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REGULARITY PROPERTIES OF RANDOM TRANSFORMS

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The purpose of this paper is to establish a number of theorems on random transforms, which, according to our definition, are slightly generalized random processes. The properties of random transforms such as for example almost sure continuity will be considered under a more general concept of regularity. It seems to be not without interest to show how, for example, a version of Doob's continuity theorem follows from our completely elementary results.

§ 1. The basic space of elementary events is the set F of all transforms $f(x)$ of a fixed abstract space $X \neq 0$ into another fixed abstract space $Y \neq 0$ and, according to Kolmogorov [1], the σ-algebra of random events will be defined as follows: Given a σ-algebra $\mathfrak{B}$ of subsets of Y the elements of which will be called Borel sets and let $\mathfrak{G}$ be a base of $\mathfrak{B}$. For each fixed $x \in X$ and each fixed $G \in \mathfrak{G}$ the set $\{f : f(x) \in G\}$ will be called $\mathfrak{G}$-interval. For $A \subset X$ we shall denote by $\mathfrak{F}(A)$ the smallest σ-algebra of subsets of F generated by the class of all $\mathfrak{G}$-intervals, where x runs over A and G runs over $\mathfrak{G}$. Instead of $\mathfrak{F}(X)$ we shall write briefly $\mathfrak{F}$ and this is our basic σ-algebra of random events.

It is easy to verify that $\mathfrak{F}(A)$ does not depend on the choice of the base $\mathfrak{G}$, i. e. we obtain the same σ-algebra $\mathfrak{F}(A)$ if we replace the $\mathfrak{G}$-intervals by the corresponding $\mathfrak{B}$-intervals. It is further clear that $\mathfrak{F}(0) = \{0, F\}$ and $\mathfrak{F}(A) \subset \subset \mathfrak{F}(B)$ for $A \subset B \subset X$.

A useful property of the σ-algebra of random events is expressed by the identity

$$\mathfrak{F} = \bigcup_{D \in \mathfrak{D}} \mathfrak{F}(D) , \tag{1}$$

where $\mathfrak{D}$ is a class of *denumerable* subsets of X which satisfies the following *two conditions*:

$$\textit{the union of all sets from } \mathfrak{D} \textit{ is equal to } X , \tag{2}$$

$$\left\{ \begin{array}{l} \textit{each denumerable union of sets from } \mathfrak{D} , \\ \textit{is contained in at least one set from } \mathfrak{D} . \end{array} \right. \tag{3}$$

For the proof of (1) we note only that, as may be easily shown, the right hand side of (1) is a σ-algebra of subsets of F containing the class of all $\mathfrak{G}$-intervals and each term of the union in (1) is contained in $\mathfrak{F}$.

The well known fact that each random event is determined by denumerably many coordinates will be expressed more precisely by

Lemma 1. *To each $E \in \mathfrak{F}$ there exists a set $D_E \in \mathfrak{D}$ such that*

$$E = \bigcup_{g \in E} \bigcap_{x \in D_E} \{f : f(x) = g(x)\} .$$

Proof: For $A \subset X$ let us denote by $\mathfrak{C}(A)$ the class of all sets $\bigcap_{x \in A} \{f : f(x) = g(x)\} \subset F$, where g runs over F. We see at once that the class $\mathfrak{A}(A)$ of all unions of sets from $\mathfrak{C}(A)$ is a σ-algebra of subsets of F. Clearly, if $x_0 \in A$, $G \in \mathfrak{G}$ and $H = \{f : f(x_0) \in G\}$, then

$$H = \bigcup_{g \in H} \bigcap_{x \in A} \{f : f(x) = g(x)\} \in \mathfrak{A}(A)$$

and hence

$$\mathfrak{F}(A) \subset \mathfrak{A}(A) \text{ for } A \subset X . \tag{4}$$

By (1) to every $E \in \mathfrak{F}$ there corresponds a $D_E \in \mathfrak{D}$ such that $E \in \mathfrak{F}(D_E)$ and therefore by (4) $E \in \mathfrak{A}(D_E)$. Using the definition of $\mathfrak{A}(D_E)$, we see that

$$\bigcap_{x \in D_E} \{f : f(x) = g(x)\} \subset E \text{ for } g \in E ,$$

hence,

$$\bigcup_{g \in E} \bigcap_{x \in D_E} \{f : f(x) = g(x)\} \subset E .$$

The opposite set-inclusion, which completes the proof, is obvious.

The notion of regularity will be introduced by a transform T of the class of all subsets of X into the class of all subsets of F which satisfies the following *conditions*:

$$\textit{if } D \in \mathfrak{D} \textit{ and } g \in T(D) \textit{ then } T(X) \cap \bigcap_{x \in D} \{f : f(x) = g(x)\} \neq 0 , \tag{5}$$

$$\textit{if } D \in \mathfrak{D} \textit{ then } T(D) \in \mathfrak{F} . \tag{6}$$

$$\textit{if } A \subset B \subset X \textit{ then } T(B) \subset T(A) , \tag{7}$$

$$T(X) \neq 0 . \tag{8}$$

The example $T(A) = F$ for $A \subset X$ shows that the above conditions can be satisfied simultaneously. Less trivial examples will be considered in §§ 2, 3 and 4.

A useful tool for proving the main theorem is

Lemma 2. *If T satisfies the sole condition* (5) *and if $T(X) \subset E \in \mathfrak{F}$ then there exists a set $D_0 \in \mathfrak{D}$ such that $T(D_0) \subset E$.*

Proof: By lemma 1 there is a set $D_0 \in \mathfrak{D}$ such that

$$\text{if } g \in F - E = E' \text{ then } \bigcap_{x \in D_0} \{f : f(x) = g(x)\} \subset E' . \qquad (9)$$

Let us suppose that the assertion of lemma 2 is not true, i. e. to each set from $\mathfrak{D}$ and therefore also to D_0 there corresponds an $h_0 \in F$ such that

$$h_0 \in E' , \qquad (10)$$

$$h_0 \in T(D_0) . \qquad (11)$$

Because of (5) and (11) there is an $f_0 \in F$ such that

$$f_0 \in T(X) , \qquad (12)$$

$$f_0 \in \bigcap_{x \in D_0} \{f : f(x) = h_0(x)\} . \qquad (13)$$

Since by hypothesis $T(X) \subset E$, we have by (12) $f_0 \in E$. On the other hand, because of (10) and (9), it follows from (13) that $f_0 \in E'$ and this is a contradiction.

In lemma 2 we have used only the property (5) of T. But from now on we shall consistently suppose that the transform T satisfies all conditions (5), (6) (7) and (8) simultaneously.

If $M \subset F$ then $M \cap \mathfrak{F}$ means the class of all sets $M \cap E$ where E runs over $\mathfrak{F}$. Clearly, the class $M \cap \mathfrak{F}$ is a σ-algebra of subsets of M. If μ is a probability measure in $\mathfrak{F}$ then $(F, \mathfrak{F}, \mu)$ is said to be a random transform or a generalized random process. It is natural to define almost sure T-regularity of a random transform as follows: The random transform $(F, \mathfrak{F}, \mu)$ is said to be almost surely T-regular or T-regular with probability unity, if there exists one and only one probability measure ν in $T(X) \cap \mathfrak{F}$ such that $\nu(T(X) \cap E) = \mu(E)$ for $E \in \mathfrak{F}$. From a well known theorem of Doob [2] it follows at once that $(F, \mathfrak{F}, \mu)$ is T-regular with probability unity if and only if $\bar{\mu}(T(X)) = 1$, where $\bar{\mu}$ denotes outer measure, and we prefer to use this equivalent definition of almost sure T-regularity.

Now we proceed to establish the main result. It is essentially contained in the following elementary

Theorem 1. *A necessary and sufficient condition for a random transform* $(F, \mathfrak{F}, \mu)$ *to be almost surely* T*-regular is that* $\mu(T(D)) = 1$ *for every* $D \in \mathfrak{D}$.

Proof: The necessity of the condition is obvious. Since utilizing (6) and (7) the assertion of lemma 2 may be completed by $T(D_0) \in \mathfrak{F}$ and $T(X) \subset T(D_0)$, hence, the sufficiency of the condition follows at once from the definition of outer measure.

Although theorem 1 holds without any additional restriction, the application of that result is not useful if the space X is denumerable.

In the rest of this paper, unless explicitly stated otherwise, we shall consistently assume that X is a separable metric space with the distance function δ, Y is a separable and complete metric space with the distance function ϱ, $\mathfrak{G}$ is a denumerable open base of Y and $\mathfrak{D}$ is the class of all denumerable subsets of X dense in X which, as may be easily shown, satisfies the conditions (2) and (3).

§ 2. We shall now give the transform T the following concrete meaning: $T(A)$ for $A \subset X$ is the set of all transforms from F which are uniformly continuous in A, i. e.

$$T(A) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ f : \sup_{v, x \in A, \delta(v,x) < \frac{1}{n}} \varrho(f(v), f(x)) < \frac{1}{m} \right\} .$$

In this particular case T-regularity means that the random transform $(F, \mathfrak{F}, \mu)$ is uniformly continuous with probability one.

If X is non-denumerable and Y contains at least two points, it follows from lemma 1 that $T(X)$ does not belong to $\mathfrak{F}$ and the application of theorem 1 is useful.

We see at once that the conditions (7) and (8) are satisfied.

Taking into account the separability of the space Y we may verify without difficulty that (6) holds. The corresponding proof will be outlined. Since the sets in $\mathfrak{D}$ are by definition denumerable, it suffices to show that for each v, $x \in X$ and each $\varepsilon > 0$ we have

$$\{f : \varrho(f(v), f(x)) < \varepsilon\} \in \mathfrak{F}(\{v, x\}) . \tag{14}$$

Let us denote by $\mathfrak{B}^2$ the σ-algebra of subsets of the cartesian power $Y \times Y = Y^2$ generated by the class $\mathfrak{G}^2$ of all rectangles with sides from $\mathfrak{G}$ and let S be a transform of F into Y^2, which involves the correspondence to each $f \in F$ of the point $S(f) = (f(v), f(x)) \in Y^2$. Clearly, S is a transform of F onto Y^2 and S^{-1} is an isomorphism between $\mathfrak{B}^2$ and $\mathfrak{F}(\{v, x\})$. It is well known that the distance function $\varrho(y, z)$, where $(y, z) \in Y^2$, is continuous in the space Y^2 with the usual product metric. Since the class $\mathfrak{G}^2$ is a denumerable open base of Y^2, hence, $\varrho(y, z)$ is $\mathfrak{B}^2$-measurable and (14) follows from the isomorphism S^{-1}.

The sole property of T which is not so easy to recognize is (5). But (5) is a consequence of the following well known

Extension theorem 1. *If the transform f of a point set A of a metric space into a complete metric space is uniformly continuous, then there exists one and only one uniformly continuous extension of f from A to the closure $\bar{A}$ of A.*

The proof of this elementary theorem may be found in many books on point set topology, for example in [3].

Applying the extension theorem 1 to all denumerable sets $D \subset X$ dense in X we see that the extension of each uniformly continuous transform f of D into Y to the whole space X is possible, hence, (5) is satisfied. We note that the uniqueness of the extension was not used.

Summarizing the above results we obtain from theorem 1

Theorem 2. *A necessary and sufficient condition for a random transform $(F, \mathfrak{F}, \mu)$ to be uniformly continuous with probability unity is that for each denumerable set $D \subset X$ dense in X the set of transforms from F which are uniformly continuous in D has probability unity.*

We see that theorem 2 is in fact an obvious generalization of Doob's theorem.

Utilizing theorem 2 we may easily obtain an equivalent version of that theorem first established by Mann [4].

Theorem 3. *The random transform $(F, \mathfrak{F}, \mu)$ is uniformly continuous with probability one, if and only if to each $\varepsilon > 0$ and $\eta > 0$ there exists an $\omega(\varepsilon, \eta) > 0$ such that*

$$\mu\{f : \max_{v, x \in S, \delta(v,x) < \omega(\varepsilon, \eta)} \varrho(f(v), f(x)) < \varepsilon\} \geqq 1 - \eta \tag{14}$$

or each finite set $S \subset X$.

We see at once that if the space X is compact and therefore separable, then each of the conditions in theorem 2 or 3 is necessary and sufficient for the almost sure continuity of the random transform $(F, \mathfrak{F}, \mu)$.

§ 3. Another example of the same type as in the preceding paragraph may be obtained if $T(A)$ for $A \subset X$ is the set of all transforms from F which satisfy the Lipschitz condition with a constant $c \geqq 0$ in A, i. e.

$$T(A) = \bigcap_{v, x \in A} \{f : \varrho(f(v), f(x)) \leqq c\, \delta(v, x)\} .$$

If the random transform $(F, \mathfrak{F}, \mu)$ is almost surely T-regular in that sense, then it is said to have almost surely the Lipschitz property with the constant c.

Clearly, the conditions (6), (7) and (8) are satisfied and (5) follows from

Extension theorem 2. *If the transform f of a point set A of a metric space into a complete metric space satisfies the Lipschitz condition with a constant $c \geqq 0$, then there exists one and only one extension of f from A to the closure $\bar{A}$ of A which also satisfies the Lipschitz condition with the same constant c.*

We omit the proof of this simple extension theorem and write only the final result which is an obvious consequence of theorem 1.

Theorem 4. *A necessary and sufficient condition for a random transform $(F, \mathfrak{F}, \mu)$ to have almost surely the Lipschitz property with the constant $c \geqq 0$ is that*

$$\mu\{f : \varrho(f(v), f(x)) \leqq c\, \delta(v, x)\} = 1 \tag{15}$$

for each $v, x \in X$.

Since the Lipschitz property is stronger than, for example, the uniform continuity, it follows from theorem 4

Theorem 5. *If* (15) *holds, then the random transform* $(F, \mathfrak{F}, \mu)$ *is uniformly continuous with probability one.*

§ 4. In this last paragraph we shall suppose that X is a Boolean σ-algebra with unity 1, $Y = R$ is the space of all real numbers and $\mathfrak{D}$ is the class of all denumerable subalgebras of X. To eliminate misunderstandigs we note that $1 \in A$ for each $A \in \mathfrak{D}$.

We see at once that (2) is satisfied and (3) follows from the well known fact that each subalgebra of X generated by a denumerable subset of X is itself denumerable.

We shall denote by O the zero of X and to denote unions and intersections in X we shall use the same symbols as for the corresponding set-operations.

Now let T means additivity, i. e.

$$T(A) = \bigcap_{v, x \in A,\ v \cup x \in A,\ v \cap x = 0} \{f : f(v \cup x) = f(v) + f(x)\}$$

for each set $A \subset X$. Clearly, T-regularity of $(F, \mathfrak{F}, \mu)$ means additivity with probability one.

We see at once that the conditions (7) and (8) are satisfied.

Since clearly

$$\{f : f(v \cup x) = f(v) + f(x)\} =$$
$$= \Big(\bigcup_{r \in R_0} (\{f : f(v \cup x) < r\} \cap \bigcup_{s \in R_0}(\{f : f(v) > -s\} \cap \{f : f(x) > s + r\}))\Big)' \cap$$
$$\cap \Big(\bigcup_{r \in R_0} (\{f : f(v \cup x) > r\} \cap \bigcup_{s \in R_0} (\{f : f(v) < s\} \cap \{f : f(x) < r - s\}))\Big)',$$

where R_0 is the set of all rational numbers, and the algebras from $\mathfrak{D}$ are denumerable, hence, (6) is satisfied.

The property (5) of T follows from the

Extension theorem 3. *If A is a subalgebra of the σ-algebra X with the same unity and the real function f is additive in A, then there exists a real additive extension of f from A to the whole σ-algebra X.*

The proof of this extension theorem may be found for example in [5].

Utilizing theorem 1 we can state

Theorem 6. *The random function* $(F, \mathfrak{F}, \mu)$ *is additive with probability one if and only if*

$$\mu\{f : f(v \cup x) = f(v) + f(x)\} = 1$$

for each $v, x \in X$, $v \cap x = 0$.

Theorem 6 has a useful application in the theory of conditional probabilities, as will be shown in another paper.

It is possible to establish a great number of theorems by a simple application of the general theorem 1. The elementary results contained in the last three paragraphs serve to illustrate by concrete examples this possibility.

REFERENCES

[1] *A. N. Kolmogorov*: Grundbegriffe der Wahrscheinlichkeitsrechnung (1933).

[2] *J. L. Doob*: Probability in function space. Bull. Amer. Math. Soc., vol. *53* (1947), pp. 15 to 30.

[3] *G. T. Whyburn*: Analytic topology (1942).

[4] *H. B. Mann*: On the realization of stochastic processes by probability distributions in function spaces. Sankhyā, vol. *11* (1951), pp. 3—8.

[5] *J. Łoś, E. Marczewsky*: Extensions of measure. Fundam. Math., vol. *36* (1949), pp. 267 to 276.

Резюме

РЕГУЛЯРНОСТЬ СЛУЧАЙНЫХ ПРЕОБРАЗОВАНИЙ

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Под случайным преобразованием $(F, \mathfrak{F}, \mu)$ мы понимаем пространство F, σ-алгебру $\mathfrak{F}$ подмножеств пространства F и меру вероятности μ в $\mathfrak{F}$. Элементами пространства F являются все преобразования f пространства $X \neq 0$ в пространство $Y \neq 0$. Дана σ-алгебра $\mathfrak{B}$ с базисом $\mathfrak{G}$ подмножеств пространства Y. Структуру пространства X мы не ограничиваем никакими предположениями. Представляется целесообразным определить $\mathfrak{F}$ как минимальную σ-алгебру подмножеств пространства F, содержащую все множества типа $\{f : f(x) \in G\}$, где x пробегает X, а G пробегает $\mathfrak{G}$. $\mathfrak{F}$ не зависит от выбора базиса $\mathfrak{G}$.

Пусть $\mathfrak{D}$ — система счетных подмножеств множества X, обладающая следующими свойствами:

(а) $\mathfrak{D}$ покрывает X,

(б) каждое счетное соединение множеств из $\mathfrak{D}$ содержится в одном множестве из $\mathfrak{D}$.

Регулярность определяется при помощи преобразования T, переводящего систему всех подмножеств множества X в систему всех подмножеств множества F, следующим образом:
если $D \in \mathfrak{D}$ и $g \in T(D)$, то

$$T(X) \cup \bigcap_{x \in D} \{f : f(x) = g(x)\} \neq 0 , \tag{1}$$

если $D \in \mathfrak{D}$, то

$$T(D) \in \mathfrak{F} , \tag{2}$$

если $A \subset B \subset X$, то

$$T(B) \subset T(A)\,, \tag{3}$$

$$T(X) \neq 0\,. \tag{4}$$

Скажем, что случайное преобразование $(F, \mathfrak{F}, \mu)$ будет почти наверное T-регулярным, если $\bar{\mu}(T(x)) = 1$, где $\bar{\mu}$ обозначает внешнюю меру вероятности; имеет место следующая главная теорема:

Случайное преобразование $(F, \mathfrak{F}, \mu)$ будет почти наверное T-регулярным тогда и только тогда, если $\mu(T(D)) = 1$ для $D \in \mathfrak{D}$.

Эту теорему можно применить к специальным свойствам регулярности. В следующих двух примерах предполагается, что X — метрическое сепарабельное пространство с метрикой δ, Y — метрическое сепарабельное и польное пространство с метрикой ϱ, $\mathfrak{G}$ — счетный открытый базис пространства Y и что $\mathfrak{B}$ есть σ-алгебра борелевских подмножеств пространства Y. Если $\mathfrak{D}$ есть система всех счетных подмножеств пространства X, плотных в X, то условия (а) и (б) выполняются.

Если $T(A)$ для $A \subset X$ является множеством всех преобразований из F, равномерно непрерывных в A, то условия (1), (2), (3) и (4) выполняются, и из первой теоремы следует:

Случайное преобразование $(F, \mathfrak{F}, \mu)$ будет почти наверное равномерно непрерывным тогда и только тогда, если для любого счетного множества $D \subset X$, плотного в X, вероятность множества всех преобразований из F равномерно непрерывных в D, равна единице.

Это так называемая теорема Дуба, которую, согласно Г. Б. Манну, можно сформулировать следующим образом:

Случайное преобразование $(F, \mathfrak{F}, \mu)$ будет почти наверное равномерно непрерывным тогда и только тогда, если для каждого $\varepsilon > 0$ и $\eta > 0$ существует $\omega\,(\varepsilon, \eta) > 0$ так, что

$$\mu\{f : \max_{v,\, x \in S,\, \delta(v,\, x) < \omega(\varepsilon,\, \eta)} \varrho(f(v), f(x)) < \varepsilon\} \geqq 1 - \eta$$

для любого конечного множества $S \subset X$.

Другой пример применения первой теоремы получится в том случае, когда $T(A)$ обозначает множество всех преобразований из F, удовлетворяющих в A т. наз. условию Липшица с постоянной $c \geqq 0$, т. е.

$$T(A) = \bigcap_{v,\, x \in A} \{f : \varrho(f(v), f(x)) \leqq c\delta(v, x)\}\,.$$

Условия (1), (2), (3), (4) опять-таки выполняются, и справедливо утверждение:

Случайное преобразование $(F, \mathfrak{F}, \mu)$ почти наверное удовлетворяет условию Липшица с постоянной $c \geqq 0$ тогда и только тогда, если для любой пары v, $x \in X$

$$\mu\{f : \varrho(f(v), f(x)) \leqq c\delta(v, x)\} = 1\,.$$

Первую теорему можно использовать при решении некоторых вопросов из теории случайных множественных функций. Пусть Y — пространство действительных чисел с обычной метрикой, $\mathfrak{B}$ есть σ-алгебра всех борелевских множеств, X есть σ-алгебра Буля, и $\mathfrak{D}$ — система всех счетных подалгебр с одним и тем же единичным элементом $1 \in X$. Система $\mathfrak{D}$ удовлетворяет условиям (а) и (б). В следующем примере T обозначает аддитивность, т. е. для $A \subset X$

$$T(A) = \bigcap_{v, x \in A,\, v \cup x \in A,\, v \cap x = 0} \{f : f(v \cup x) = f(v) + f(x)\} ,$$

где $\cup$ и $\cap$ — теоретико-структурные операции, а 0 — нулевой элемент в X. Условия (1), (2), (3) и (4) выполняются и, используя первую теорему, получим следующий простой результат:

Случайная функция $(F, \mathfrak{F}, \mu)$ *будет почти наверное аддитивной тогда и только тогда, если*

$$\mu\{f : f(v \cup x) = f(v) + f(x)\} = 1$$

для любой пары v, $x \in X$, $v \cap x = 0$.

Приведенные теоремы являются лишь простыми примерами применения главной теоремы. Аналогичным образом можно найти ряд дальнейших результатов.