Vlastimil Pták

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WEAK COMPACTNESS IN CONVEX TOPOLOGICAL LINEAR SPACES

VLASTIMIL PTÁK, Praha

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The author proves the following theorem: Let X be a complete convex topological linear space. Let $M \subset X$ be pseudocompact in the weak topology. In such a case, the bipolar set M^{**} is weakly compact.

It is the purpose of the present paper to prove a theorem which generalizes two well known results of the theory of normed linear spaces.

M. Krein and V. Šmulian have shown in [5] that the closed convex envelope of a weakly compact subset of a Banach (i. e. complete normed) space is weakly compact. In 1947, W. F. Eberlein [1] has obtained the following important and beautiful result. Every weakly closed and weakly countably compact subset of a complete normed space is weakly compact.*)

In the present remark the more general notion of a pseudocompact set is considered. We prove a theorem which represents a simultaneous generalization of the results of Krein, Šmulian and Eberlein.

Let X be a complete convex topological linear space. Let $M \subset X$ be pseudocompact in the weak topology of the space X. In these conditions, the closed symmetrical convex envelope of M is weakly compact.

The proof is based on the fact that pseudocompact spaces can be shown to possess an interesting property which does not seem to be quite superficial. [See lemma (1,2) of the present remark.] Using this result the theorem mentioned above can be proved without difficulties.

§ 1.

First of all we are going to recapitulate some notations introduced in [6]. If T is a given completely regular topological space, we shall denote by $C(T)$

*) Added while reading the proofs. Countable compactness in linear spaces has been discussed recently by *A. Grothendieck,* Critères de compacité dans les espaces fonctionnels généraux, Am. Journ. of Math., *74* (1952), 168—186 and *J. Dieudonné,* Sur un théorème de Šmulian, Archiv der Math., *3* (1952), 436—440.

the space of all realvalued continuous functions on T. Let K be an arbitrary compact subset of T, ε an arbitrary positive number. Put

$$U(K, \varepsilon) = \mathop{\mathrm{E}}_{x} [x \in C(T), |x(K)| \leqq \varepsilon] .$$

The topology of $C(T)$ will be defined by the postulate that the system of all sets $U(K, \varepsilon)$ be a complete system of neighbourhoods of zero. It is easy to see that, for every $t \in T$, the function which assigns to every $x \in C(T)$ its value at the point t is a linear functional on $C(T)$. We shall denote it by $\varphi(t)$. We have thus for every $x \in C(T)$ and every $t \in T$

$$x \,.\, \varphi(t) = x(t) .$$

We have thus obtained a mapping φ of T into the space Y dual to $C(T)$. The space T being completely regular, the mapping φ is one to one. Now let Y be taken in the weak topology. In this way, a topology is introduced into $\varphi(T)$ as well. It is easy to see that the mapping φ is a homeomorphism.

First of all, let us show that for every $M \subset T$

$$\varphi(T) \cap \overline{\varphi(M)} \subset \varphi(\overline{M}) .$$

Let $\varphi(t) \in \overline{\varphi(M)}$ and suppose that t non $\in \overline{M}$. It follows that there is an $x \in C(T)$ such that $x(M) = 0$ and $x(t) = 1$. We have then $x \,.\, \varphi(M) = 0$ and $x \,.\, \varphi(t) = 1$, which is a contradiction since $\varphi(t) \in \overline{\varphi(M)}$.

Let us show now that

$$\varphi(\overline{M}) \subset \overline{\varphi(M)}$$

for every $M \subset T$. To see that, let us take an arbitrary $t \in \overline{M}$ and suppose that $\varphi(t)$ non $\in \overline{\varphi(M)}$. It follows that there exist $x_1, \ldots, x_n \in C(T)$ such that the following implication holds

$$y \in Y \,, \quad |x_i(y - \varphi(t))| \leqq 1 \Rightarrow y \text{ non} \in \varphi(M) .$$

We see that for every $m \in M$ an index i can be found such that

$$|x_i\,(\varphi(m) - \varphi(t))| > 1 .$$

Let us put for $s \in T$

$$w(s) = \sum_{i=1}^{n} |x_i(s) - x_i(t)| .$$

Clearly we have $w(s) \in C(T)$ and $w(t) = 0$. On the other hand, we have $w(m) > 1$ for every $m \in M$. It follows that t non $\in \overline{M}$ which is a contradiction.

In this way the space T is homeomorphically mapped into Y. We shall adopt the usual convention of not distinguishing between a point $t \in T$ and its image in Y so that we shall be able to perform algebraic operations on points of T.

Now let us denote by $L(T)$ the substace of Y consisting of all linear combinations

$$\lambda_1 t_1 + \ldots + \lambda_n t_n$$

where $t_i \in T$. Clearly $L(T)$ is dense in Y. It is possible to show, however, that $L(T)$ is dense in every set U^*.

To see that, we note first that the closure of $L(T) \cap U^*$ is the set $(L(T) \cap \cap U^*)^{**}$. The inclusion

$$(L(T) \cap U^*)^{**} \supset U^*$$

is equivalent to the inclusion

$$(L(T) \cap U^*)^* \subset U .$$

Now let $U = U(K, \varepsilon)$. Clearly, for every $t \in K$ we have $\frac{1}{\varepsilon} t \in L(T) \cap U^*$. If $x \in (L(T) \cap U^*)^*$, we have

$$\left| x \cdot \frac{1}{\varepsilon} t \right| \leqq 1$$

for every $t \in K$, so that $x \in U(K, \epsilon)$. This proves our assertion.

First of all we shall need a remark concerning topological spaces. Following E. Hewitt [2], we shall say that a completely regular topological space is pseudocompact if every continuous function on T is bounded on T. In [2], Hewitt gives a simple characterization of pseudocompact spaces based on the properties of βT. For our purpose it will be convenient to use another characterization which will be formulated as a simple lemma.

(1,1) *A completely regular topological space T is pseudocompact if and only if the following condition is fulfilled:*

for every countable $S \subset C(T)$ and every $s \in \beta T$ there exists a point $t \in T$ such that

$$x(t) = x(s)$$

for every $x \in S$.

Proof: First of all, let T be a pseudocompact space. Let us take an arbitrary countable $S \subset C(T)$ and an arbitrary $s \in \beta T$. Let x_n be a sequence which contains all functions of the set S. For every $t \in T$ let us put

$$w(t) = \sum_n \frac{1}{2^n \alpha_n} |x_n(t) - x_n(s)|$$

where

$$\alpha_n = 1 + \sup_{t \in T} |x_n(t)| .$$

Clearly $w(t)$ is a continuous function on T. Suppose that no point $t \in T$ exists such that $x(t) = x(s)$ for every $x \in S$. It follows that $w(t) > 0$ for every $t \in T$. On the other hand we have clearly $\inf\limits_{t \in T} w(t) = 0$. The function $\frac{1}{w(t)}$ is thus a continuous function on T which is not bounded on T. The contradiction obtained proves the existence of a point $t \in T$ with the required properties.

On the other hand, suppose that T is not pseudocompact. In this case a continuous function $v(t)$ can be found such that $v(t)$ is not bounded on T. It follows that the function

$$z(t) = \frac{1}{1 + |v(t)|}$$

is a bounded continuous function on T which fulfills $z(t) > 0$ for every $t \in T$. Since v is not bounded on T, we have $\inf z(t) = 0$. The function $z(t)$, being bounded on T, can be extended over the whole of βT. The function z attains its minimum value in some point $s \in \beta T$. We have thus $z(s) = 0$. The relation $z(t) = z(s)$ cannot be fulfilled for any $t \in T$ since $z(t)$ is positive for every $t \in T$.

Now we are coming to the proof of an important property of pseudocompact spaces. This property forms the most essential point in the proof of our theorem.

(1,2) *Let T be a pseudocompact completely regular topological space. Let Y be the space dual to $C(\beta T)$. Let us denote by $L(T)$ the subspace of Y consisting of all linear combinations $\lambda_1 t_1 + \ldots + \lambda_n t_n$, where $t_i \in T$. The weak topology on $C(T)$ corresponding to the space $L(T)$ will be called the point topology of $C(T)$. Let $B \subset C(T)$ be symmetrical convex and compact in the point topology. Suppose that the set B is equibounded on T. Then B is compact in the weak topology corresponding to the space Y.*

Proof: First of all it is easy to see that the set B is closed in the weak topology corresponding to Y. Since B is convex, it follows that B is closed as a subset of the (normed) space $C(\beta T)$. The space $C(\beta T)$ being complete, it is sufficient to prove that every sequence $x_n \in B$ has at least one limit point in B taken in the weak topology of $C(\beta T)$. Our assertion will then follow from the theorem of Eberlein [1]. Let us take an arbitrary sequence $x_n \in B$. Then there exists a $u \in B$ such that u is a limit point of the sequence x_n in the point topology. For every natural m and every $s \in \beta T$ let us take the set

$$G(m,s) = \mathop{\mathrm{E}}_{t}\left[t \in \beta T,\ |x_i(t) - x_i(s)| < \frac{1}{m},\ i = 1, 2, \ldots, m\right].$$

For every natural m the sets $G(m, s)$ form a covering of the compact space βT. This covering contains a finite subcovering consisting of sets $G(m, p)$ where p runs over a finite set $P_m \subset \beta T$. For every $p \in P_m$ let us choose a point $q \in T \cap G(m, p)$. We have thus obtained a finite set $Q_m \subset T$. The union of all sets Q_m is a countable set $Q \subset T$. By means of the diagonal process we form now a subsequence x'_n of x_n such that

$$\lim_n x'_n(q) = u(q)$$

for every $q \in Q$.

We are going to show that, in these conditions,

$$\lim_n x'_n(s) = u(s)$$

for every $s \in \beta T$.

For this purpose, we shall prove first the following proposition.

Let z be a limit point of x_n in the point topology. Suppose that, for some points t_m, t_0 of the space T the relation

$$\lim_m x_i(t_m) = x_i(t_0)$$

holds for every i. Then

$$\lim_m z(t_m) = z(t_0) .$$

In fact, suppose we have a sequence $t_m \in T$ and a point $t_0 \in T$ such that $\lim_m x_i(t_m) = x_i(t_0)$ for every i while $|z(t_m) - z(t_0)| \geqq \sigma > 0$ for infinitely many m. Clearly we can suppose that $|z(t_m) - z(t_0)| \geqq \sigma$ for all m. The sequence t_m has a limit point $s \in \beta T$. Clearly we shall have

$$\lim_m x_i(t_m) = x_i(s)$$

for every i, and at the same time

$$|z(s) - z(t_0)| \geqq \sigma .$$

According to the preceding lemma, there exists a point $t \in T$ such that $z(t) = z(s)$ and $x_i(t) = x_i(s)$ for every i. We have thus

$$x_i(t) = x_i(t_0)$$

for every i and at the same time

$$|z(t) - z(t_0)| \geqq \sigma .$$

Now z is a limit point of x_n in the point topology. It follows that, for a suitable x_n, we have simultaneously

$$|x_n(t) - z(t)| < \tfrac{1}{4}\sigma$$
$$|x_n(t_0) - z(t_0)| < \tfrac{1}{4}\sigma .$$

Since $x_n(t) = x_n(t_0)$ we obtain a contradiction. The proof of our proposition is this concluded.

Using the preceding results, we shall show now that the point u is the unique limit point of x'_n in the point topology. First of all it is easy to see that, if z is an arbitrary limit point of x'_n in the point topology, we have $z(q) = u(q)$ for every $q \in Q$.

Now let z be an arbitrary limit of x'_n in the point topology and let t be an arbitrary point of the space T. For every natural m there exists a $p_m \in P_m$

such that $t \in G(m, p_m)$. If we take the corresponding $q_m \in T \cap G(m, p_m)$, we obtain the following estimates valid for $m \geqq i$

$$|x_i(q_m) - x_i(p_m)| < \frac{1}{m},$$

$$|x_i(t) - x_i(p_m)| < \frac{1}{m}.$$

It follows that, for a fixed i, we have

$$m \geqq i \Rightarrow |x_i(q_m) - x_i(t)| < \frac{2}{m}$$

so that

$$\lim_m x_i(q_m) = x_i(t)$$

for every i.

The point z being a limit point of x'_n in the point topology, we shall have

$$\lim_m z(q_m) = z(t)$$

according to the proposition proved above. For the same reason, we have

$$\lim_m u(q_m) = u(t) .$$

At the same time, we know that $z(q_m) = n(t_m)$ for every m. It follows that $z(t) = u(t)$ for every $t \in T$.

Now we are able to show that

$$\lim x'_n(s) = u(s)$$

for every $s \in \beta T$.

To see that, let us take a fixed $s \in \beta T$. First of all, we find a $t \in T$ such that $u(s) = u(t)$ and at the same time $x_i(s) = x_i(t)$ for every i. It will be sufficient to prove that

$$\lim_n x'_n(t) = u(t) .$$

Suppose that this is not true. Then it is possible to define a subsequence x''_n such that

$$\lim_n x''_n(t) = \alpha \neq u(t) .$$

Since all $x''_n \in B$, there exists a point $v \in B$ such that v is a limit point of the sequence x''_n in the point topology. Clearly we have $v(t) = \alpha$, so that v is different from u. The point v, being a limit point of x''_n, is at the same time a limit point of x'_n. We have just proved, however, that the sequence x'_n has only one limit point in the point topology. This contradiction proves that the sequence x'_n converges to u in every point of βT. Now we are going to show that for every $y \in Y$

$$\lim_n x'_n y = uy .$$

To see that it is sufficient to recall the well known result of Kakutani [3] according to which every $y \in Y$ can be expressed as an integral over βT. The assertion mentioned above follows immediately from the convergence of the functions x_n to u by means of well known properties of the integral. We have thus shown that the point u is a limit point of the sequence x_n also in the weak topology corresponding to the space Y. This concludes the proof.

The question arises how far the assumption that T is pseudocompact is essential. In fact, if T is an arbitrary completely regular topological space, space, we can form the space $C(\beta T)$. In the space Y dual to $C(\beta T)$ the space $L(T)$ consisting of all linear combinations $\lambda_1 t_1 + \ldots + \lambda_n t_n$, $(t_i \in T)$ can be defined in the same manner as in the preceding lemma. We can define the point topology in $C(\beta T)$ as the weak topology corresponding to $L(T)$. We can ask now whether every equibounded symmetrical convex and pointcompact subset B of $C(\beta T)$ is already weakly compact. We are going to show that this question has to be answered in the negative.

Let us denote by R the normed space of all continuous functions $r(p)$ defined on $\langle 0, 1\rangle$. Let us denote by T the closed unit sphere of the space R. The space T is thus completely regular. Take $C(\beta T)$. Let us denote by S the space of all linear functionals on R. Every element $s \in S$ is continuous and bounded on T and can be therefore considered as an element of $C(\beta T)$. We have thus $S \subset$ $\subset C(\beta T)$. Let us denote by B the intersection of the closed unit sphere of $C(\beta T)$ with the space S. Since the norm of an element $s \in S$ as a functional on R coincides with its norm as an element of $C(\beta T)$, we see that B is equal to the closed unit sphere of the space S. According to a wellknown theorem the set B is compact in the weak topology corresponding to R, so that B as a subset of $C(\beta T)$ is pointcompact in the sense of our definition.

It is easy to see that, for every $p \in \langle 0, 1\rangle$, the mapping $p(t)$ which assigns to the function $t \in T$ its value at the point p, is an element of B. It follows that B contains all linear combinations

$$\lambda_1 p_1 + \ldots + \lambda_n p_n$$

where p_i are points of the interval $\langle 0, 1\rangle$ and $\Sigma|\lambda_i| \leqq 1$. On the other hand, if a linear combination

$$b = \lambda_1 p_1 + \ldots + \lambda_n p_n$$

(where the points p_i are different from each other) belongs to B, we must have $\Sigma|\lambda_i| \leqq 1$. This is an easy consequence of the fact that a $t \in T$ exists so that

$$p_i(t) = t(p_i) = \operatorname{sign} \lambda_i \,.$$

Since $t \in T$, $b \in B$, we must have $|b(t)| \leqq 1$. Now

$$b(t) = \Sigma\lambda_i p_i(t) = \Sigma\lambda_i t(p_i) = \Sigma\lambda_i \operatorname{sign} \lambda_i = \Sigma|\lambda_i| \,.$$

The subspace of S consisting of all linear combinations $\lambda_1 p_1 + \cdots + \lambda_n p_n$ $(p_i \in \langle 0, 1\rangle)$ will be denoted by P.

Let Y be the space dual to $C(\beta T)$. Take a $p_0 \in \langle 0, 1\rangle$. Let us define a linear function z on P by the postulate that $zp_0 = 1$ and $zp = 0$ for all $p \in \langle 0, 1\rangle$. $p \neq p_0$. It follows from the above remark that the norm of z on P is exactly 1. Now z can be extended to the whole of $C(\beta T)$ without disturbing its norm, so that z becomes an element of the space Y.

Suppose now that the set B is compact in the weak topology corresponding to Y. It follows that z is continuous on B in the point topology. Clearly, if z is taken as a linear function on the space S dual to R, it becomes an almost continuous linear functional on S. Since R is complete, there exists an $r \in R$ such that

$$zs = rs$$

for every $s \in S$. Especially, for every $p \in \langle 0, 1\rangle$, we obtain

$$z(p) = r(p) .$$

Now r is a continuous function on $\langle 0, 1\rangle$ and at the same time we have $r(p) = 0$ for every $p \neq p_0$. It follows that $r(p_0) = 0$. On the other hand, we have $z(p_0) = 1$. The contradiction obtained proves that the set B cannot be compact in the weak topology of the space $C(\beta T)$.

§ 2.

Now we are able to prove the main theorem. The most essential part of the proof is contained in the preceding lemma.

(2,1) *Let X be a complete convex topological linear space. Let $B \subset X$ be pseudocompact in the weak topology. Then B^{**} is weakly compact.*

Proof: According to a well-known theorem*) it is sufficient to show that the set B^* is a neighbourhood of zero in the minimal topology of the space Y dual to X. To see that, let us take a linear function r defined on Y and such that

$$|rB^*| \leqq 1 .$$

Our assertion will be proved if we show that r is a linear functional on Y. Since X is complete, it is sufficient to show that r is almost continuous. Let us take an arbitrary U and an arbitrary positive number ε. Consider the space $C(B)$, the set B being taken in the weak topology so that B is pseudocompact. For every $y \in Y$, denote by $\varphi(y)$ its restriction on B, so that $\varphi(y) \in C(B)$.

Let us denote by $M(B)$ the space dual to $C(B)$. Let us take the subspace $L(B)$ of $M(B)$.

Consider the set $\varphi(U^*) \subset C(B)$. Since U^* is compact in the weak topology of Y, it is easy to see that $\varphi(U^*)$ is compact in the weak topology of $C(B)$

*) See, e. g. theorem (3,5) of [4].

corresponding to $L(B)$, in other words, that $\varphi(U^*)$ is compact in the point topology.

The set $B \subset X$ is pseudocompact in the weak topology. It follows that the set By is bounded for every y, so that B is bounded. It follows that a positive number σ can be found such that $B \subset \sigma U$. In other words, for every $b \in B$ and every $y \in U^*$, we have $|by| \leqq \sigma$. This inequality clearly asserts that the set $\varphi(U^*)$ is equibounded on B.

Since $\varphi(U^*)$ is clearly symmetrical and convex, we infer from (1,2) that $\varphi(U^*)$ is compact in the weak topology of the (normed) space $C(\beta B)$. Let us denote by Z the space dual to $C(\beta B)$. It follows that the polar set $(\varphi(U^*))^Z$ will be a neighbourhood of zero in the minimal topology of the space Z.

Let us show now that, for every $y \in Y$, the following inequality holds:

$$|ry| \leqq |\varphi(y)| \, .$$

Here, of course, $|\varphi(y)|$ means the usual norm in the space $C(\beta B)$.

To see that, suppose that $|\varphi(y)| < |ry|$ for some $y \in Y$. It follows that, for a suitable $y' = \alpha' y$, the following relations would be fulfilled simultaneously

$$|\varphi(y')| \leqq 1 \, ,$$
$$|ry'| > 1 \, .$$

This is, however, a contradiction, since the inequality $|\varphi(y')| \leqq 1$ is equivalent to the inclusion $y' \in B^*$. Now let y_1 and y_2 be two elements of Y such that $\varphi(y_1) = \varphi(y_2)$. We obtain the following estimate

$$|ry_1 - ry_2| = |r(y_1 - y_2)| \leqq |\varphi(y_1 - y_2)| = |\varphi(y_1) - \varphi(y_2)| = 0$$

so that $ry_1 = ry_2$.

It is therefore possible to define a linear function r on the space $\varphi(Y) \subset C(\beta B)$ by means of the relation

$$r\varphi(y) = ry \, .$$

At the same time, the estimate $|ry| \leqq |\varphi(y)|$ shows that r is a continuous functional on $\varphi(Y)$ and that its norm does not exceed 1. Hence it is possible to extend r to the whole of $C(\beta B)$ without disturbing its norm. In this way r becomes a functional on $C(\beta B)$ in other words, a point of Z.

Since $L(B)$ is dense in Z, a point $q \in L(B)$ can be found so that $q - r \in \varepsilon(\varphi(U^*))^Z$. We have $q = \lambda_1 b_1 + \ldots + \lambda_n b_n$, where $b_i \in B$. It follows that

$$|(\lambda_1 b_1 + \ldots + \lambda_n b_n - r)\, y| \leqq \varepsilon$$

for every $y \in U^*$.

We have thus shown that r can be approximated on U^* by means of a continuous functional with an error $\leqq \varepsilon$. This proves that r is an almost continuous functional.

§ 3.

Added, January 16, 1954. The assumption that B is equibounded is not necessary in (1,2). In fact we have the following simple lemma.

(3,1) *Let T be a pseudocompact completely regular topological space. Let $B \subset C(T)$ be symmetrical convex and compact in the point topology. Then there exists a number $\sigma < 0$ such that $|b(t)| \leqq \sigma$ for every $b \in B$ and every $t \in T$.*

Proof: We shall use the following abbreviations. For any $x \in C(T)$, let $|x|_+ = \max\limits_{t \in T} x(t)$. For any $t \in T$, let $\beta(t) = \max\limits_{b \in B} b(t)$.

To prove our theorem, it is sufficient to show that

$$\sup_{b \in B} |b|_+ < \infty .$$

Suppose that $\sup\limits_{b \in B} |b|_+ = \infty$. Then there exists a $b_1 \in B$ and a point $t_1 \in T$ such that $b_1 t_1 > 1$. We shall denote by B_1 the set

$$B_1 = \mathop{\mathrm{E}}_{b} [b \in B, bt_1 > 1] .$$

We have thus $B_1 \neq 0$. Suppose now that the points $t_1, \ldots, t_n$ have been already constructed so that the set

$$B_n = \mathop{\mathrm{E}}_{b} [b \in B, bt_i > i, \ 1 \leqq i \leqq n]$$

contains at least one point b_n.

Suppose that $|B_n|_+ \leqq n + 1$. Choose a real number λ so that

$$1 > \lambda > \max \frac{i + \beta(t_i)}{b_n t_i + \beta(t_i)} .$$

Take an arbitrary $b \in B$. We have then

$$\lambda b_n + (1 - \lambda)\, b \in B$$

and at the same time, for every i $(1 \leqq i \leqq n)$

$$\lambda b_n t_i + (1 - \lambda)\, bt_i \geqq \lambda b_n t_i - (1 - \lambda)\, \beta(t_i) > i$$

so that $\lambda b_n + (1 - \lambda)\, b \in B_n$. It follows that

$$|\lambda b_n + (1 - \lambda)\, b|_+ \leqq n + 1 .$$

On the other hand, we have

$$b = \frac{1}{1 - \lambda} ((\lambda b_n + (1 - \lambda)\, b) - \lambda b_n)$$

whence

$$|b|_+ \leqq \frac{1}{1 - \lambda} (n + 1 + |b_n|_+) .$$

Now b was an arbitrary element of B. This contradiction shows that the inequality $|B_n|_+ \leqq n + 1$ is impossible. We have thus proved the existence of a point $b_{n+1} \in B_n$ and a point t_{n+1} such that $b_{n+1}t_{n+1} > n + 1$. We have then

$$b_{n+1} \in B_{n+1} = \underset{b}{\mathrm{E}} [b \in B, bt_i > i, \ 1 \leqq i \leqq n + 1] .$$

Put

$$C_n = \underset{b}{\mathrm{E}} [b \in B, bt_i \geqq i, \ 1 \leqq i \leqq n] .$$

Clearly the sets C_n are closed in the point topology and $C_n \supset C_{n+1}$ for every n. The sets C_n are not empty since $C_n \supset B_n$. It follows that there exists a point $b \in B$ which lies in every C_n. For such a point b, we have $bt_i \geqq i$ for every i, which is a contradiction, the function b being bounded on T.

This concludes the proof.

We need not emphasise that the convexity of B is essential in the preceding lemma. The most trivial examples show that the lemma does not remain true if this assumption is dropped.

Some of the results of the present paper admit further interesting generalizations. We intend to return to these questions in another communication.

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Резюме

СЛАБАЯ КОМПАКТНОСТЬ В ТОПОЛОГИЧЕСКИХ ЛИНЕЙНЫХ ПРОСТРАНСТВАХ

ВЛАСТИМИЛ ПТАК (Vlastimil Pták), Прага.

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Целью настоящей работы является доказательство теоремы, которая представляет собой обобщение двух известных теорем теории нормированных пространств.

М. Крейн и В. Шмульян показали в [5], что замкнутая выпуклая оболочка слабо компактного подмножества любого пространства Банаха является слабо компактной. В 1947 г. В. Ф. Эберлейн [1] получил следующий важный результат. Пусть M слабо замкнутое и слабо счетно компактное подмножество некоторого пространства Банаха. Тогда M является слабо компактным.

В настоящей заметке мы занимаемся более общим понятием псевдокомпактного множества. Мы доказываем теорему, которая является одновременно обобщением результатов Крейна, Шмульяна и Эберлейна.

Пусть X — полное топологическое линейное пространство. Пусть M — псевдокомпактно в слабой топологии пространства X. Тогда замкнутая симметричная выпуклая оболочка множества M является слабо компактной.

Доказательство основывается на том обстоятельстве, что псевдокомпактные пространства обладают одним замечантельным свойством, которое не кажется вполне поверхностным. [См. лемму (1,2) настоящей заметки.] Пользуясь этим результатом, высказанную сверху теорему можем доказать без трудностей.