$. The solution is desired in the form $x = x(t, \varepsilon)$, $\varphi = \omega(\varepsilon)\, t + \varepsilon\, \Phi(t, \varepsilon)$ the functions $x(t, \varepsilon)$ and $\Phi(t, \varepsilon)$ being periodic in $t$ of period $T(\varepsilon) = \dfrac{2\pi N}{\omega(\varepsilon)}$ ($N$ being a positive integer).

To investigate the stability, a general theorem on the stability of the periodic solution of the second kind of an autonomous system has been derived.

The motion of a motor-driven mechanical system was lately investigated in papers [1] and [2], assuming that the motor cannot be considered a "hard" source of energy, i. e. that the influence of the mechanical system's motion on the motor action cannot be neglected.

In paper [1] the system

$$\ddot{x} + \omega^2 x = -\ \varepsilon\delta\dot{x} + \varepsilon\varkappa \cos \varphi\ ,$$
$$\ddot{\varphi} = \varepsilon(\mu_0 - \mu_1\dot{\varphi}) + \varepsilon^2\varkappa\varrho \sin \varphi(\cos \varphi - x)\ , \qquad (0,1)$$

($\varepsilon$, $\delta$, $\varkappa$, $\varrho$, $\mu_0$, $\mu_1$, $\omega$ being all positive constants and $\varepsilon$ being a "small" parameter) and in paper [2] the system

$$M\ddot{x} - m\varrho \sin \varphi\, \ddot{\varphi} - m\varrho \cos \varphi\, \dot{\varphi}^2 + \alpha\dot{x} + cx = 0\ ,$$
$$I\ddot{\varphi} - m\varrho \sin \varphi\, \ddot{x} + q(\dot{\varphi}) = L(\dot{\varphi}) \qquad (0,2)$$

($c$, $m$, $I$, $M$, $\alpha$, $\varrho$ being positive constants, $\alpha$, $m$, and $\varrho$ being small in comparison with $I$ and $M$, $q$ and $L$ being continuously differentiable functions) are investigated.

In both cases $x$ determines the position of the mechanical system and $\varphi$ is the motor's angle of rotation.

The equation (0,1) and (0,2) have been investigated in the quoted papers with the aid of Krylov-Bogoljubov asymptotic method.

However, the existence of a solution in the form of an asymptotic series for equations of the given kind has not yet been proved exactly and also the investigation on the stability of this solution cannot be based on any exactly established theorem. From the practical point of view using the asymptotic method is relatively easy, but in determining the accuracy with which the approximate solution approaches the exact one one cannot again rely upon any known method of estimating the error.

It has therefore been considered useful to prove the existence of the periodic solution of the second kind in the form of convergent series in $\varepsilon$ in the analytic case and, in the nonanalytic case, to put it roughly, as a continuously differentiable function of $\varepsilon$ and $t$ for $0 \leqq \varepsilon \leqq \varepsilon_0$ ($\varepsilon_0 > 0$) and $0 \leqq t < \infty$ as well as to investigate rigorously its stability. To prove the existence of the desired solution the Coddington-Levinson method [3] will be used as a basis.

Considering the relative magnitudes of the coefficients in equations (0,1) and (0,2), both systems are included in the following general form:

$$\ddot{x} + \varkappa^2 x = \varepsilon\, f(x, \dot{x}, \varphi, \dot{\varphi}, \varepsilon)\,,$$
$$\ddot{\varphi} = \varepsilon\, M(\dot{\varphi}) + \varepsilon^2\, g(x, \dot{x}, \varphi, \dot{\varphi}, \varepsilon)\,,$$

where $f$ and $g$ are $2\pi$-periodic in $\varphi$. ($M$ is the so-called characteristic of the motor.) Without loss of generality one can assume $\varkappa = 1$.

## 1. The Existence of a Periodic Solution of the Second Kind

Hence, let us consider the system

$$\ddot{x} + x = \varepsilon\, f(x, \dot{x}, \varphi, \dot{\varphi}, \varepsilon)\,,$$
$$\ddot{\varphi} = \varepsilon\, M(\dot{\varphi}) + \varepsilon^2\, g(x, \dot{x}, \varphi, \dot{\varphi}, \varepsilon)\,. \tag{1,1}$$

Let $f$ and $g$ be $2\pi$-periodic in $\varphi$ (let us assume that at least one of them depends on $\varphi$ explicitly) and let $f$, $g$ and $M$ together with their first partial derivatives with respect to the variables $x$, $\dot{x}$, $\varphi$ and $\dot{\varphi}$ be continuous in $x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$ and $\varepsilon$ for $|\varepsilon| \leqq \varepsilon_0$ ($\varepsilon_0 > 0$) and for $x$, $\dot{x}$, $\varphi$ and $\dot{\varphi}$ in some domain $G$, which will be more precisely described later. Hence, through each point of $G$ passes just one solution of the system (1,1), which is (as far as it stays in $G$) continuous in $t$ and $\varepsilon$ for $0 \leqq t < \infty$ and $|\varepsilon| \leqq \varepsilon_0$ and depends continuously on the initial conditions.

Because of the nature of the solution of (1,1) for $\varepsilon = 0$, it will be sought in the following form

$$x = x(t, \varepsilon)\,, \quad \varphi = \omega(\varepsilon)\, t + \varepsilon\, \Phi(t, \varepsilon)\,, \tag{1,2}$$

where the functions $x(t, \varepsilon)$ and $\Phi(t, \varepsilon)$ are of the same period $T(\varepsilon)$ in $t$ (i. e. the solution is a periodic vector function of the second kind). Having in mind the original physical meaning of $\omega(\varepsilon)$ only the case $\omega(\varepsilon) > 0$ will be considered, so that $\varphi(t) \to +\infty$ for $t \to +\infty$.

Let $\varphi(0, \varepsilon) = 0$; thus $\Phi(0, \varepsilon) = 0$.

Let the following variables be introduced:

$$y = -\frac{\partial x}{\partial t}, \quad \Psi = \frac{\partial \Phi}{\partial t}.$$

The system (1.1) can be rewritten in the form

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x - \varepsilon f(x, -y, \varphi, \omega + \varepsilon\Psi, \varepsilon), \\ \dot{\Psi} &= M(\omega + \varepsilon\Psi) + \varepsilon g(x, -y, \varphi, \omega + \varepsilon\Psi, \varepsilon), \\ \dot{\varphi} &= \omega + \varepsilon\Psi. \end{aligned} \tag{1,3}$$

We seek such a solution of (1,3), for which the functions $x(t, \varepsilon)$, $y(t, \varepsilon)$, $\Psi(t, \varepsilon)$ and $\Phi(t, \varepsilon) = \int_0^t \Psi(\tau, \varepsilon)\, d\tau$ are in $t$ of the same period $T(\varepsilon)$.

The functions $T(\varepsilon)$ and $\omega(\varepsilon)$ are meanwhile undetermined. Let us in the first place determine their mutual relation. Let us assume there exists a solution of the type in question, for which $x(t)$,[1] $y(t)$, $\Psi(t)$ and $\Phi(t)$ are $T$-periodic in $t$. Then the functions $f(x(t), -y(t), \varphi(t), \omega + \varepsilon\Psi(t), \varepsilon)$ and $g(x(t), -y(t), \varphi(t), \omega + \varepsilon\Psi(t), \varepsilon)$ must also be $T$-periodic in $t$.

Considering that $f$ and $g$ are $2\pi$-periodic in $\varphi$, it is natural (and almost necessary) to require

$$\varphi(t + T) - \varphi(t) = 2\pi N \quad (N \text{ being a positive integer})$$

or, considering the assumption $\Phi(t + T) = \Phi(t)$,

$$\omega(\varepsilon)\, T(\varepsilon) = 2\pi N \quad (N \text{ being a positive integer}). \tag{1,4}$$

The function $T(\varepsilon)$ and $\omega(\varepsilon)$ will be determined in the following considerations. The values of $T(0) = T_0$ and $\omega(0) = \omega_0$ can be easily determined according to the natural requirement that the period $T(\varepsilon)$ of the desired solution of the system (1,3) be continuously dependent on $\varepsilon$ for sufficiently small $\varepsilon$.

Hence, the desired solution of the system (1,3) is to converge to the solution of the following system

$$\begin{aligned} \dot{x}^{(0)} &= -y^{(0)}, & \dot{\Psi}^{(0)} &= M(\omega_0), \\ \dot{y}^{(0)} &= x^{(0)}, & \dot{\varphi}^{(0)} &= \omega_0, \end{aligned} \tag{1,3$_0$}$$

for which $x^{(0)}(t)$, $y^{(0)}(t)$, $\Psi^{(0)}(t)$ and $\Phi^{(0)}(t)$ are periodic of period $T_0 = \dfrac{2\pi N}{\omega_0}$.

[1]) For the sake of brevity the notation of dependence on $\varepsilon$ is omitted everywhere, where no misunderstanding can arise.

The general solution of the system $(1{,}3_0)$ is

$$x^{(0)}(t) = x_0^{(0)} \cos t - y_0^{(0)} \sin t\,, \quad y^{(0)}(t) = x_0^{(0)} \sin t + y_0^{(0)} \cos t\,,$$
$$\Psi^{(0)}(t) = \Psi_0^{(0)} + t\,M(\omega_0)\,, \qquad \varphi^{(0)}(t) = \varphi_0 + \omega_0 t\,,$$

while

$$\Phi^{(0)}(t) = t\Psi_0^{(0)} + \tfrac{1}{2}t^2\,M(\omega_0)\,.$$

The necessary and sufficient condition for $\Psi^{(0)}(t)$ to be $T_0$-periodic is evidently

$$T_0\,M(\omega_0) = \frac{2\pi N}{\omega_0}\,M(\omega_0) = 0\,. \tag{1,5}$$

This is the desired condition for $\omega_0$. Suppose (1,5) has at least one real positive solution.

The remaining conditions expressing $x^{(0)}(t)$, $y^{(0)}(t)$ and $\Phi^{(0)}(t)$ to be $T_0$-periodic can always be fulfilled. It is however necessary to distinguish between two cases:

1. $\omega_0 = \dfrac{N}{n}$ ($n$ being a positive integer). Then $\Psi_0^{(0)} = 0$ and $x_0^{(0)}, y_0^{(0)}$ are arbitrary constants (the resonant case).

2. $\omega_0 \neq \dfrac{N}{n}$ ($n$ being a positive integer). Then $\Psi_0^{(0)} = x_0^{(0)} = y_0^{(0)} = 0$ (the nonresonant case).

Before getting on to further computations let us introduce the following notation:

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 \\ -f \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Further we shall restrict our considerations on $\varepsilon$ in the interval $|\varepsilon| \leqq \varepsilon_1$ $(\leqq \varepsilon_0)$ $(\varepsilon_1 > 0)$ and such a domain $G_0 \subset G$ of initial conditions that all solutions starting for $t = 0$ in $G_0$ lie for $0 \leqq t \leqq T(\varepsilon)$ in $G$.

Thus, if $x, y, \varphi$ and $\Psi$ attain for $t = 0$ the values $x(0, \varepsilon)$, $y(0, \varepsilon)$, $0$ and $\Psi(0, \varepsilon)$ such that $(x(0, \varepsilon), y(0, \varepsilon), 0, \omega_0 + \varepsilon\,\Psi(0, \varepsilon)) \in G_0$ and if $|\varepsilon| \leqq \varepsilon_1$, it can easily be found with the aid of the variation-of-constants method, that the components $x, y$ and $\Psi$ of the general solution of (1,3) fulfil the following integral equations:

$$\mathbf{u}(t, \varepsilon) = e^{t\mathbf{S}}\,\mathbf{u}(0, \varepsilon) + \varepsilon \int_0^t e^{(t-s)\mathbf{S}}\,\mathbf{U}(x(s, \varepsilon), \ldots)\,\mathrm{d}s\,,$$
$$\Psi(t, \varepsilon) = \Psi(0, \varepsilon) + \int_0^t [M(\omega + \varepsilon\,\Psi(s, \varepsilon)) + \varepsilon\,g(x(s, \varepsilon), \ldots]\,\mathrm{d}s\,, \tag{1,6a}$$

and

$$\varphi(t, \varepsilon) = \omega(\varepsilon)\,t + \varepsilon\,\Phi(t, \varepsilon)\,, \tag{1,6b}$$

where

$$\Phi(t, \varepsilon) = \int_0^t \Psi(\tau, \varepsilon)\,\mathrm{d}\tau\,. \tag{1,6c}$$

Conversely, every solution of these integral equations fulfils (1,5) including the initial conditions in question.

The equations (1,6a, c) yield the necessary and sufficient conditions for functions $x$, $y$, $\Psi$ and $\Phi$ to be of period $T(\varepsilon) = \dfrac{2\pi N}{\omega(\varepsilon)}$:

$$(e^{T(\varepsilon)\mathbf{S}} - \mathbf{E})\, \mathbf{u}(0, \varepsilon) + \varepsilon \int_0^{T(\varepsilon)} e^{(T(\varepsilon)-s)\mathbf{S}}\, \mathbf{U}(x(s, \varepsilon), \ldots)\, \mathrm{d}s = 0\,,$$

$$\int_0^{T(\varepsilon)} [M(\omega(\varepsilon) + \varepsilon\, \Psi(s, \varepsilon)) + \varepsilon\, g(x(s, \varepsilon), \ldots)]\, \mathrm{d}s = 0\,, \tag{1,7}$$

$$\int_0^{T(\varepsilon)} \{\Psi(0, \varepsilon) + \int_0^t [M(\omega(\varepsilon) + \varepsilon\, \Psi(s, \varepsilon)) + \varepsilon\, g(x(s, \varepsilon), \ldots)]\, \mathrm{d}s\}\, \mathrm{d}t = 0\,.$$

To fulfil these equations there are the functions $x(0, \varepsilon)$, $y(0, \varepsilon)$, $\Psi(0, \varepsilon)$ and $\omega(\varepsilon)$ at our disposal.

In the first place let us examine the case $\omega_0 \neq \dfrac{N}{n}$. In (1,7) let $\varepsilon \to 0$. In this way we obtain the necessary conditions that $x(0, 0) = x_0$, $y(0, 0) = y_0$, $\Psi(0, 0) = \Psi_0$ and $\omega(0) = \omega_0$ must fulfil:

$$(e^{\frac{2\pi N}{\omega_0}\mathbf{S}} - \mathbf{E})\, \mathbf{u}_0 = 0\,, \quad \frac{2\pi N}{\omega_0} M(\omega_0) = 0\,, \quad \frac{2\pi N}{\omega_0} \Psi_0 = 0\,. \tag{1,7$_0$}$$

As according to assumption $M(\omega_0) = 0$ has at least one real positive root $\omega_0^*$, the system (1,$7_0$) has a real solution $x_0 = x_0^* = 0$, $y_0 = y_0^* = 0$, $\Psi_0 = \Psi_0^* = 0$, $\omega_0 = \omega_0^*$. If the Jacobian of (1,$7_0$) with respect to $x_0$, $y_0$, $\Psi_0$ and $\omega_0$ is nonvanishing at the point $(0, 0, 0, \omega_0^*)$, the sufficient conditions for the existence of a solution of (1,7) can easily be obtained by the theorem on implicit functions.

Thus the following theorem will be proved:

**Theorem 1.1.** *Let*

(a) *$f$ and $g$ be $2\pi$-periodic in the variable $\varphi$;*

(b) *the equation $M(\omega_0) = 0$ have at least one real positive root $\omega_0^* \neq \dfrac{N}{n}$ ($N$ and $n$ being natural numbers);*

(c) *for $|\varepsilon| \leqq \varepsilon_0$ ($\varepsilon_0 > 0$) and for $x$, $\dot{x}$, $\varphi$ and $\dot{\varphi}$ in the domain $G$ of the space $(x, \dot{x}, \varphi, \dot{\varphi})$ which is defined as the neighbourhood of the set $(0, 0, \omega_0^* t, \omega_0^*)$ where $t$ attains values in the interval $\left\langle 0, \dfrac{2\pi N}{\omega_0^*} \right\rangle$, the functions $f$, $g$, $M$ including their partial derivatives of the first order with respect to the variables $x$, $\dot{x}$, $\varphi$ and $\dot{\varphi}$ be continuous in $x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$ and $\varepsilon$;*

(d) *$M'(\omega_0^*) \neq 0$ (i. e. the root $\omega_0^*$ is simple).*

*Then there exists only one solution of* (1,1) *having the form* (1,2), *for which the functions $x^*(t, \varepsilon)$, $y^*(t, \varepsilon)$, $\Psi^*(t, \varepsilon)$ and $\Phi^*(t, \varepsilon)$ are periodic in $t$ of period*

$T^*(\varepsilon) = \dfrac{2\pi N}{\omega^*(\varepsilon)}$, *are continuous in* $t$ *and* $\varepsilon$ *for any* $t$ *and sufficiently small* $\varepsilon$, $\omega^*(\varepsilon)$ (*and consequently* $T^*(\varepsilon)$) *is continuous for sufficiently small* $\varepsilon$ *and* $x^*(t, 0) = y^*(t, 0) = \Psi^*(t, 0) = 0$, $\omega^*(0) = \omega_0^*$ *holds.*

Proof. The existence and uniqueness of the solution of (1,3) in the interval $\langle 0, T^*(\varepsilon)\rangle$ for sufficiently small $\varepsilon$ and for initial values $(x(0, \varepsilon), y(0, \varepsilon), \Psi(0, \varepsilon))$ having sufficiently small deviations from the values (0, 0, 0) (and of course $\varphi(0, \varepsilon) = 0$) is secured by the assumption (c).

According to (b) $(1,7_0)$ is soluble. Since again according to (b) $\det (e^{T_0 \mathbf{S}} - \mathbf{E}) \neq 0$, $x_0^* = y_0^* = 0$ holds. According to (d) $M'(\omega^*) \neq 0$ and therefore the Jacobian of $(1,7_0)$, which equals $\det (e^{T_0 \mathbf{S}} - \mathbf{E}) \cdot \left(\dfrac{2\pi N}{\omega_0^*}\right)^2 \cdot M'(\omega^*)$ is nonvanishing at the point $x_0^* = y_0^* = \Psi_0^* = 0$, $\omega_0 = \omega_0^*$.

The Jacobian of (1,7) is therefore also nonvanishing at the point $x_0^* = y_0^* = \Psi_0^* = \varepsilon = 0$ and $\omega_0 = \omega_0^*$, and according to (c) the system (1,7) fulfils all assumptions of the theorem on implicit functions. Hence the theorem follows easily.

Let us now examine the resonant case, i. e. $\omega_0^* = \dfrac{N}{n}$. For the sake of brevity let

$$\tau(\varepsilon) = 2\pi n \frac{\omega_0^* - \omega(\varepsilon)}{\omega(\varepsilon)} ; \tag{1,8}$$

then

$$T(\varepsilon) = 2\pi n + \tau(\varepsilon) ,$$

where

$$\tau(\varepsilon) \to 0 \quad \text{for} \quad \varepsilon \to 0 .$$

With respect to

$$e^{(2\pi n + \tau)\mathbf{S}} - \mathbf{E} = e^{2\pi n \mathbf{S}} - \mathbf{E} + e^{2\pi n \mathbf{S}} (e^{\tau \mathbf{S}} - \mathbf{E})$$

and

$$e^{2\tau n \mathbf{S}} = \mathbf{E} ,$$

the first equation $(1,7_0)$ can for $\varepsilon \neq 0$ be rewritten as

$$\frac{1}{\tau} (e^{\tau \mathbf{S}} - \mathbf{E})\, \mathbf{u}(0, \varepsilon) \frac{\tau}{\varepsilon} + \int_0^{2\pi n + \tau} e^{(2\pi n + \tau - s)\mathbf{S}}\, \mathbf{U}(x(s, \varepsilon), \ldots)\, \mathrm{d}s = 0 . \tag{1,9}$$

Letting $\varepsilon \to 0$, we get

$$\mathbf{S}\, \mathbf{u}_0 \lim_{\varepsilon \to 0} \frac{\tau}{\varepsilon} + \int_0^{2\pi n} e^{(2\pi n - s)\mathbf{S}}\, \mathbf{U}(x^{(0)}(s), - y^{(0)}(s), \omega_0^* s, \omega_0^*, 0)\, \mathrm{d}s = 0 , \tag{$1,9_0$}$$

where

$$x^{(0)}(t) = x\ (t, 0) \quad \text{and} \quad y^{(0)}(t) = y(t, 0) .$$

If $x^*(0, 0) = x_0^*$ and $y^*(0, 0) = y_0^*$ are components of the sought solution of $(1,7_0)$ and $\mathbf{S}u_0^* \neq 0$, i. e. $|x_0^*| + |y_0^*| \neq 0$, then from (1,9) it is clear that for this solution there exists $\lim_{\varepsilon \to 0} \frac{\tau}{\varepsilon}$ and consequently there exists $\lim_{\varepsilon \to 0} \frac{\omega_0^* - \omega(\varepsilon)}{\varepsilon}$.

The third equation of (1,7) can be written as

$$T(\varepsilon)\, \Psi(0, \varepsilon) + \int_0^{T(\varepsilon)} (T(\varepsilon) - s)[M(\omega(\varepsilon) + \varepsilon\, \Psi(s, \varepsilon)) + \varepsilon\, g(x(s, \varepsilon), \ldots)]\, ds = 0. \tag{1,10}$$

This equation gives for $\varepsilon = 0$

$$2\pi n \Psi_0 = 0\,. \tag{1,10$_0$}$$

Finally, let us investigate the second equation of (1,7). This equation is for $\varepsilon = 0$ with respect to $M(\omega_0^*) = 0$ fulfilled identically; let it be divided by $\varepsilon$ ($\varepsilon \neq 0$) and let the integral be replaced by two integrals with the limits $0, T_0^*$ and $T_0^*, T(\varepsilon)$. In the first integral the first term of the expression under the integral sign be modified according to the mean-value theorem of the differential calculus:

$$M(\omega + \varepsilon\Psi) = M(\omega + \varepsilon\Psi) - M(\omega_0^*) = M'(a)[\omega - \omega_0^* + \varepsilon\Psi]\,, \tag{1,11}$$

where

$$a = \omega_0^* + \vartheta(\varepsilon, t)[\omega - \omega_0^* + \varepsilon\, \Psi(t, \varepsilon)] \quad \text{and} \quad |\vartheta(\varepsilon, t)| < 1\,.$$

Let the second integral be modified according to the mean-value theorem of the integral calculus. Thus

$$\int_{T_0^*}^{T(\varepsilon)} [M(\omega(\varepsilon) + \varepsilon\, \Psi(s, \varepsilon)) + \varepsilon\, g(x(s, \varepsilon), \ldots)]\, ds =$$
$$= (T(\varepsilon) - T_0^*)[M(\omega(\varepsilon) + \varepsilon\, \Psi(\sigma, \varepsilon)) + \varepsilon\, g(x(\sigma, \varepsilon), \ldots)]\,, \tag{1,12}$$

where $\sigma \in (T_0^*, T(\varepsilon))$.

Now, the considered equation has the following form

$$\int_0^{T_0^*} \left[\frac{1}{\varepsilon} M'(a)\, (\omega(\varepsilon) - \omega_0^* + \varepsilon\, \Psi(s, \varepsilon)\right] + g(x(s, \varepsilon), \ldots) +$$
$$+ \frac{2\pi n}{\omega(\varepsilon)} \frac{\omega_0^* - \omega(\varepsilon)}{\varepsilon} [M(\omega(\varepsilon) + \varepsilon\, \Psi(\sigma, \varepsilon)) + \varepsilon g(x(\sigma, \varepsilon), \ldots)] = 0\,. \tag{1,13}$$

Let $\varepsilon \to 0$. As in the integral the limiting process may be performed under the integral sign (making use of $M(\omega_0^*) = 0$ and of the fact that by $(1,10_0)$ and the second equation (1,6a) $\Psi(t, \varepsilon) \to 0$ for $\varepsilon \to 0$ uniformly with respect to $t$) (1,13) results in

$$T_0^*\, M'(\omega_0^*) \lim_{\varepsilon \to 0} \frac{\omega(\varepsilon) - \omega_0^*}{\varepsilon} + \int_0^{T_0^*} g(x^{(0)}(s), -y^{(0)}(s), \omega_0^* s, \omega_0^*, 0)\, ds = 0\,. \tag{1,14}$$

Consequently there exists $\lim\limits_{\varepsilon\to 0} \dfrac{\omega(\varepsilon) - \omega_0^*}{\varepsilon}$ if $M'(\omega_0^*) \neq 0$.

In connection with previous results we know that $\lim\limits_{\varepsilon\to 0} \dfrac{\omega(\varepsilon) - \omega_0^*}{\varepsilon}$ exists, if $|x_0^*| + |y_0^*| + |M'(\omega_0^*)| \neq 0$. Suppose further that this inequality is fulfilled. From the existence of $\lim\limits_{\varepsilon\to 0} \dfrac{\omega(\varepsilon) - \omega_0^*}{\varepsilon}$ there follows that for sufficiently small $\varepsilon$ there exists a function $\Omega(\varepsilon)$ (continuous at $\varepsilon = 0$) such that

$$\omega(\varepsilon) = \omega_0^* + \varepsilon\, \Omega(\varepsilon)\,. \tag{1,15}$$

Then, of course (let us introduce the notation $\Omega(0) = \Omega_0$)

$$\lim_{\varepsilon\to 0} \frac{\omega(\varepsilon) - \omega_0^*}{\varepsilon} = \Omega_0\,, \quad \lim_{\varepsilon\to 0} \frac{\tau}{\varepsilon} = \frac{2\pi n}{\omega_0^*}\, \Omega_0 = -\frac{2\pi n^2}{N}\, \Omega_0\,.$$

The equations $(1,9_0)$ can be now written in more detail:

$$\begin{aligned} &2\pi n \frac{\Omega_0}{\omega_0^*} y_0 - \int_0^{2\pi n} f(x_0 \cos s - y_0 \sin s, -x_0 \sin s - y_0 \cos s, \omega_0^* s, \omega_0^*, 0) \sin s \,\mathrm{d}s = 0\,, \\ &-2\pi n \frac{\Omega_0}{\omega_0^*} x_0 - \int_0^{2\pi n} f(x_0 \cos s - y_0 \sin s, -x_0 \sin s - y_0 \cos s, \omega_0^* s, \omega_0^*, 0) \cos s \,\mathrm{d}s = 0\,. \end{aligned} \tag{1,9'_0}$$

To be able to replace the second equation of (1,7) by an equivalent condition that would not be identically fulfilled for $\varepsilon = 0$ let us for $0 < |\varepsilon| \leqq \varepsilon_0$, $|\Lambda| \leqq \Lambda_0$ ($\Lambda_0 > 0$) define the function $\mu(\Lambda, \varepsilon)$, where $\Lambda$ and $\varepsilon$ are independent variables, by means of the relation

$$\mu(\Lambda, \varepsilon) = \frac{1}{\varepsilon} M(\omega_0^* + \varepsilon\Lambda)\,.$$

As by the mean-value theorem

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} M(\omega_0^* + \varepsilon\Lambda) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} [M(\omega_0^* + \varepsilon\Lambda) - M(\omega_0^*)] = \\ &= \lim_{\varepsilon\to 0} M'(a)\, \Lambda = M'(\omega_0^*)\, \Lambda\,, \end{aligned}$$

$$(a = \omega_0^* + \vartheta(\Lambda, \varepsilon)\, \varepsilon\Lambda,\ |\vartheta| < 1)$$

it is possible to extend the function $\mu(\Lambda, \varepsilon)$ continuously onto the segment $\varepsilon = 0$, $|\Lambda| \leqq \Lambda_0$. Let us keep for this extended function the same notation $\mu(\Lambda, \varepsilon)$. Since for $\varepsilon \neq 0$

$$\frac{\partial\mu(\Lambda, \varepsilon)}{\partial\Lambda} = \frac{1}{\varepsilon} M'(\omega_0^* + \varepsilon\Lambda)\, \varepsilon = M'(\omega_0^* + \varepsilon\Lambda)$$

and for $\varepsilon = 0$

$$\frac{\partial\mu(\Lambda, 0)}{\partial\Lambda} = M'(\omega_0^*)\,,$$

there exists the partial derivative $\frac{\partial \mu}{\partial \Lambda}$ which is continuous for all $\Lambda$ and $\varepsilon$ in the rectangle $(-\Lambda_0, \Lambda_0) \times (-\varepsilon_0, \varepsilon_0)$.

Hence

$$\frac{1}{\varepsilon} M(\omega_0^* + \varepsilon\Omega + \varepsilon \Psi(t, \varepsilon)) = \mu(\Omega(\varepsilon) + \Psi(t, \varepsilon), \varepsilon)$$

can be written for all $\varepsilon$, $0 < \varepsilon \leqq \varepsilon_0$, and it can easily be found that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} M(\omega_0^* + \varepsilon\, \Omega(\varepsilon) + \varepsilon\, \Psi(t, \varepsilon)) = \mu(\Omega_0 + \Psi(t, 0), 0) = M'(\omega_0^*)[\Omega_0 + \Psi(t, 0)]\,.$$

Making use of the previous notations, the third equation of the system (1,7), after having been divided by $\varepsilon$, can be written in the form

$$\int_0^{T(\varepsilon)} [\mu(\Omega(\varepsilon) + \Psi(s, \varepsilon), \varepsilon) + g(x(s, \varepsilon), \ldots)]\, ds = 0\,. \tag{1,16}$$

Letting $\varepsilon \to 0$ results in essentially (1,14), only the notation being somewhat different (the use of $\Psi(t, 0) = 0$ is made again):

$$2\pi n\, M'(\omega_0^*)\, \Omega_0 + \int_0^{2\pi n} g(x^{(0)}(s), -y^{(0)}(s), \omega_0^* s, \omega_0^*, 0)\, ds = 0\,. \tag{$1,16_0$}$$

The equations $(1,9_0')$, $(1,16_0)$ and $(1,10_0)$ form a system of necessary conditions which $x_0$, $y_0$, $\Psi_0$ and $\Omega_0$ must fulfil in order that the functions $x(t, \varepsilon)$, $y(t, \varepsilon)$, $\Psi(t, \varepsilon)$ and $\Phi(t, \varepsilon)$ should be $T(\varepsilon)$-periodic in $t$. This system of equations allows us to derive the sufficient conditions for existence of the periodic functions in question.

First let us write the Jacobian of $(1,9_0')$, $(1,16_0)$ and $(1,10_0)$ at the point $x_0 = x_0^*$, $y_0 = y_0^*$, $\Omega_0 = \Omega_0^*$, $\Psi_0 = \Psi_0^* = 0$:

$$\begin{vmatrix} \int_0^{2\pi n} F_1^*(s) \cos s\, ds\,, & \int_0^{2\pi n} F_2^*(s) \cos s\, ds + \frac{2\pi n \Omega_0^*}{\omega_0^*}\,, & \frac{2\pi n y_0^*}{\omega_0^*}\,, & 0 \\ \int_0^{2\pi n} F_1^*(s) \sin s\, ds - \frac{2\pi n \Omega_0^*}{\omega_0^*}\,, & \int_0^{2\pi n} F_2^*(s) \sin s\, ds, & -\frac{2\pi n x_0^*}{\omega_0^*}\,, & 0 \\ \int_0^{2\pi n} G_1^*(s)\, ds, & \int_0^{2\pi n} G_2^*(s)\, ds, & 2\pi n\, M'(\omega_0^*)\,, & 0 \\ 0, & 0, & 0, & 2\pi n \end{vmatrix}, \tag{1,17}$$

where

$$F_1^* = -\left(\frac{\partial f}{\partial x}\right)^* \cos s + \left(\frac{\partial f}{\partial \dot{x}}\right)^* \sin s\,, \quad F_2^* = \left(\frac{\partial f}{\partial x}\right)^* \sin s + \left(\frac{\partial f}{\partial \dot{x}}\right)^* \cos s\,,$$

$$G_1^* = \left(\frac{\partial g}{\partial x}\right)^* \cos s - \left(\frac{\partial g}{\partial \dot{x}}\right) \sin s\,, \quad G_2^* = -\left(\frac{\partial g}{\partial x}\right)^* \sin s - \left(\frac{\partial g}{\partial \dot{x}}\right)^* \cos s\,.$$

From the third column it is evident that as soon as this determinant is non-vanishing, $x_0^* = y_0^* = M'(\omega_0^*) = 0$ cannot hold simultaneously.

Let the following theorem now be proved:

**Theorem 1.2.** *Let*

(a) *the functions $f$ and $g$ be $2\pi$-periodic in $\varphi$;*

(b) *the equation $M(\omega_0) = 0$ have at least one real positive root $\omega_0 = \omega_0^* = \frac{N}{n}$ ($N$ and $n$ being natural numbers);*

(c) *the system $(1{,}9_0')$ and $(1{,}16_0)$ have a real solution $x_0 = x_0^*$, $y_0 = y_0^*$ and $\Omega_0 = \Omega_0^*$;*

(d) *for $|\varepsilon| \leqq \varepsilon_0$ ($\varepsilon_0 > 0$) and for $x$, $\dot{x}$, $\varphi$ and $\dot{\varphi}$ in the domain $G$ which is defined as the neighbourhood of the set ($x_0^* \cos t - y_0^* \sin t$, $- x_0^* \sin t - y_0^* \cos t$, $\omega_0^* t$, $\omega_0^*$), where $t$ attains values in the interval $\langle 0, 2\pi n\rangle$, the functions $f$, $g$ and $M$ including their partial derivatives with respect to $x$, $\dot{x}$, $\varphi$ and $\dot{\varphi}$ be continuous in $x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$ and $\varepsilon$;*

(e) *the Jacobian of $(1{,}9_0')$ and $(1{,}16_0)$ with respect to $x_0$, $y_0$ and $\Omega_0$ at the point $x_0 = x_0^*$, $y_0 = y_0^*$ and $\Omega_0 = \Omega_0^*$ be nonvanishing.*

*Then there exists for sufficiently small $\varepsilon$ just one solution of $(1{,}1)$ in the desired form $(1{,}2)$, for which the functions $x^*(t, \varepsilon)$, $y^*(t, \varepsilon)$, $\Psi^*(t, \varepsilon)$ and $\Phi^*(t, \varepsilon)$ are periodic in $t$ of period $T^*(\varepsilon) = \frac{2\pi N}{\omega^*(\varepsilon)}$, where $\omega^*(\varepsilon) = \omega_0^* + \varepsilon\, \Omega^*(\varepsilon)$, with a continuous function $\Omega^*(\varepsilon)$ and they are continuous in $t$ and $\varepsilon$ for all $t \geqq 0$, while*

$$x^*(t, 0) = x_0^* \cos t - y_0^* \sin t\,, \quad y^*(t, 0) = x_0^* \sin t + y_0^* \cos t\,,$$

$$\Psi^*(t, 0) = 0\,, \quad \Omega^*(0) = \Omega_0^*\,.$$

Proof. The existence and uniqueness of the solution of $(1{,}3)$ on the interval $\langle 0, T^*(\varepsilon)\rangle$ is secured for sufficiently small $\varepsilon$ and for initial values of $x$, $y$ and $\Psi$ having sufficiently small deviations from the initial values $x_0^*$, $y_0^*$ and $\Psi_0^*$ (of course, $\varphi(0, \varepsilon) = 0$) by the assumption (d). As we have demonstrated above, by assumption (e) $|x_0^*| + |y_0^*| + |M(\omega_0^*)| \neq 0$, so that $\lim\limits_{\varepsilon \to 0} \frac{\omega(\varepsilon) - \omega_0^*}{\varepsilon}$ exists and $\omega(\varepsilon)$ is therefore at the point $\varepsilon = 0$ not only continuous but also differentiable and can be written in the form $\omega(\varepsilon) = \omega_0^* + \varepsilon\, \Omega(\varepsilon)$, where $\Omega(\varepsilon)$ is continuous at $\varepsilon = 0$.

So, the assumption (b) being fulfilled, the system of necessary and sufficient conditions for $x(t, \varepsilon)$, $y(t, \varepsilon)$, $\Psi(t, \varepsilon)$ and $\Phi(t, \varepsilon)$ to be periodic of period $T(\varepsilon) = \frac{2\pi N}{\omega(\varepsilon)}$, that the functions $x(0, \varepsilon)$, $y(0, \varepsilon)$, $\Psi(0, \varepsilon)$ and $\Omega(\varepsilon)$ must fulfil, is given by the equations $(1{,}7)$, where $\omega(\varepsilon)$ has been replaced by $\omega_0^* + \varepsilon\Omega(\varepsilon)$. It remains to prove that this system has a real solution $x^*(0, \varepsilon)$, $y^*(0, \varepsilon)$, $\Psi^*(0, \varepsilon)$ and $\Omega^*(\varepsilon)$ for sufficiently small $\varepsilon$.

Since according to the above considerations there exists for sufficiently small $\varepsilon$ the partial derivative of the first order of the function $\mu(\omega_0^* + \varepsilon\Lambda, \varepsilon)$ with respect to $\Lambda$, there exists the Jacobian of the system (1,9), (1,10) and (1,16) (where $\omega(\varepsilon) = \omega_0^* + \varepsilon\Omega(\varepsilon)$ and it is by (d) continuous for sufficiently small $\varepsilon$ and for $x(0, \varepsilon)$, $y(0, \varepsilon)$, $\Psi(0, \varepsilon)$ and $\Omega(\varepsilon)$ having sufficiently small deviations from $x_0^*$, $y_0^*$, $\Psi_0^*$ and $\Omega_0^*$. As this Jacobian reduces for $\varepsilon = 0$ to the Jacobian of $(1,9_0')$, $(1,10_0)$ and $(1,16_0)$ and the latter, being $2\pi n$ times the Jacobian of $(1,9_0')$ and $(1,16_0)$, is according to (e) nonvanishing, the system (1,9), (1,10) and (1,16) fulfils according to (d) all assumptions of the theorem on implicit functions. Thus there exists for sufficiently small $\varepsilon$ the solution of the system (1,9), (1,10) and (1,16) continuous in $\varepsilon$ which for $\varepsilon = 0$ reduces to the solution of the system $(1,9_0')$, $(1,10_0)$ and $(1,16_0)$.

It is immediately evident that the solution of (1,9), (1,10) and (1,16) is simultaneously the solution of (1,7). From this the theorem readily follows.

For the analytic case let us state the following theorem:

**Theorem 1.3.** *Let the functions $f$, $g$ and $M$ fulfil either the conditions* (a)—(d) *of the theorem* 1,1 *or the conditions* (a)—(e) *of the theorem* 1,2 *and moreover let them be analytic in $x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$ and $\varepsilon$ for $(x, \dot{x}, \varphi, \dot{\varphi}) \in G$ and $|\varepsilon| \leqq \varepsilon_0$ $(\varepsilon_0 > 0)$.*

*Then the corresponding solutions with the corresponding functions $\omega(\varepsilon)$ or $\Omega(\varepsilon)$ (and also $T(\varepsilon)$) are analytic in $\varepsilon$ for all $t \geqq 0$ and for sufficiently small $\varepsilon$.*

Proof. According to the existence theorem the solutions of (1,1) (as well as those of (1,3)) are for the assumptions mentioned above for sufficiently small $\varepsilon$ analytic functions of the initial conditions and of the parameter $\varepsilon$. Further, according to the theorem on implicit functions in the analytic case the solution of (1,7) (i. e. the initial conditions of the functions $x(t, \varepsilon)$, $y(t, \varepsilon)$ and $\Psi(t, \varepsilon)$ and $\omega(\varepsilon)$ or $\Omega(\varepsilon)$) is analytic in $\varepsilon$ for sufficiently small $\varepsilon$. With respect to the fact that $\varphi(0, \varepsilon) = 0$ and $\Phi(0, \varepsilon) = 0$ are fixed, and that for analytic $\omega(\varepsilon)$, $\omega(0) \neq 0$ $T(\varepsilon) = \dfrac{2\pi N}{\omega(\varepsilon)}$ in the neighbourhood of $\varepsilon = 0$ is also analytic, the assertion of the theorem follows immediately.

It could easily be proved that the functions $x^*(0, \varepsilon)$, $y^*(0, \varepsilon)$, $\Psi^*(0, \varepsilon)$ as well as $\omega(\varepsilon)$ or $\Omega(\varepsilon)$ could be found with the aid of recurrent formulas.

Remark 1. The system (1,1) can be generalized in different fashions. In the first place the case of more mechanical systems with different eigenfrequencies $\varkappa_i$ being driven by one motor can be considered. Thus the system

$$\ddot{x}_i + \varkappa_i^2 x_i = \varepsilon f_i(x_j, \dot{x}_j, \varphi, \dot{\varphi}, \varepsilon) ,$$
$$\ddot{\varphi} = \varepsilon M(\dot{\varphi}) + \varepsilon^2 g(x_j, \dot{x}_j, \varphi, \dot{\varphi}, \varepsilon) \quad (i, j = 1, 2, \ldots, n)$$

is arrived at. This generalization brings no new aspects; it just increases the troubles of a technical character.

On the other hand, if the characteristic $M$ of the motor is considered to depend on $\varphi$ too, or the action of the motor to be influenced by a mechanical system with members of order $\varepsilon$, i. e. in (1,1) $\ddot{\varphi} = \varepsilon\, h(x, \dot{x}, \varphi, \dot{\varphi}, \varepsilon)$ our problem leads to the finding of a periodic solution in the neighbourhood of the periodic solution of a certain nonlinear system. This problem is rather involved and it will be examined in some future paper.

Remark 2. In paper [4] a more detailed analysis of the conditions for the existence and stability of the solution in the desired form of (0,2) has been performed. The nonautonomous system for the functions $x$, $y$, $\Psi$ and $\Phi$ has been made use of.

Remark 3. In practice, the characteristic function of motor $M$ is not known quite precisely. Therefore the root of the equation $M(\omega_0) = 0$ is not known precisely either. According to preceding calculations if $\omega_0^* = \frac{N}{n}$ and one seeks the solution of the period $T = \frac{2\pi N}{\omega}$, the first approximation of the amplitude of oscillations of $x$ (as well as $y$) has in general a nonvanishing value. On the contrary if $\omega_0^* \neq \frac{N}{n}$, the first approximation is equal to zero. Hence it could seem that a minute change in the determination of the root $\omega_0^*$ can cause a large change in the determination of the amplitude. But it is not so. If the root $\tilde{\omega}_0^* \neq \frac{N}{n}$ of the equation $\tilde{M}(\omega_0) = 0$ approximates closely the root $\omega_0^* = \frac{N}{n}$ of the equation $M(\omega_0) = 0$, then the value of the determinant $|e^{\tilde{T}_0 \mathbf{S}} - \mathbf{E}|$ for $\tilde{T}_0 = \frac{2\pi N}{\omega_0^*}$ differs little from zero and the magnitude of further approximations of $x_0$ and $y_0$ (and so of the amplitude) rapidly increases. (In addition the successive approximations converge only within small range of $\varepsilon$.)

## 2. A Theorem on the Stability of the Periodic Solution of the Second Kind and Some Lemmas

The stability of the solution which was found in the former paragraph can easily be investigated with the aid of a modified Andronov-Vitt theorem [3], [5] which deals with the stability of the periodic solution of a general autonomous system. Slightly altering the proof performed in [3] we can modify this theorem to investigate the stability of a periodic solution of the second kind of autonomous systems.

**Theorem 2,1.** *Let the system*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2,1}$$

*be given.* $\mathbf{x}$ *is a vector with components* $x_1, x_2, \ldots, x_n$ *and* $\mathbf{f}$ *is a vector function with components* $f_1, f_2, \ldots, f_n$. *Let* $\mathbf{p}$ *be an n-dimensional vector with some components equal to one and with the remaining components equal to zero. Let the function* $\mathbf{f}$ *fulfil the relation*

$$\mathbf{f}(\mathbf{x} + 2\pi\mathbf{p}) = \mathbf{f}(\mathbf{x}) \,. \tag{2,2}$$

*Let* (2,1) *have a solution in the form*

$$\mathbf{x} = \omega t\mathbf{p} + \boldsymbol{\psi}(t) = \boldsymbol{\varphi}(t) \,, \tag{2,3}$$

*where the vector function* $\boldsymbol{\psi}(t)$ *is periodic in* $t$ *of period* $T = \dfrac{2\pi N}{\omega}$ *(N being a natural number). Let* $\mathbf{f}$ *be defined and have continuous partial derivatives of the first order with respect to* $x_1, x_2, \ldots, x_n$ *in the neighbourhood* $V$ *of* $\boldsymbol{\varphi}(t)$ *for* $t \geqq 0$.

*Let the equation of first variation*

$$\dot{\mathbf{y}} = \mathbf{f}_{\mathbf{x}}(\boldsymbol{\varphi}(t))\, \mathbf{y} \tag{2,4}$$

*of the system* (2,1) *with respect to the solution* (2,3) *(which has T-periodic coefficients) have* $n - 1$ *characteristic exponents with negative real parts.*

*Then there exists a number* $\eta > 0$ *such that if a solution* $\boldsymbol{\xi}(t)$ *of* (2,1) *satisfies the inequality* $\|\boldsymbol{\xi}(t_1) - \boldsymbol{\varphi}(t_0)\| \leqq \eta$ *for some* $t_1$ *and* $t_0$ *there exists a number c such that*

$$\lim_{t \to \infty} \|\boldsymbol{\xi}(t) - \boldsymbol{\varphi}(t + c)\| = 0 \,, \tag{2,5}$$

*(i. e. the solution* $\boldsymbol{\varphi}(t)$ *has asymptotical orbital stability and any solution which is sufficiently near to it has an asymptotic phase).*

*On the contrary if at least one of the characteristic exponents of* (2,4) *has a positive real part, the solution* $\boldsymbol{\varphi}(t)$ *is orbitally unstable.*

The proof will not be performed in detail here; it will be confined to the modifications of the cited Andronov-Vitt theorem necessary for obtaining our theorem.

The $T$-periodicity of the coefficients of (2,4) follows from the fact that

$$\mathbf{f}_{\mathbf{x}}\left(\omega\Big(t + \frac{2\pi N}{\omega}\Big)\mathbf{p} + \boldsymbol{\psi}\left(t + \frac{2\pi N}{\omega}\right)\right) = \mathbf{f}_{\mathbf{x}}(\omega t\,\mathbf{p} + 2\pi N\mathbf{p} + \boldsymbol{\psi}(t)) = \mathbf{f}_{\mathbf{x}}(\omega t\,\mathbf{p} + \boldsymbol{\psi}(t)) \,.$$

It is also immediately clear that the function $\dot{\boldsymbol{\varphi}}$ is $T$-periodic. Since $\dot{\boldsymbol{\varphi}}$ is, as is well-known, a solution of (2,4), at least one of its characteristic exponents has a vanishing real part.

To perform the proof it is necessary to investigate the equations

$$\dot{\mathbf{z}} = \mathbf{f}(\boldsymbol{\varphi}(t) + \mathbf{z}) - \mathbf{f}(\boldsymbol{\varphi}(t)) = \mathbf{f}_{\mathbf{x}}(\boldsymbol{\varphi}(t))\, \mathbf{z} + \mathbf{F}(t, \mathbf{z}) \,, \tag{2,6}$$

where $\mathbf{F}(t, \mathbf{z}) = o\,(\|\mathbf{z}\|)$ uniformly with respect to $t$.

Without loss of generality a translation of the coordinate system can be assumed which makes $\boldsymbol{\varphi}(0) = 0$, because a translation does not change the nature of the components of the vector $\boldsymbol{\varphi}$.

As far as (2,6) is investigated the coordinate system can be rotated ($\mathbf{z} \to \tilde{\mathbf{z}}$) in such a way ($\tilde{\boldsymbol{\varphi}}(t)$ is the solution after the rotation) that besides $\tilde{\boldsymbol{\varphi}}(0) = 0$ also $\dot{\tilde{\boldsymbol{\varphi}}}(0) = \lambda \boldsymbol{p}$ ($\lambda \neq 0$) holds. It is evident namely that the rotation changes neither the periodicity of the coefficients and the characteristic exponents of the linearised system (2,6) (i. e. of the system (2,4)) nor the smallness of the vector $\boldsymbol{F}(t, \mathbf{z})$ and these are the qualities that the proof which is made use of is based on.

For the transformed coordinates $\tilde{\mathbf{z}}$ we can by [3] suppose as proved the existence of a $(n - 1)$-dimensional surface $\tilde{S}_0$ such that all the solutions which start on it at $t = 0$ converge to $\dot{\boldsymbol{\varphi}}(t)$ for $t \to \infty$. Having rotated back ($\tilde{\mathbf{z}} \to \mathbf{z}$) we get a surface $S_0$ having the same quality with respect to the solution $\boldsymbol{\varphi}(t)$. The vector $\boldsymbol{\varphi}(0)$ intersects the surface $S_0$ at a nonvanishing angle since $\tilde{\boldsymbol{\varphi}}(0)$ has this quality with respect to $\tilde{S}_0$.

Denoting the equation of the surface $S_0$ in the space $z$ $\Sigma_0$ in the space $x$, again $\Sigma_0(x) = S_0(x)$ holds.

As the solution $\boldsymbol{\varphi}$ does not return back on the surface $\Sigma_0$ after the time interval $T$, the sequence of surfaces $\Sigma_k$ ($k = 1, 2, \ldots$) passing through the points $2\pi N k \boldsymbol{p}$ ($k = 1, 2, \ldots$) (i. e. the points corresponding to $t = kT$) and such that all the solutions starting from $\Sigma_k$ converge to $\boldsymbol{\varphi}_k(t) = \boldsymbol{\varphi}(t + kT)$ must be constructed in this case. We easily get these surfaces by translating $\Sigma_0$ in the direction of the vector $\boldsymbol{p}$ through $2\pi N k$.

Since it is easy to verify that

$$\boldsymbol{f}(\boldsymbol{\varphi}_k + \mathbf{z}) - \boldsymbol{f}(\boldsymbol{\varphi}_k) = \boldsymbol{f}(\boldsymbol{\varphi} + \mathbf{z}) - \boldsymbol{f}(\boldsymbol{\varphi}),$$

we get according to (2,6) for all $\boldsymbol{\varphi}_k$ in the space $z$ the same surface $S_0$. Hence, with respect to $\mathbf{x} = \mathbf{z} + \boldsymbol{\varphi}_k$ and to $\boldsymbol{\varphi}_k(0) = 2\pi N k \boldsymbol{p}$, there follows that the equations of the surfaces $\Sigma_k$ are

$$\Sigma_k(\mathbf{x}) = \Sigma_0(\mathbf{x} - 2\pi N k \boldsymbol{p}) = 0 .$$

The solution $\boldsymbol{\varphi}(t)$ intersects the surfaces $\Sigma_k$ because of the periodicity of $\dot{\boldsymbol{\varphi}}(t)$ at the same angle as the surface $\Sigma_0$.

By the relation (2,2) and by the assumed form of solution (2,3) there follows that on a certain neighbourhood $M$ of the solution $\boldsymbol{\varphi}(t)$: $\|\mathbf{x} - \boldsymbol{\varphi}(t)\| \leqq \varrho$, $t \geqq 0$ a common Lipschitz constant $L$ exists, so that for two solutions $\boldsymbol{\xi}^{(1)}(t)$ and $\boldsymbol{\xi}^{(2)}(t)$ of (2,1) lying in the set $M$ for which

$$\|\boldsymbol{\xi}^{(1)}(t_0) - \boldsymbol{\xi}^{(2)}(t_0)\| \leqq \delta ,$$

$$\|\boldsymbol{\xi}^{(1)}(t) - \boldsymbol{\xi}^{(2)}(t)\| \leqq \delta(e^{L(t-t_0)} - 1)$$

holds for $t \geqq t_0$.

Thus, if for some solution $\xi(t)$ of (2,1) $\|\xi(t_1) - \varphi(t_0)\| \leqq \eta$ holds for some $t_1$ and $t_0$, while $kT \leqq t_0 < (k+1)\,T$ ($k$ being a natural number), $\|\xi(t - t_0 + t_1) - \varphi(t)\|$ remains for $t_0 \leqq t \leqq t_0 + T$ uniformly small with respect to $t_0$. For sufficiently small $\eta$, $\xi(t - t_0 + t_1)$ intersects the surface $\Sigma_{k+1}$ for some $\bar{t}$, $t_0 \leqq \bar{t} \leqq t_0 + T$. Then the solution $\bar{\xi}(t) = \xi(t - t_0 + t_1 + t)$ has $\bar{\xi}(0)$ on $\Sigma_{k+1}$ and consequently $\bar{\xi}(t) - \varphi_{k+1}(t) \to 0$ holds.

Therefore $\xi(t - t_0 + t_1 + \bar{t} + (k+1)\,T) - \varphi(t) \to 0$ for $t \to \infty$ and it is sufficient to set $c = t_0 - t_1 - t - (k+1)\,T$ in order to prove the first statement of theorem 2,1.

To prove the statement about instability, which is almost obvious, would be rather lengthy and for this reason this proof will not be performed here.

Let us now prove some lemmas of an algebraic character.

**Lemma 2,1.** *Let us consider the expression*

$$R \equiv \varrho^4 - \varrho^3(2 + 2\cos T(\varepsilon) + \varepsilon\, C_1(\varepsilon)) + \varrho^2(2 + 4\cos T(\varepsilon) + \varepsilon\, C_2(\varepsilon)) - \\ - \varrho(2 + 2\cos T(\varepsilon) + \varepsilon\, C_3(\varepsilon)) + 1 + \varepsilon\, C_4(\varepsilon)\,, \qquad (2,7)$$

*where* $T(\varepsilon)$ *and* $C_i(\varepsilon)$ $(i = 1, 2, 3, 4)$ *are continuous functions of* $\varepsilon$ *in the neighbourhood of* $\varepsilon = 0$ *and* $T(0) \neq 2\pi\nu$, $\nu$ *being an integer.*

*Then* $R$ *can for sufficiently small* $\varepsilon$ *be written as the product of two quadratic expressions*

$$R \equiv [\varrho^2 - 2\varrho(\cos T(\varepsilon) + \varepsilon\, P_1(\varepsilon)) + 1 + \varepsilon\, Q_1(\varepsilon)][\varrho^2 - 2\varrho(1 + \varepsilon\, P_2(\varepsilon)) + \\ + 1 + \varepsilon\, Q_2(\varepsilon)]\,, \qquad (2,8)$$

*where* $P_i$ *and* $Q_i$ *are continuous functions.*

Proof. Performing the multiplication in (2,8) and equating the coefficients at like powers $\varrho$ we get after some manipulation the following system for $P_i$ and $Q_i$ $(i = 1, 2)$:

$$2P_1 + 2P_2 = C_1\,,$$
$$Q_1 + Q_2 + 4P_1 + 4\cos T(\varepsilon)\, P_2 + 4\varepsilon P_1 P_2 = C_2\,,$$
$$2P_1 + 2P_2 + 2Q_1 + 2\cos T(\varepsilon)\, Q_2 + \varepsilon(P_1Q_2 + P_2Q_1) = C_3\,,$$
$$Q_1 + Q_2 + \varepsilon Q_1 Q_2 = C_4\,. \qquad (2,9)$$

For $\varepsilon = 0$ the system (2,9) reduces to a linear system for $P_i(0)$ and $Q_i(0)$ the Jacobian of which equals to $-16(1 - \cos T(0))^2$ and is by the assumption $T(0) \neq 2\pi\nu$ ($\nu$ being an integer) nonvanishing. The system (2,9) fulfils thus all assumptions of the theorem on implicit functions and so our above statement is proved.

**Lemma 2,2.** *Let us consider the algebraic equation*

$$\varrho^2 - 2(\cos T(\varepsilon) + \varepsilon\, P(\varepsilon)) + 1 + \varepsilon\, Q(\varepsilon) = 0\,, \qquad (2,10)$$

*where* $T(\varepsilon)$, $P(\varepsilon)$ *and* $Q(\varepsilon)$ *are, for sufficiently small* $\varepsilon$, *continuous functions of* $\varepsilon$ *and* $T(0) \neq 2\pi\nu$ *(*$\nu$ *being an integer).*

*Then,* (2,10) *has for sufficiently small* $\varepsilon \neq 0$ *of the opposite* (*the same*) *sign as* $Q(0)$ *both roots in absolute values less* (*more*) *than one.*

Proof. The roots of (2,13) are given by the formula

$$\varrho_{1,2} = \cos T(\varepsilon) + \varepsilon P(\varepsilon) \pm \sqrt{(\cos T(\varepsilon) + \varepsilon P(\varepsilon))^2 - 1 - \varepsilon Q(\varepsilon)} .$$

Since for sufficiently small $\varepsilon \cos T(\varepsilon) \neq 1$ the expression under the square-root sign is for sufficiently small $\varepsilon$ negative.

Then $|\varrho_{1,2}| = \sqrt{1 + \varepsilon Q(\varepsilon)}$ and $|\varrho_{1,2}| < 1$ or $|\varrho_{1,2}| > 1$ if $\varepsilon Q(\varepsilon) < 0$ or $\varepsilon Q(\varepsilon) > 0$. With respect to $Q(\varepsilon)$ being a continuous function, the statement of the lemma is evident.

**Lemma 2,3.** *Let us consider the algebraic equation*

$$\varrho^2 - 2\varrho(1 + \varepsilon P(\varepsilon) + 1 + \varepsilon Q(\varepsilon) = 0 , \tag{2,11}$$

*where* $P(\varepsilon)$ *and* $Q(\varepsilon)$ *are continuous functions in the neighbourhood of* $\varepsilon = 0$. *Let one of its roots equal one.*

*Then, for sufficiently small* $\varepsilon \neq 0$ *of the opposite* (*the same*) *sign as* $P(0)$ *the remaining root is in absolute value less* (*more*) *than one.*

Proof. As $\varrho = 1$ is a root of (2,11),

$$1 - 2(1 + \varepsilon P(\varepsilon)) + 1 + \varepsilon Q(\varepsilon) = \varepsilon(- 2P(\varepsilon) + Q(\varepsilon)) = 0 \tag{2,12}$$

holds.

The roots of (2,12) are consequently:

$$\varrho_1 = 1 , \quad \varrho_2 = 1 + 2\varepsilon P(\varepsilon) .$$

Hence the statement follows immediately in consequence of $P(\varepsilon)$ being continuous.

**Lemma 2,4.** *Let us consider the expression*

$$S \equiv \sigma^3 + \sigma^2\varepsilon P_1(\varepsilon) + \sigma\varepsilon^2 P_2(\varepsilon) + \varepsilon^3 P_3(\varepsilon) , \tag{2,13}$$

*where* $P_i(\varepsilon)$ *are continuous functions in the neighbourhood of* $\varepsilon = 0$.

*Then* (2,16) *can be written in the following form*

$$S \equiv \prod_{i=1}^{3}(\sigma - \varepsilon \alpha_i(\varepsilon)) , \tag{2,14}$$

*where* $\alpha_i(\varepsilon)$ *are* (*in general complex*) *functions continuous in the neighbourhood of* $\varepsilon = 0$.

Proof. Equating like powers of $\sigma$ in (2,13) and (2,14) we get after having divided by appropriate powers of $\varepsilon$ the following system of equations for $\alpha_i(\varepsilon)$:

$$\alpha_1 + \alpha_2 + \alpha_3 = - P_1 , \quad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = P_2 ,$$
$$\alpha_1\alpha_2\alpha_3 = - P_3 . \tag{2,15}$$

The functions $\alpha_i$ are evidently for every constant $\varepsilon$ the roots of the algebraic equation

$$\tau^3 + P_1(\varepsilon)\,\tau^2 + P_2(\varepsilon)\,\tau + P_3(\varepsilon) = 0\,.$$

According to the well-known theorem on the continuous dependence of the roots of the algebraic equation on its coefficients and, in consequence of $P_i(\varepsilon)$ being continuous, the functions $\alpha_i(\varepsilon)$ are for sufficiently small $\varepsilon$ continuous functions of $\varepsilon$.

### 3. The Stability of the Found Solution

We are now able to proceed to the investigation of the stability of the found solution of (1,3). Suppose the assumptions of theorem 1,1, 1,2 respectively, to be fulfilled so that such a solution really exists.

Let the system (1,1) be replaced by an equivalent system

$$\begin{aligned}
\dot{x} &= -\,y\,,\\
\dot{y} &= x - \varepsilon\, f(x, -y, \varphi, \psi, \varepsilon)\,,\\
\dot{\psi} &= \varepsilon\, M(\psi) + \varepsilon^2\, g(x, -y, \varphi, \psi, \varepsilon)\,,\\
\dot{\varphi} &= \psi\,.
\end{aligned} \tag{3,1}$$

This system has consequently the solution

$$\begin{gathered}
x = x^*(t, \varepsilon)\,, \quad y = y^*(t, \varepsilon)\,,\\
\varphi = \omega^*(\varepsilon)\, t + \varepsilon\, \Phi^*(t, \varepsilon)\,, \quad \psi = \omega^*(\varepsilon) + \varepsilon\, \Psi^*(t, \varepsilon)\,,
\end{gathered} \tag{3,2}$$

where $x^*(t, \varepsilon)$, $y^*(t, \varepsilon)$, $\Phi^*(t, \varepsilon)$ and $\Psi^*(t, \varepsilon)$ are periodic functions of period $T^*(\varepsilon) = \dfrac{2\pi N}{\omega^*(\varepsilon)}$. The system (3,1) and its solution (3,2) fulfil evidently all assumptions of the theorem 2,1, while $n = 4$ and the role of the vector $\boldsymbol{p}$ is played in the space $(x, y, \varphi, \psi)$ by the vector $(0, 0, 1, 0)$.

Denoting $u_1$, $u_2$, $u_3$ and $u_4$ the variations of the variables $x$, $y$, $\psi$ and $\varphi$, the equation of first variation of (3,1) with respect to the solution (3,2) is given by

$$\begin{aligned}
\dot{u}_1 &= -\,u_2\,,\\
\dot{u}_2 &= u_1 + \varepsilon[Au_1 + Bu_2 + Cu_3 + Du_4]\,,\\
\dot{u}_3 &= \varepsilon m u_3 + \varepsilon^2[Eu_1 + Fu_2 + Gu_3 + Hu_4]\,,\\
\dot{u}_4 &= u_3\,,
\end{aligned} \tag{3,3}$$

where

$$\begin{gathered}
A = A(t, \varepsilon) = -\left(\frac{\partial f}{\partial x}\right)^*, \quad B = B(t, \varepsilon) = \left(\frac{\partial f}{\partial \dot{x}}\right)^*,\\
C = C(t, \varepsilon) = -\left(\frac{\partial f}{\partial \dot{\varphi}}\right)^*, \quad D = D(t, \varepsilon) = -\left(\frac{\partial f}{\partial \varphi}\right)^*,\\
E = E(t, \varepsilon) = \left(\frac{\partial g}{\partial x}\right)^*, \quad F = F(t, \varepsilon) = \left(\frac{\partial g}{\partial \dot{x}}\right)^*,\\
G = G(t, \varepsilon) = \left(\frac{\partial g}{\partial \dot{\varphi}}\right)^*, \quad H = H(t, \varepsilon) = \left(\frac{\partial g}{\partial \varphi}\right)^*,\\
m = m(t, \varepsilon) = M'^* = M'(\omega^*(\varepsilon) + \varepsilon\, \Psi^*(t, \varepsilon))\,.
\end{gathered} \tag{3,4}$$

The asterisks at partial derivatives denote that $x, y, \psi$ and $\varphi$ have been substituted according to (3,2), so that all these functions are $T^*(\varepsilon)$-periodic functions of the time.

Our task consists now in determining three characteristic exponents of (3,3) with such an accuracy that the sign of their real parts — at least for sufficiently small $\varepsilon$ — be fixed (the fourth characteristic exponent has, as we know, a vanishing real part).

Since the characteristic exponents of (3,3) are for $\varepsilon = 0$ $\pm i$, 0, 0, we must find more accurate values of all the three characteristic exponents in question. To this end we shall make use of the method of § 10, Chapter III, MALKIN'S Monograph [6]. This method consists in investigating the equation

$$D(\varepsilon) \equiv \det\,(\mathbf{U}(T^*(\varepsilon), \varepsilon) - \varrho \mathbf{E}) = 0\,, \tag{3,5}$$

where $\mathbf{U}(t, \varepsilon)$ is a matrix solution of (3,3) with $\mathbf{U}(0, \varepsilon) = \mathbf{E}$, the zero points of which being characteristic roots $\varrho_k$ related to characteristic exponents $\lambda_k$ by

$$\lambda_k = \frac{1}{T^*(\varepsilon)} \lg \varrho_k \quad (k = 1, 2, 3, 4)\,. \tag{3,6}$$

Thus if the three characteristic exponents are to have a nonvanishing real part, the absolute values of the three characteristic roots must be different from one.

Let the stability of the found solution in the nonresonant case be investigated in the first place so that the solution from the theorem 1,1 is inserted into (3,2) and (3,4).

The matrix solution of (3,3) can be written as

$$\mathbf{U}(t, \varepsilon) = \mathbf{V}(t) + \varepsilon\, \mathbf{W}(t, \varepsilon)\,, \tag{3,7}$$

while $\mathbf{V}$ and $\mathbf{W}$ fulfil the following differential equations and initial conditions:

$$\dot{\mathbf{V}} = \mathbf{A}\mathbf{V}\,, \quad \mathbf{V}(0) = \mathbf{E}\,,$$

$$\dot{\mathbf{W}} = (\mathbf{A} + \varepsilon \mathbf{B})\, \mathbf{W} + \mathbf{B}\mathbf{V}\,, \quad \mathbf{W}(0, \varepsilon) = 0\,,$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{T} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{B} = \mathbf{B}(t, \varepsilon) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & B & C & D \\ \varepsilon E & \varepsilon F & m + \varepsilon G & \varepsilon H \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can easily be verified that

$$\mathbf{V}(t) = \begin{pmatrix} e^{t\mathbf{S}} & 0 \\ 0 & e^{t\mathbf{T}} \end{pmatrix}, \quad e^{t\mathbf{S}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad e^{t\mathbf{T}} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

$$\mathbf{W}(t, \varepsilon) = \int_0^t e^{(t-s)\mathbf{A}}\, \mathbf{B}(s, \varepsilon)[e^{s\mathbf{A}} + \varepsilon\, \mathbf{W}(s, \varepsilon)]\, \mathrm{d}s\,,$$

$$\mathbf{W}(t, 0) = \int_0^t e^{(t-s)\mathbf{A}} \mathbf{B}(s, 0)\, e^{s\mathbf{A}}\, \mathrm{d}s = \int_0^t \begin{pmatrix} \mathfrak{A}(s) & \bullet \\ 0 & \mathfrak{M}(s) \end{pmatrix} \mathrm{d}s\,,$$

$$\mathfrak{A}(s) = \begin{pmatrix} \cos(t-s) & -\sin(t-s) \\ \sin(t-s) & \cos(t-s) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A_0 & B_0 \end{pmatrix} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix}, \quad \mathfrak{M} = \begin{pmatrix} m_0 & 0 \\ \bullet & 0 \end{pmatrix}, \tag{3,8}$$

where $A_0$, $B_0$, $m_0$ respectively, are notations for the functions $A(s, 0)$, $B(s, 0)$, $m(s, 0)$ respectively and the terms having no significance for further calculations are indicated by dots.

Let (3,8) be inserted into (3,5). Since for $\varepsilon = 0$ the characteristic roots of (3,5) are in this case $\cos T_0^* \pm \sin T_0^*$ $\left(T_0^* = \dfrac{2\pi N}{\omega_0^*} \neq 2\pi\nu,\ \nu \text{ being an integer}\right)$, 1, 1, the equation (3,5) fulfils all assumptions of the lemma 2,1 so that (3,5) can be written in the form mentioned in lemma 2,1. In consequence we can write

$$D(\varepsilon) \equiv [\varrho^2 - 2\varrho(\cos T(\varepsilon) + \varepsilon\, P_1(\varepsilon)) + 1 + \varepsilon\, Q_1(\varepsilon)]\,. \\ .\,[\varrho^2 - 2\varrho(1 + \varepsilon\, P_2(\varepsilon)) + 1 + \varepsilon\, Q_2(\varepsilon)] = 0\,, \tag{3,5'}$$

where as it is easy to find

$$2P_1(0) = \int_0^{T_0^*} [-A_0 \sin T_0^* + B_0 \cos T_0^*]\, \mathrm{d}s =$$

$$= \int_0^{T_0^*} \left[\left(\frac{\partial f}{\partial x}\right)^* \sin T_0^* + \left(\frac{\partial f}{\partial \dot{x}}\right) \cos T_0^*\right] \mathrm{d}s\,,$$

$$Q_1(0) = \int_0^{T_0^*} B_0\, \mathrm{d}s = \int_0^{T_0^*} \left(\frac{\partial f}{\partial \dot{x}}\right)^* \mathrm{d}s\,,$$

$$2P_2(0) = Q_2(0) = \int_0^{T_0^*} m_0\, \mathrm{d}s = T_0^*\, M'(\omega_0^*)\,. \tag{3,9}$$

It is evident that for sufficiently small $\varepsilon$ only the second quadratic expression can have the root $\varrho = 1$ and thus

$$-\,2P_2(\varepsilon) + Q_2(\varepsilon) = 0\,. \tag{3,10}$$

Making use of the lemmas 2,2 and 2,3 and the theorem 2,1 the following theorem follows immediately from (3,5) and (3,9):

**Theorem 3,1.** *Let the assumptions of the theorem* 1,1 *be fulfilled. Then the periodic solution of the second kind, the existence of which follows from this theorem, has asymptotic orbital stability, if for sufficiently small* $\varepsilon \neq 0$

$$\operatorname{sign} \varepsilon\,.\int_0^{T_0^*} \left(\frac{\partial f}{\partial \dot{x}}\right)^* \mathrm{d}s < 0\,, \quad \operatorname{sign} \varepsilon\,.\, M'(\omega_0^*) < 0\,, \tag{3,11}$$

*holds, and it is orbitally unstable, if any of the inequalities* (3,11) *holds with the opposite sign.*

To investigate the stability of the resonant case lemma 2,1 obviously cannot be made use of because the determinant of the system (2,9) is vanishing for $T_0^* = 2\pi n$.

To simplify further calculations let the time be transformed:

$$\vartheta = \omega^*(\varepsilon) \frac{n}{N} t \,. \tag{3,12}$$

(This transformation does not influence the investigation of stability for sufficiently small $\varepsilon$ or to put it more accurately as long as $\varepsilon \frac{n}{N} \Omega^*(\varepsilon) > -1$.)

Simultaneously, wherever convenient

$$\omega^*(\varepsilon) = \frac{N}{n} + \varepsilon\, \Omega^*(\varepsilon) \tag{3,13}$$

will be put.

Finally let the following notation be introduced

$$\Gamma(\varepsilon) = \frac{\Omega^*(\varepsilon)}{\omega^*(\varepsilon)}\,, \quad \nu(\varepsilon) = \frac{N}{n\,\omega^*(\varepsilon)}\,, \quad m(\vartheta, \varepsilon) = \nu(\varepsilon)\, M'(\omega_0^* + \varepsilon\, \Psi^*(\nu(\varepsilon)\, \vartheta, \varepsilon))\,,$$

$$A(\vartheta, \varepsilon) = -\nu(\varepsilon) \left(\frac{\partial f}{\partial x}\right)^*, \quad B(\vartheta, \varepsilon) = \nu(\varepsilon) \left(\frac{\partial f}{\partial \dot{x}}\right)^*, \quad C(\vartheta, \varepsilon) = -\nu(\varepsilon) \left(\frac{\partial f}{\partial \dot{\varphi}}\right)^*,$$

$$D(\vartheta, \varepsilon) = -\nu(\varepsilon) \left(\frac{\partial f}{\partial \varphi}\right)^*, \quad E(\vartheta, \varepsilon) = \nu(\varepsilon) \left(\frac{\partial g}{\partial x}\right)^*, \quad F(\vartheta, \varepsilon) = \nu(\varepsilon) \left(\frac{\partial g}{\partial \dot{x}}\right)^*,$$

$$G(\vartheta, \varepsilon) = \nu(\varepsilon) \left(\frac{\partial g}{\partial \dot{\varphi}}\right)^*, \quad H(\vartheta, \varepsilon) = \nu(\varepsilon) \left(\frac{\partial g}{\partial \varphi}\right)^*.$$

The asterisks by the partial derivatives denote that for $x$, $y$ and $\Phi$ and $\Psi$ in $\varphi$ and $\psi$ the solution from the theorem 1,2 has been inserted, in which the transformation of time (3,12) had been performed.

The system (3,3) can be now replaced by an equivalent system ($'$ denotes the derivation with respect to $\vartheta$):

$$\begin{aligned}
u_1' &= -u_2 + \varepsilon \Gamma u_2\,,\\
u_2' &= u_1 - \varepsilon \Gamma u_1 + \varepsilon[A(\vartheta, \varepsilon)\, u_1 + B(\vartheta, \varepsilon)\, u_2 + C(\vartheta, \varepsilon)\, u_3 + D(\vartheta, \varepsilon)\, u_4]\,,\\
u_3' &= \varepsilon m(\vartheta, \varepsilon)\, u_3 + \varepsilon^2[E(\vartheta, \varepsilon)\, u_1 + F(\vartheta, \varepsilon)\, u_2 + G(\vartheta, \varepsilon)\, u_3 + H(\vartheta, \varepsilon)\, u_4]\,,\\
u_4' &= u_3 - \varepsilon \Gamma u_3\,,
\end{aligned} \tag{3,14}$$

the right sides of which are $2\pi n$-periodic in $\vartheta$.

Let now the following notation be introduced

$$\mathbf{B}(\vartheta, \varepsilon) = \begin{pmatrix} 0 & \Gamma & 0 & 0 \\ A - \Gamma & B & C & D \\ 0 & 0 & m & 0 \\ 0 & 0 & -\Gamma & 0 \end{pmatrix}, \quad \mathbf{C}(\vartheta, \varepsilon) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & F & G & H \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3,15}$$

while the matrices $\mathbf{A}$, $\mathbf{S}$ and $\mathbf{T}$ have the same meaning as before.

The matrix solution $\mathbf{U}(\vartheta, \varepsilon)$ with $\mathbf{U}(0, \varepsilon) = \mathbf{E}$ of (3,14) can be written in the form

$$\mathbf{U}(\vartheta, \varepsilon) = \mathbf{V}(\vartheta) + \varepsilon\, \mathbf{W}(\vartheta, \varepsilon) + \varepsilon^2\, \mathbf{Z}(\vartheta, \varepsilon)\,, \tag{3,16}$$

where $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{Z}$ fulfil the following differential equations and initial conditions:

$$\begin{aligned}
&\mathbf{V}' = \mathbf{A}\mathbf{V}\,, \quad \mathbf{V}(0) = \mathbf{E}\,,\\
&\mathbf{W}' = \mathbf{A}\mathbf{W} + \mathbf{B}(\vartheta, \varepsilon)\,\mathbf{V}\,, \quad \mathbf{W}(0, \varepsilon) = 0\,,\\
&\mathbf{Z}' = (\mathbf{A} + \varepsilon\, \mathbf{B}(\vartheta, \varepsilon) + \varepsilon^2\, \mathbf{C}(\vartheta, \varepsilon)\, \mathbf{Z} + (\mathbf{B}(\vartheta, \varepsilon) + \varepsilon\, \mathbf{C}(\vartheta, \varepsilon))\, \mathbf{W} + \mathbf{C}(\vartheta, \varepsilon)\, \mathbf{V}\,,\\
&\qquad \mathbf{Z}(0, \varepsilon) = 0\,.
\end{aligned} \tag{3,17}$$

There holds:

$$\begin{aligned}
\mathbf{V}(\vartheta) &= e^{\vartheta \mathbf{A}}\,,\\
\mathbf{W}(\vartheta, \varepsilon) &= \int^{\vartheta} e^{(\vartheta - s)\mathbf{A}}\, \mathbf{B}(s, \varepsilon)\, e^{s\mathbf{A}}\, \mathrm{d}s\,,\\
\mathbf{Z}(\vartheta, \varepsilon) &= \int_0^{\vartheta} e^{(\vartheta - s)\mathbf{A}}[\varepsilon(\mathbf{B}(s, \varepsilon) + \varepsilon^2\, \mathbf{C}(s, \varepsilon))\, \mathbf{Z}(s, \varepsilon) + (\mathbf{B}(s, \varepsilon) + \varepsilon\, \mathbf{C}(s, \varepsilon))\, \mathbf{W}(s, \varepsilon) +\\
&\quad + \mathbf{C}(s, \varepsilon)\, e^{s\mathbf{A}}]\, \mathrm{d}s\,.
\end{aligned} \tag{3,18}$$

$v_{ij}$, $w_{ij}$, ${w_{ij}}^{(0)}$, $z_{ij}$ and $z_{ij}^{(0)}$ denoting the elements of matrices $\mathbf{V}(2\pi n)$, $\mathbf{W}(2\pi n, \varepsilon)$, $\mathbf{W}(2\pi n, 0)$, $\mathbf{Z}(2\pi n, \varepsilon)$ and $\mathbf{Z}(2\pi n, 0)$ it can easily be verified that

$$\begin{aligned}
&v_{11} = v_{22} = v_{33} = v_{44} = 1\,, \quad v_{43} = 2\pi n \text{ and all remaining } v_{ij} = 0\,,\\
&w_{31}^{(0)} = w_{32}^{(0)} = w_{34}^{(0)} = w_{41}^{(0)} = w_{42}^{(0)} = w_{43}^{(0)} = 0\,.
\end{aligned} \tag{3,19}$$

Let us now insert into (3,5) $\mathbf{U}$ according to (3,16) and put $T = 2\pi n$. Let $\varrho - 1 = \sigma$. With the aid of a more detailed calculation we can easily find that the biquadratic equation for $\sigma$ has the following form:

$$\begin{aligned}
&\sigma^4 - \sigma^3\, \varepsilon[w_{11}^{(0)} + w_{22}^{(0)} + w_{33}^{(0)} + P_1(\varepsilon)] +\\
&+ \sigma^2\, \varepsilon^2[2\pi n z_{34}^{(0)} + w_{11}^{(0)}w_{22}^{(0)} - w_{12}^{(0)}w_{21}^{(0)} + (w_{11}^{(0)} + w_{22}^{(0)})\, w_{33}^{(0)} + P_2(\varepsilon)] +\\
&+ \sigma\, \varepsilon^3[-\, 2\pi n(w_{22}^{(0)}z_{34}^{(0)} - w_{24}^{(0)}z_{32}^{(0)} + w_{11}^{(0)}z_{34}^{(0)} - w_{14}^{(0)}z_{31}^{(0)}) + w_{11}^{(0)}w_{22}^{(0)}w_{33}^{(0)} + P_3(\varepsilon)] +\\
&+ \varepsilon^4\, Q(\varepsilon) = 0\,,
\end{aligned} \tag{3,20}$$

where $P_i(\varepsilon)$ ($i = 1, 2, 3$) and $Q(\varepsilon)$ are continuous functions of $\varepsilon$ in the neighbourhood of $\varepsilon = 0$ and $P_i(0) = 0$. Since, as we know, (3,20) must have for sufficiently small $\varepsilon$ one root equal to zero, $Q(\varepsilon) = 0$ holds for sufficiently small $\varepsilon$.

Let us now take notice of the fact, that if $\varrho = 1 + \sigma$ where $\sigma = \varepsilon(a(\varepsilon) + i\, b(\varepsilon))$ and $a(\varepsilon)$ and $b(\varepsilon)$ are real continuous functions in the neighbourhood of $\varepsilon = 0$, then evidently for sufficiently small $\varepsilon \neq 0$ $|\varrho| < 1$ ($|\varrho| > 1$), if and only if $a(0)$ sign $(\varepsilon) < 0$ ($a(0)$ sign $\varepsilon > 0$). Since according to lemma 2,4 the expression in (3,20) can be written in the form $\sigma \prod_{t=1}^{3}(\sigma - \varepsilon\, \alpha_2(\varepsilon))$, the condition for the three characteristic roots $\varrho_k$ for small $\varepsilon \neq 0$ to have their absolute values less than one is equivalent to the condition that the three roots $\varepsilon\, \alpha_i(\varepsilon)$ have for small $\varepsilon \neq 0$ negative real parts. Analogously the condition for at least one characteristic root to have for small $\varepsilon \neq 0$ its absolute value more than one is equivalent to the condition that at least one root $\varepsilon\alpha_i(\varepsilon)$ of (3,20) have for small $\varepsilon \neq 0$ a positive real part.

Making use of the Hurwitz criterion it is easy to see that the conditions for the equation (3,20) to have for sufficiently small $\varepsilon \neq 0$, apart from the vanishing root, three roots with negative real parts are following:

$$(\text{sign}\ \varepsilon)\ (w_{11}^{(0)} + w_{22}^{(0)} + w_{33}^{(0)}) < 0\ ,$$

$$(\text{sign}\ \varepsilon)\ \{[w_{11}^{(0)} + w_{22}^{(0)} + w_{33}^{(0)}]\,[2\pi n z_{34}^{(0)} + w_{11}^{(0)} w_{22}^{(0)} - w_{12}^{(0)} w_{21}^{(0)} + (w_{11}^{(0)} + w_{22}^{(0)})\, w_{33}^{(0)}] -$$
$$- [- 2\pi n(w_{22}^{(0)} z_{34}^{(0)} - w_{24}^{(0)} z_{32}^{(0)} + w_{11}^{(0)} z_{34}^{(0)} - w_{14}^{(0)} z_{31}^{(0)}) + w_{11}^{(0)} w_{22}^{(0)} w_{33}^{(0)}]\} < 0\ ,$$

$$(\text{sign}\ \varepsilon)\ [- 2\pi n(w_{22}^{(0)} z_{34}^{(0)} - w_{24}^{(0)} z_{32}^{(0)} + w_{11}^{(0)} z_{34}^{(0)} - w_{14}^{(0)} z_{31}^{(0)} + w_{11}^{(0)} w_{22}^{(0)} w_{33}^{(0)}] < 0\ .$$

Hence inserting for $w_{ij}^{(0)}$ and $z_{ij}^{(0)}$ from (3,18) and (3,13) and performing some arithmetical operations yields the theorem.

**Theorem 3,2.** *Let the assumptions of the theorem* 1,2 *be fulfilled. Then the periodic solution of the second kind, the existence of which follows from this theorem, has asymptotic orbital stability, if for sufficiently small* $\varepsilon \neq 0$

$$0 > (\text{sign}\ \varepsilon) \left[ \int_0^{2\pi n} \left(\frac{\partial f}{\partial \dot{x}}\right)^* \mathrm{d}t + 2\pi n\, M'(\omega_0^*) \right],$$

$$0 > (\text{sign}\ \varepsilon) \Bigg\{ M'(\omega_0^*) \Bigg[ \left( 2\pi n \Gamma_0 + \frac{1}{2} \int_0^{2\pi n} \left(\frac{\partial f}{\partial x}\right)^* \mathrm{d}t \right)^2 - \frac{1}{4} \left( \int_0^{2\pi n} \left[ \left(\frac{\partial f}{\partial x}\right)^* \sin 2t + \right.\right.$$
$$\left.\left. + \left(\frac{\partial f}{\partial \dot{x}}\right) \cos 2t \right] \mathrm{d}t \right)^2 - \frac{1}{4} \left( \int_0^{2\pi n} \left[ \left(\frac{\partial f}{\partial x}\right)^* \cos 2t - \left(\frac{\partial f}{\partial \dot{x}}\right)^* \sin 2t \right] \mathrm{d}t \right)^2 +$$
$$+ \frac{1}{4} \left( \int_0^{2\pi n} \left(\frac{\partial f}{\partial \dot{x}}\right)^* \mathrm{d}t \right)^2 \Bigg] - \int_0^{2\pi n} \left(\frac{\partial f}{\partial x}\right)^* \mathrm{d}t \int_0^{2\pi n} \left(\frac{\partial g}{\partial \dot{\varphi}}\right)^* \mathrm{d}t - \tag{3,21}$$
$$- \int_0^{2\pi n} \left(\frac{\partial f}{\partial \varphi}\right)^* \cos t\, \mathrm{d}t \int_0^{2\pi n} \left[ - \left(\frac{\partial g}{\partial x}\right)^* \sin t + \left(\frac{\partial g}{\partial \dot{x}}\right)^* \cos t \right] \mathrm{d}t -$$
$$- \int_0^{2\pi n} \left(\frac{\partial f}{\partial \varphi}\right)^* \sin t\, \mathrm{d}t \int_0^{2\pi n} \left[ \left(\frac{\partial g}{\partial x}\right)^* \cos t + \left(\frac{\partial g}{\partial \dot{x}}\right)^* \sin t \right] \mathrm{d}t \Bigg\},$$

$$0 > (\text{sign}\ \varepsilon) \Bigg\{ - 2\pi n \Bigg[ 2\pi n\, M'(\omega_0^*) \int_0^{2\pi n} \left(\frac{\partial g}{\partial \varphi}\right)^* \mathrm{d}t -$$

$$
- \int_0^{2\pi n} \left(\frac{\partial f}{\partial \varphi}\right)^* \sin t \, dt \int_0^{2\pi n} \left[\left(\frac{\partial g}{\partial x}\right)^* \cos t + \left(\frac{\partial g}{\partial \dot{x}}\right)^* \sin t\right] dt -
$$

$$
- \int_0^{2\pi n} \left(\frac{\partial f}{\partial \varphi}\right)^* \cos t \, dt \int_0^{2\pi n} \left[- \left(\frac{\partial g}{\partial x}\right)^* \sin t + \left(\frac{\partial g}{\partial \dot{x}}\right)^* \cos t\right] dt \Bigg] +
$$

$$
+ \int^{2\pi n} \left(\frac{\partial f}{\partial \dot{x}}\right)^* dt \left[\left(2\pi n \Gamma_0 + \frac{1}{2} \int_0^{2\pi n} \left(\frac{\partial f}{\partial x}\right)^* dt\right)^2 - \frac{1}{4} \left(\int_0^{2\pi n} \left[\left(\frac{\partial f}{\partial x}\right)^* \sin 2t + \right.\right.\right.
$$

$$
\left.\left. + \left(\frac{\partial f}{\partial \dot{x}}\right) \cos 2t\right] dt\right)^2 - \frac{1}{4} \left(\int_0^{2\pi n} \left[\left(\frac{\partial f}{\partial x}\right)^* \cos 2t - \left(\frac{\partial f}{\partial \dot{x}}\right) \sin 2t\right] dt\right)^2 +
$$

$$
\left. + \frac{1}{4} \left(\int_0^{2\pi n} \left(\frac{\partial f}{\partial \dot{x}}\right)^* dt\right)^2 + 2\pi n \, M'(\omega_0^*) \left[2\pi n \, M'(\omega_0^*) + \int_0^{2\pi n} \left(\frac{\partial f}{\partial \dot{x}}\right)^* dt\right]\right\}
$$

*holds and is orbitally unstable if any of the inequalities* (3,21) *holds with the opposite sign.*

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## Резюме

# О СУЩЕСТВОВАНИИ И УСТОЙЧИВОСТИ ПЕРИОДИЧЕСКОГО РЕШЕНИЯ 2-ГО РОДА ОПРЕДЕЛЕННОЙ МЕХАНИЧЕСКОЙ СИСТЕМЫ


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В работе исследуется движение слабо нелинейного осцилятора, возбуждаемого мотором, который неможем считать твердым источником энергии, так что нельзя пренебречь обратным воздействием движения осцилятора на ход мотора. (Исследование частных случаев см. [1], [2] и [4]).

Первый параграф работы посвящен исследованию существования решения системы (1,1) в виде (1,2), где функции $x$, $\dot{x} = -y$, $\Phi$ и $\dot{\Phi} = \Psi$ являются периодическими функциями с периодом $\frac{2\pi N}{\omega}$ ($N$ — натуральное число).

Доказаны следующие три теоремы:

**Теорема 1,1.** *Пусть*

(a) *$f$ и $g$ — $2\pi$-периодические функции по переменному $\varphi$;*

(b) *уравнение $M(\omega_0) = 0$ имеет по крайней мере один положительньй корень $\omega_0^* \neq \frac{N}{n}$ ($n$ — натуральное число);*

(c) *для $|\varepsilon| \leqq \varepsilon_0$ ($\varepsilon_0 > 0$) и для $x$, $\dot{x}$, $\varphi$ и $\dot{\varphi}$ из области $G$ пространства ($x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$), определенной как окрестность множества $(0, 0, \omega^* t, \omega_0^*)$, причем $t$ принимает значения из интервала $\left\langle 0, \frac{2\pi N}{\omega_9^*}\right\rangle$, функции $f$, $g$ и $M$, равно как их частные производные первого порядка по переменным $x$, $\dot{x}$, $\varphi$ и $\dot{\varphi}$ непрерывны в $x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$ и $\varepsilon$;*

(d) $M'(\omega_0^*) \neq 0$.

*Тогда существует одно и только одно решение системы* (1,1) *вида* (1,2), *для которого функции $x^*(t, \varepsilon)$, $y^*(t, \varepsilon)$, $\Psi^*(t, \varepsilon)$ и $\Phi^*(t, \varepsilon)$ периодичны по $t$ с периодом $\frac{2\pi N}{\omega^*(\varepsilon)}$; кроме того, они непрерывны по $t$ и $\varepsilon$ для всех $t$ и для достаточно малых $\varepsilon$; $\omega^*(\varepsilon)$ непрерывна для достаточно малых $\varepsilon$ и $x^*(t, 0) = y^*(t, 0) = \Psi^*(t, 0) = 0, \omega^*(0) = \omega_0^*$.*

**Теорема 1,2.** *Пусть*

(a) *функции $f$ и $g$ $2\pi$-периодичны по переменной $\varphi$;*

(b) *уравнение $M(\omega_0) = 0$ имеет по крайней мере один действительный положительный корень $\omega_0^* = \frac{N}{n}$ ($n$ — натуральное число);*

(c) *система уравнений $(1,9_0')$ и (1,16) имеет действительное решение $x_0^*$, $y_0^*$ и $\Omega_0^*$;*

(d) *для $|\varepsilon| \leqq \varepsilon_0$ ($\varepsilon_0 > 0$) и для $x$, $\dot{x}$, $\varphi$ и $\dot{\varphi}$ из области $G$, определенной как окрестность множества*

$$(x_0^* \cos t - y_0^* \sin t\,, \quad - x_0^* \sin t - y_0^* \cos t\,, \quad \omega_0^* t\,, \quad \omega_0^*)\,,$$

*где $t$ принимает значения из интервала $\langle 0, 2\pi n\rangle$, функции $f$, $g$ и $M$, а также и их частные производные первого порядка по $x$, $\dot{x}$ $\varphi$ и $\dot{\varphi}$ непрерывны в $x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$ и $\varepsilon$;*

(e) *определитель Якоби системы уравнений $(1,9_0')$ и $(1,16_0)$ относительно $x_0$, $y_0$ и $\Omega_0$ не равен в точке $(x_0^*, y_0^*, \Omega_0^*)$ нулю.*

*Тогда для достаточно малых $\varepsilon$ существует одно и только одно решение системы* (1,1) *вида* (1,2), *для которого функции* $x^*(t, \varepsilon)$, $y^*(t, \varepsilon)$, $\Psi^*(t, \varepsilon)$ *и* $\Phi^*(t, \varepsilon)$ *периодичны по* $t$ *с периодом* $T^*(\varepsilon) = \frac{2\pi N}{\omega^*(\varepsilon)}$, *где* $\omega^*(\varepsilon) = \omega_0^* + \varepsilon\Omega^*(\varepsilon)$, *причем функция* $\Omega^*(\varepsilon)$ *непрерывна; кроме того, эти функции непрерывны для всех* $t$ *и достаточно малых* $\varepsilon$, *причем*

$$x^*(t, 0) = x_0^* \cos t - y_0^* \sin t\,, \quad y_0^*(t, 0) = - x_0^* \sin t - y_0^* \cos t\,,$$

$$\psi^*(t, 0) = 0\,, \quad \Omega^*(0) = \Omega_0^*\,.$$

**Теорема 1,3.** *Пусть функции* $f$, $g$ *и* $M$ *удовлетворяют условиям* (a)—(d) *теоремы* (1,1) *или условиям* (a)—(e) *теоремы* (1,2) *и пусть они аналитичны по* $x$, $\dot{x}$, $\varphi$, $\dot{\varphi}$ *и* $\varepsilon$ *для* $(x, \dot{x}, \varphi, \dot{\varphi}) \in G$ *и* $|\varepsilon| \leqq \varepsilon_0$.

*Тогда соответствующие решения, равно как и соответствующие функции* $\omega(\varepsilon)$ *или* $\Omega(\varepsilon)$ (*и, следовательно, также* $T(\varepsilon)$) *аналитичны для всех* $t$ *и для достаточно малых* $\varepsilon$.

Во втором параграфе доказывается в первую очередь следующая модификация теоремы Андронова-Витта:

**Теорема 2,1.** *Пусть задана система* (2,1), *где* $\mathbf{x}$ *и* $\mathbf{f}$ *представляют собой* $n$*-мерные векторы. Пусть* $\mathbf{p}$ — $n$*-мерный вектор, некоторые из составляющих которого равны единице, а остальные нулю. Пусть функция* $f$ *удовлетворяет соотношению* (2,2).

*Пусть* (2,1) *имеет решение вида* (2,3), *где векторная функция* $\boldsymbol{\psi}(t)$ *периодична по* $t$ *с периодом* $T = \frac{2\pi N}{\omega}$ ($N$ — *натуральное число*).

*Пусть функция* $\mathbf{f}$ *определена и имеет непрерывные частные производные первого порядка по* $x$ *в окрестности* $V$ *функции* $\boldsymbol{\varphi}(t)$ *для* $t \geqq 0$.

*Пусть уравнения в вариациях* (2,4) *имеют* $n - 1$ *характеристических показателей с отрицательной действительной частью.*

*Тогда существует число* $\eta > 0$ *такое, что удовлетворяет ли произвольное решение* $\boldsymbol{\xi}(t)$ *системы* (2,1) *неравенству* $\|\boldsymbol{\xi}(t_1) - \boldsymbol{\varphi}(t_0)\| \leqq \eta$ *для некоторых* $t_0$ *и* $t_1$, *то существует число* $c$ *так, что имеет место соотношение* (2,5) (*т. е. решение является асимптотически орбитально устойчивым имеет асимптотическую фазу.*).

*Наоборот, если по крайней мере один характеристический показатель системы* (2,4) *имеет положительную действительную часть, то решение* $\boldsymbol{\varphi}(t)$ *орбитально неустойчиво.*

При помощи этой теоремы производится в третьем параграфе исследование устойчивости решений, найденных в первом параграфе, и доказываются следующие теоремы:

**Теорема 3,1.** *Пусть выполнены условия теоремы* 1,1. *Тогда решение, существование которого гарантировано этой теоремой, является асимптотически орбитально устойчивым, если для достаточно малых* $\varepsilon \neq 0$ *справедливы неравенства* (3,11); *если же в некотором из этих неравенств имеет место обратный знак неравенства, то упомянутое решение является орбитально неустойчивым.*

**Теорема 3,2.** *Пусть выполнены условия теоремы* 1,2. *Тогда решение, существование которого гарантировано этой теоремой, является асимптотически орбитально устойчивым, если для достаточно малых* $\varepsilon \neq 0$ *справедливы неравенства* (3,21); *если же в некотором из этих неравенств имеет место обратный знак неравенства, то упомянутое решение является орбитально неустойчивым.*