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ON SEMIGROUPS HAVING A KERNEL

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The author investigates the structure of left and two-sided ideals in semigroups containing a minimal two-sided ideal, which is called a kernel of the semigroups. Further, he passes to the discussion of the structure of simple semigroups with a kernel. Finally some semigroups are considered the study of which may be reduced to the study of simple left or two-sided ideals and simple semigroups.

The purpose of this paper is to find a number of new theorems concerning the theory of semigroups. At the same time we generalise some results previously found by SUSCHKEWITSCH (Сушкевич), REES, CLIFFORD and the author.

By a semigroup we mean a non-vacuous set S of elements $a, b, c, \dots$ closed under an associative univalent operation: $(ab)c = a(bc)$.

In such a system we introduce in the usual way the notion of left (right, two-sided) ideals. The non-vacuous subset L (R) of S is called a left (right) ideal if — in the sense of the calculus of complexes — the relation $SL \subseteqq L$ ($RS \subseteqq R$) holds. A two — sided ideal is a subset which is both left and right ideal.

The intersection (if it is non-vacuous) and the union of two left (right, two-sided) ideals is a left (right, two-sided) ideal.

A semigroup S can contain (but needs not contain) at most one element z with the property: $az = za = z$ for every $a \in S$. Such an element is termed a „zero element".

A left (right) ideal of S is called a minimal left (right) ideal of S if it does not contain any proper subset which is itself a left (right) ideal of S. A semigroup needs not contain naturally minimal left (right) ideals. Two minimal left (right) ideals have no element in common.

A semigroup has at most one minimal two-sided ideal $\mathfrak{n}$. The set $\mathfrak{n}$ is then contained in every two-sided ideal of S. Hence $\mathfrak{n}$ can be described as the intersection of all two-sided ideals of S. The ideal $\mathfrak{n}$ is called the „kernel of the semigroup S". (It is often called also „Suschkewitch kernel" after SUSCHKEWITCH [1] who described first its structure in the finite case.) If S has a zero element z, it has also a kernel and it is clear that $\mathfrak{n}$ equals the set $\{z\}$ consisting of z alone.

Clifford [3] gave a very general condition that assures the existence of a kernel. It is easy to show that if S contains at least one minimal left ideal, then it has a kernel. The kernel is then the sum of all minimal left ideals of S. If S has moreover at least one minimal right ideal, then $\mathfrak{n}$ is a „completely simple semigroup" in the sense of Rees [1].

Rees [1], [2] and Clifford [2], [3] call a semigroup S simple if it does not contain any proper two-sided subideal of S except possibly the zero ideal $\{z\}$. In his paper [3] Clifford studies first the structure of a simple semigroup S assuming that S has no zero element. In his paper [4] he is studying semigroups having a zero element. In this connection it is necessary to alter the definition of minimal ideals and to mean by them minimal but $\neq \{z\}$. Clifford found here, under some conditions, the structure of simple semigroups and simple two-sided ideals of a semigroup having a zero element. (Simple — in the sense of the above definition.)

In this paper we shall study semigroups having a kernel (which can reduce of course in special cases to $\{z\}$ alone). We give an other definition of the simplicity of a semigroup. We shall mean by it a semigroup having a kernel but no other two-sided ideal $\neq S$. If S has a zero element z, our definition coincides with the original definition of Rees and Clifford. But if S has no zero element the definitions differ even in the simplest case of finite semigroups. We introduce new concepts of simple left (right, two-sided) ideals by meaning by them left (right, two — sided) ideals of S containing $\mathfrak{n}$ but no other proper subideals of S containing $\mathfrak{n}$. One of the purposes of this paper is to study the structure of simple (left, right, two-sided) ideals defined in such a way.

A few of the results of our paper can be obtained from Clifford's results [4] using the notion of the „difference-semigroup" originally introduced by Rees [1] and recently used by Clifford [5] in another problem. We prefer to give direct proofs, for — as it seems to us — they enable a clearer insight in the state of things. Moreover, the methods of our proofs are different from those of Clifford and have some contact with my paper [2], though this last deals only with semigroups without a zero element. I introduced the notion of simple semigroups and simple ideals in the meaning used in this paper yet more years ago when studying the structure of commutative semigroups ([1], pages 51—61). Hence the results of the present paper generalise also these older results of mine. For the special case of commutative semigroups the results found in this paper have also some loose connection with the recent papers of Ляпин [1], [2], [3] (Liapin [1], [2], [3]).

At the beginning of every section a short characterization of its content is given, so that we can omit here to give a brief account of all the paper.

Remark. In what follows we use the following notations. The symbol $A \subset B$ (in contrary to $A \subseteqq B$) means always that A is a proper subset of B. The sum of two sets A, B is offen denoted by (A, B). The symbol $A \oplus B$ is used only for a special type of sums. The symbol $A - B$ means the set-theoretical difference of sets. Other notations have the usual meaning.

1. Simple ideals.

In this section we give some preliminary definitions and results concerning semigroups having a kernel.

Let S be a semigroup having the kernel $\mathfrak{n}$. The product $\mathfrak{n}S$ is a two-sided ideal of S contained in $\mathfrak{n}$, hence (with respect to the minimality of $\mathfrak{n}$) $\mathfrak{n}S = \mathfrak{n}$. Analogously $S\mathfrak{n} = \mathfrak{n}$.

If S' is any subset of S, $S' \subset S$, then for every $n \in \mathfrak{n}$ there holds $nS' \subseteqq \mathfrak{n}S' \subseteqq \mathfrak{n}S = \mathfrak{n}$, i. e. $nS' \subseteqq \mathfrak{n}$. Analogously $S'n \subseteqq \mathfrak{n}$.

Especially for every element $x \in S$ there holds $x\mathfrak{n} \subseteqq \mathfrak{n}$, $\mathfrak{n}x \subseteqq \mathfrak{n}$.

The set $\mathfrak{n}$ and every element $n \in \mathfrak{n}$ have analogous properties as the zero element: any subset of S multiplied by any element of $\mathfrak{n}$ belongs to $\mathfrak{n}$.

There exist semigroups for which $S = \mathfrak{n}$. Such a semigroup has none two-sided ideal $\neq S$. A simple example of such a semigroup is the semigroup $S = \{e_1, e_2, e_3, \ldots\}$ with the multiplication defined by $e_i e_k = e_k$ for every $i, k = 1, 2, 3, \ldots$ In this paper we shall not be interested in such semigroups. We shall study only semigroups with $\mathfrak{n} \subset S$. To be clear enough we shall often use (if it will be necessary) the words: S has a proper kernel.

In the introduction we defined the notion of minimal (left, right, two-sided) ideals. In this section we introduce a new concept of simple (left, right, two-sided) ideals and prove some of their rather elementary properties.

Definition 1,1. *Let S be a semigroup having a kernel $\mathfrak{n}$. A left ideal L of S is called a simple left ideal of S if $\mathfrak{n} \subset L \subseteqq S$ but there does not exist a left ideal L' of S with $\mathfrak{n} \subset L' \subset L \subseteqq S$.*

Theorem 1,1. *Let S be a semigroup having a proper kernel $\mathfrak{n}$. The intersection of two different simple left ideals of S equals $\mathfrak{n}$.*

Proof. Let $L_1 \neq L_2$ be two simple left ideals of S. It is $\mathfrak{n} \subseteqq L_1 \cap L_2 \subseteqq L_1$, $\mathfrak{n} \subseteqq L_1 \cap L_2 \subseteqq L_2$. The intersection $L_1 \cap L_2$ is a left ideal of S. If it would contain any element non $\in \mathfrak{n}$ we would have (with respect to the simplicity of L_1, L_2) $L_1 \cap L_2 = L_1 = L_2$, which contradicts the supposition.

Remark. According to the definition every simple left ideal of S contains $\mathfrak{n}$ and hence — if S has minimal left ideals — every minimal left ideal of S. (See the introduction.) The definition of simplicity of left ideals seems therefore at this stage to be somewhat artificial. But in studying two-sided ideals we shall see that this definition is quite natural.

Theorem 1,2. *For every simple left ideal L of a semigroup S with kernel $\mathfrak{n}$ the following relations hold*:

$$L\mathfrak{n} = \mathfrak{n}L = \mathfrak{n}.$$

Proof. a) $L\mathfrak{n}$ is a two sided ideal of S contained in $\mathfrak{n}$, hence equal to $\mathfrak{n}$.

b) Since $\mathfrak{n}$ is a two-sided ideal of S, there holds $\mathfrak{n}L \subseteq \mathfrak{n}S \subseteq \mathfrak{n}$. On the other side $\mathfrak{n}L \supseteq \mathfrak{n}\mathfrak{n} = \mathfrak{n}^2 = \mathfrak{n}$. Hence $\mathfrak{n}L = \mathfrak{n}$.

Theorem 1,3. *Let S be a semigroup having a proper kernel $\mathfrak{n}$. Let L_1, L_2 be two (non necessarily different) simple left ideals of S. Then it is either $L_1L_2 = \mathfrak{n}$ or $L_1L_2 = L_2$.*

Proof. L_1L_2 is a left ideal of S contained in L_2. Since $\mathfrak{n} \subset L_1$, $\mathfrak{n} \subset L_2$ we have $\mathfrak{n}^2 = \mathfrak{n} \subseteq L_1L_2$. Hence $\mathfrak{n} \subseteq L_1L_2 \subseteq L_2$. With respect to the simplicity of L_2 it is either $L_1L_2 = L_2$ or $L_1L_2 = \mathfrak{n}$.

Putting $L_1 = L_2 = L$ we get:

Corollary 1,3. *For every simple left ideal L there holds: either $L^2 = L$ or $L^2 = \mathfrak{n}$.*

Theorem 1,4. *Let L be a simple left ideal of a semigroup S having a kernel $\mathfrak{n}$. Let c be any element of S. Then the set $\mathfrak{n} + Lc$ is either a simple left ideal of S or equals $\mathfrak{n}$.*

Proof. It is either $Lc \subseteq \mathfrak{n}$ or there exists at least one element $a \in Lc$ with a non $\in \mathfrak{n}$.

In the first case the theorem is trivially true.

In the second case we prove that $\mathfrak{n} + Lc$ is a simple left ideal. The proof follows indirectly. Suppose that L^* is a proper left subideal of S: $\mathfrak{n} \subset L^* \subset \mathfrak{n} + Lc$. Let L_1 be the set of all element $a \in L$ such that $ac \in L^*$ holds. If $s \in S$, it is $sac \in sL^* \subseteq L^*$, so that $sa \in L_1$. Hence L_1 is a left ideal of S contained in L. Since L is simple it is either $L_1 = L$ or $L_1 \subseteq \mathfrak{n}$. The second alternative implies $L^* = \mathfrak{n} + L_1c \subseteq \mathfrak{n} + \mathfrak{n}c \subseteq \mathfrak{n}$, which contradicts $\mathfrak{n} \subset L^*$. The first alternative gives $L_1 = L$, $L^* = \mathfrak{n} + Lc$. This proves Theorem 1,3.

Remark. If we would write only Lc (in place of $\mathfrak{n} + Lc$) we would obtain again a left ideal but we cannot say that it contains all elements of $\mathfrak{n}$. (Remember that $\mathfrak{n}c \subseteq \mathfrak{n}$ but not necessarily $\mathfrak{n}c = \mathfrak{n}$.)

Theorem 1,5. *If the sum of all simple left ideals of a semigroup S having a kernel $\mathfrak{n}$ is non-vacuous, it is a two-sided ideal of S.*

Proof. Let $M = \sum_{\alpha} L_\alpha$ be the class sum of all simple left ideals of S. M is clearly a left ideal. We prove that it is also a right ideal. Let s be any element of S. It is $Ms = \sum_{\alpha}(L_\alpha s)$. Every summand $L_\alpha s$ is contained in the set $\mathfrak{n} + L_\alpha s$, which itself-being $\mathfrak{n}$ or a simple left ideal — is already contained in M. Hence $Ms \subseteqq M$ for any $s \in S$. This proves our theorem.

Needless to say that analogously as in Definition 1,1 we introduce the notion of simple right ideals and that theorems analogous to Theorems 1,1—1,5 hold.

In a semigroup having a kernel $\mathfrak{n}$ every two-sided ideal contains $\mathfrak{n}$. Hence it is natural to give the following definition:

Definition 1,2. *Let S be a semigroup with a kernel $\mathfrak{n}$. A two-sided ideal M of S is called simple if $\mathfrak{n} \subset M \subseteqq S$ holds but there does not exist a two-sided ideal M' of S with $\mathfrak{n} \subset M' \subset M \subseteqq S'$.*

Remark. If S has a zero element z then $\mathfrak{n} = \{z\}$ and our definition coincides with that of CLIFFORD [4], who calls of course such an ideal a „minimal two-sided ideal".

Theorem 1,6. *Let S have a kernel $\mathfrak{n}$. For any two different simple two-sided ideals M_1, M_2 it is always*

$$M_1 M_2 = M_1 \cap M_2 = \mathfrak{n}.$$

Proof. It is evidently $\mathfrak{n} \subseteqq M_1 M_2 \subseteqq M_1$, $\mathfrak{n} \subseteqq M_1 M_2 \subseteqq M_2$. Hence $\mathfrak{n} \subseteqq M_1 M_2 \subseteqq M_1 \cap M_2$. But analogously as in Theorem 1,1 $M_1 \cap M_2 = \mathfrak{n}$. Hence $\mathfrak{n} = M_1 M_2 = M_1 \cap M_2$.

If $M_1 = M_2$ we have only the relation $\mathfrak{n} \subseteqq M^2 \subseteqq M$. This gives — with respect to the simplicity of M — the result:

Corollary 1,6. *For every simple two-sided ideal M it is either $M^2 = M$ or $M^2 = \mathfrak{n}$.*

2. The class sum of all simple left (right, two-sided) ideals.

In this section we find conditions under which a semigroup having a proper kernel can be written as the class sum of its simple (left, right, two-sided) ideals.

Theorem 2,1. *Let S be a semigroup with a proper kernel $\mathfrak{n}$. If S is the class sum of its simple left (right, two-sided) ideals this decomposition is uniquely determined.*

Proof. We prove it (for instance) for two-sided ideals. Let be

$$S = \sum_{\lambda} M_\lambda, \tag{1}$$

$$S = \sum_{\varkappa} M'_\varkappa, \tag{2}$$

where M_λ, $M'_\varkappa$ are simple two-sided ideals of S. It is sufficient to show that every M_λ is contained among the summands of (2) and every M'_χ among the summands of (1). Let M'_α be any summand from the right hand side of (2). Let us form the intersection

$$S \cap M'_\alpha = (\sum_\lambda M_\lambda) \cap M'_\alpha.$$

$$M'_\alpha = \sum_\lambda (M_\lambda \cap M'_\alpha). \tag{3}$$

The intersection $M_\lambda \cap M'_\alpha$ is either $\mathfrak{n}$ or $M_\lambda \cap M'_\alpha = M_\lambda = M'_\alpha$. Since on the left hand side of (3) the ideal M'_α is $\supset \mathfrak{n}$ the second alternative must occur for at least one λ. Hence for at least one λ $M'_\alpha = M_\lambda$ holds. We prove similarly that on the other side every M_β belonging to the sum (1) equals some of the ideals $M'_\varkappa$. This proves our theorem.

The proof clearly holds if we mean by M simple left (or right) ideals.

Theorem 2,2. *Let S be a semigroup with a proper kernel $\mathfrak{n}$. Then S is the class sum of its simple left ideals if and only if the following condition holds: every relation*

$$a = xb, \quad a, b \text{ non} \in \mathfrak{n}$$

implies a relation

$$\bar{x}a = b$$

with some

$$\bar{x} \text{ non} \in \mathfrak{n}.$$

Proof. 1. The condition is necessary. According to the supposition every element $a \in S - \mathfrak{n}$ is contained in some simple left ideal. This ideal is necessarily $L_a = (\mathfrak{n}, a, Sa)$. Let be $b \neq a$, $b \in S - \mathfrak{n}$. The set $L_b = (\mathfrak{n}, b, Sb)$ is again a simple left ideal of S. According to Theorem 1,1 it is either $(\mathfrak{n}, a, Sa) \cap (\mathfrak{n}, b, Sb) = \mathfrak{n}$ or $(\mathfrak{n}, a, Sa) = (\mathfrak{n}, b, Sb)$. Since $a \neq b$, $a \in S - \mathfrak{n}$, $b \in S - \mathfrak{n}$ the relations $a \in Sb$, $b \in Sa$ are simultaneously satisfied. I. e. if there exists an $x \in S$ with $a = xb$ there exists also an $\bar{x} \in S$ with $b = \bar{x}a$. Further, it is $\bar{x} \in S - \mathfrak{n}$ since $\bar{x} \in \mathfrak{n}$ would imply $b = \bar{x}a \in \mathfrak{n}a \subseteq \mathfrak{n}$, contrary to the hypothesis.

2. The condition is sufficient. We show: if S satisfies the condition given in our theorem, then every element $a \in S - \mathfrak{n}$ is contained in some simple left ideal of S. This will prove our theorem.

Consider the element $a \in S - \mathfrak{n}$ and the ideal $L_a = (\mathfrak{n}, a, Sa)$. We shall show: there does not exist a left ideal L with $\mathfrak{n} \subset L \subset L_a$, i. e. L_a is a simple left ideal of S. We prove it indirectly. Let L be such an ideal and b any element of L, $b \in L - \mathfrak{n}$, $b \neq a$. Then $L_b = (\mathfrak{n}, b, Sb)$ clearly satisfies

$$\mathfrak{n} \subset L_b \subseteq L \subset L_a. \tag{4}$$

Since it is $b \in L_a$, b non $\in \mathfrak{n}$, $b \neq a$ we have $b = xa$ with some $x \in S - \mathfrak{n}$.

According to the supposition this relation implies a relation $a = \bar{x}b$ with $\bar{x}$ non ϵ $\mathfrak{n}$. Hence it is $a \in Sb$ and successively

$$a \in (\mathfrak{n}, b, Sb),$$
$$Sa \subseteqq (\mathfrak{n}, Sb, S^2b) = (\mathfrak{n}, Sb),$$
$$(\mathfrak{n}, a, Sa) \subseteqq (\mathfrak{n}, b, Sb),$$
$$L_a \subseteqq L_b.$$

The last relation — together with (4) — gives $L = L_a = L_b$, contrary to $L \subset L_a$. This proves our theorem.

The right dual theorem, which can be proved by an analogous argument, is the following:

Theorem 2,3. *Let S be a semigroup with a proper kernel $\mathfrak{n}$. Then S is the class sum of its simple right ideals if and only if the following condition holds: every relation*

$$a = by, \ a, b \in S - \mathfrak{n}$$

implies a relation

$$b = a\bar{y}$$

with some

$$\bar{y} \in S - \mathfrak{n}.$$

Definition 2,1. *We shall say that S is the direct sum of two-sided ideals if S can be written as the class sum of its simple two-sided ideals.*

Remark. The notion „direct sum" is justified by the following fact: If $S = \sum_{\varkappa} M_{\varkappa}$ ($M_{\varkappa}$ simple two-sided) then for every couple of summands $M_{\varkappa}, M_{\lambda}, \varkappa \neq \lambda, M_{\varkappa}M_{\lambda} = \mathfrak{n}$ holds. In this case we shall use the notation $S = \sum_{\varkappa} \oplus M_{\varkappa}$.

Theorem 2,4. *Let S be a semigroup with a kernel $\mathfrak{n}$. Then S is a direct sum of two-sided ideals if and only if the following condition holds. Let a, b be two elements $\in S - \mathfrak{n}$. If a can be written in at least one of the forms*

$$a = bx_1 \ \text{or} \ a = x_2b \ \text{or} \ a = x_3bx_4, \tag{5}$$

then b can be written in at least one of the forms

$$b = a\bar{x}_1 \ \text{or} \ b = \bar{x}_2a \ \text{or} \ b = \bar{x}_3a\bar{x}_4. \tag{6}$$

Proof. 1. The condition is necessary. Let S be the class sum of its simple two-sided ideals. Then every element $a \in S - \mathfrak{n}$ is contained in a simple two-sided ideal M_a. If a belongs to M_a so does aS, Sa, SaS. Hence $M_a = (\mathfrak{n}, a, Sa, aS, SaS) = (a, Sa, aS, SaS)$.[1])

Let $b \neq a$ be another element $\in S - \mathfrak{n}$. The simple ideal to which b

[1]) It is not necessary to write $\mathfrak{n}$, since $\mathfrak{n} \subseteqq S\,a\,S$.

belongs is necessarily $M_b = (b, Sb, bS, SbS)$. It is either $M_a \cap M_b = \mathfrak{n}$ or $M_a = M_b$. The second alternative

$$(a, Sa, aS, SaS) = (b, Sb, bS, SbS)$$

implies at least one of the relations $a \in Sb$, $a \in bS$, $a \in SbS$, i. e. the existence of at least one of the relations $a = bx_1$, $a = x_2b$, $a = x_3bx_4$. But then b is contained in at least one of the sets Sa, aS, SaS, i. e. at least one of the relations $b = a\bar{x}_1$, $b = \bar{x}_2a$, $b = \bar{x}_3a\bar{x}_4$ holds.

2. The condition is sufficient. It is again sufficient to prove that under the conditions of our theorem every element $a \in S$ is contained in some simple two-sided ideal of S.

Let be $a \in S - \mathfrak{n}$. The „least" possible two-sided ideal of S, to which a belongs, is $M_a = (a, Sa, aS, SaS)$. We show that M_a is simple. This follows again indirectly. Suppose there exists a two-sided ideal M of S with $\mathfrak{n} \subset M \subset M_a$. Let be $b \in M - \mathfrak{n}$, $b \neq a$. Then $M_b = (b, Sb, bS, SbS)$ is clearly a two-sided ideal of S contained in M. Hence

$$\mathfrak{n} \subset M_b \subseteqq M \subset M_a. \tag{7}$$

Now it is $b \in M_a$. Hence there holds at least one of the relations

$$b \in Sa, \quad b \in aS, \quad b \in SaS.$$

According to the supposition there holds therefore at least one of the relations $a \in Sb$, $a \in bS$, $a \in SbS$.

a) Let be $a \in Sb$. Then $aS \subseteqq SbS$, $Sa \subseteqq S^2b \subseteqq Sb$, $SaS \subseteqq SbS$. Hence

$$M_a = (a, Sa, aS, SaS) \subseteqq (Sb, SbS) \subseteqq (b, bS, Sb, SbS) = M_b.$$

With respect to (7) we get a contradiction to the supposition.

b) Let be $a \in bS$. We prove similarly $M_a \subseteqq M_b$, which gives again a contradiction.

c) Let be at last $a \in SbS$. Then $aS \subseteqq SbS$, $Sa \subseteqq SbS$, $SaS \subseteqq SbS$. Hence $M_a \subseteqq (b, SbS) \subseteqq M_b$, which is again a contradiction.

Any ideal M_a is a simple two-sided ideal of S, which proves our theorem.

To give important corollaries of Theorems 2,2—2,4 we introduce the following definitions.

Definition 2,2. *Let S be a semigroup with a kernel $\mathfrak{n}$. The totality $\mathfrak{n}_r$ of all elements $a \in S$ with $Sa \subseteqq \mathfrak{n}$ is called the right annihilator of S. The totality $\mathfrak{n}_l$ of all elements $a \in S$ with $aS \subseteqq \mathfrak{n}$ is called the left annihilator of S.*

We prove some properties of the sets $\mathfrak{n}_r$ and $\mathfrak{n}_l$.

Theorem 2,5. *The annihilators defined in Definition 2,2 have the following properties*:

a) $\mathfrak{n} \subseteqq \mathfrak{n}_l$, $\mathfrak{n} \subseteqq \mathfrak{n}_r$,

b) $\mathfrak{n}_l$, $\mathfrak{n}_r$ *are two-sided ideals of S*,

c) $S\mathfrak{n}_r = \mathfrak{n}$, $\mathfrak{n}_l S = \mathfrak{n}$,

d) $\mathfrak{n}_l^2 = \mathfrak{n}$, $\mathfrak{n}_r^2 = \mathfrak{n}$,

e) $\mathfrak{n}_l \mathfrak{n}_r = \mathfrak{n}$.

Proof. a) Since $S\mathfrak{n} = \mathfrak{n}S = \mathfrak{n}$, it is clearly $\mathfrak{n} \subseteqq \mathfrak{n}_r$, $\mathfrak{n} \subseteqq \mathfrak{n}_l$.

b) We show that $\mathfrak{n}_l$ is a two-sided ideal of S. Let be $a \in \mathfrak{n}_l$, $s \in S$, s arbitrary. Then $(sa)S = s(aS) \subseteqq s \cdot \mathfrak{n} \subseteqq \mathfrak{n}$. Therefore $sa \in \mathfrak{n}_l$, $s\mathfrak{n}_l \subseteqq \mathfrak{n}_l$ for every $s \in S$. The set $\mathfrak{n}_l$ is a left ideal of S. On the other side $(as)S = a(sS) \subseteqq aS \subseteqq \mathfrak{n}$. Hence $as \subseteqq \mathfrak{n}_l$, $\mathfrak{n}_l s \subseteqq \mathfrak{n}_l$ for every $s \in S$. The set $\mathfrak{n}_l$ is a right ideal.

Analogously it follows that $\mathfrak{n}_r$ is a two-sided ideal.

c) According to the definition $\mathfrak{n}_l S \subseteqq \mathfrak{n}$. But with respect to b) $\mathfrak{n}_l S$ is a two-sided ideal of S. Hence (with respect to the minimality of $\mathfrak{n}$) $\mathfrak{n}_l S = \mathfrak{n}$. Similarly $S\mathfrak{n}_r = \mathfrak{n}$.

d) It is $\mathfrak{n}_l^2 \subseteqq \mathfrak{n}_l S = \mathfrak{n}$. Since $\mathfrak{n}_l^2$ is a two-sided ideal of S, we get again $\mathfrak{n}_l^2 = \mathfrak{n}$. Analogously $\mathfrak{n}_r^2 = \mathfrak{n}$.

e) $\mathfrak{n}_l \mathfrak{n}_r$ is clearly a two-sided ideal of S. It is contained in $\mathfrak{n}_l S = \mathfrak{n}$, hence equal to $\mathfrak{n}$.

Corollary 2,2. *Let S be a semigroup with a kernel $\mathfrak{n}$. Let S be the class sum of its simple left ideals. Then to every $a \in S - \mathfrak{n}_r$ there exists an $e \in S - \mathfrak{n}_l$ such that $a = ea$.*

Proof. According to Theorem 2,2 the left ideal $L_a = (\mathfrak{n}, a, Sa)$ is simple. The set $(\mathfrak{n}, Sa)$ is clearly again a left ideal of S. Since a does not belong to the right annihilator $\mathfrak{n}_r$ of S, the set Sa is not entirely contained in $\mathfrak{n}$, hence $(\mathfrak{n}, Sa) \supset \mathfrak{n}$. But then — with respect to the simplicity of L_a — there must hold $(\mathfrak{n}, a, Sa) = (\mathfrak{n}, Sa)$. Since a non $\in \mathfrak{n}$, it is $a \in Sa$, hence there exists an e with $a = ea$. The element e does not belong to $\mathfrak{n}_l$, since otherwise there would be $a = ea \in \mathfrak{n}_l S = \mathfrak{n}$, contrary to the supposition.

Analogously:

Corollary 2,3. *If the semigroup S having a kernel $\mathfrak{n}$ is the class sum of its simple right ideals, then to every $a \in S - \mathfrak{n}_l$ there exists an element $f \in S - \mathfrak{n}_r$ satisfying the relation $a = af$.*

We introduce a further annihilator (containing $\mathfrak{n}_l$ and $\mathfrak{n}_r$ as subsets):

Definition 2,3. *The set of all elements $a \in S$ satisfying the relation $SaS = \mathfrak{n}$ will be denoted by $\mathfrak{n}_o$.*

Theorem 2,6. *The set $\mathfrak{n}_o$ introduced in the Definition 2,3 has the following properties:*

a) *$\mathfrak{n}_o$ is a two-sided ideal of S,*

b) $\mathfrak{n}_0^2 \subseteqq \mathfrak{n}_r$, $\mathfrak{n}_0^2 \subseteqq \mathfrak{n}_l$,

c) $\mathfrak{n}_0^3 = \mathfrak{n}$.

Proof. a) Follows as in Theorem 2,5.

b) It is $\mathfrak{n} = S\mathfrak{n}_o S \supseteqq \mathfrak{n}_0^2 S$. Since $\mathfrak{n}_0^2 S$ is a two-sided ideal of S, it is $\mathfrak{n} = \mathfrak{n}_0^2 S$. Hence $\mathfrak{n}_0^2 \subseteqq \mathfrak{n}_l$. Analogously $\mathfrak{n}_0^2 \subseteqq \mathfrak{n}_r$.

c) It is $\mathfrak{n} = S\mathfrak{n}_o S \supseteqq \mathfrak{n}_o\mathfrak{n}_o\mathfrak{n}_o = \mathfrak{n}_0^3$. Since again $\mathfrak{n}_0^3$ is a two-sided ideal of S, we have (with respect to the minimality of $\mathfrak{n}$) $\mathfrak{n}_0^3 = \mathfrak{n}$.

Remark. It is easy to derive further relation between the annihilators $\mathfrak{n}_l$, $\mathfrak{n}_r$, $\mathfrak{n}_o$ and to prove the following table:

$$\mathfrak{n} = \begin{Bmatrix} \mathfrak{n}_l\mathfrak{n}_o \\ \mathfrak{n}_l^2 \\ \mathfrak{n}_l\mathfrak{n}_r \\ \mathfrak{n}_o\mathfrak{n}_r \\ \mathfrak{n}_r^3 \\ \mathfrak{n}_0^3 \end{Bmatrix} \subseteqq \mathfrak{n}_r\mathfrak{n}_l \begin{Bmatrix} \subseteqq \mathfrak{n}_o\mathfrak{n}_e \\ \subseteqq \mathfrak{n}_r\mathfrak{n}_o \end{Bmatrix} \subseteqq \mathfrak{n}_0^2 \subseteqq \mathfrak{n}_l \cap \mathfrak{n}_r \begin{Bmatrix} \subseteqq \mathfrak{n}_r \\ \subseteqq \mathfrak{n}_l \end{Bmatrix} \subseteqq \mathfrak{n}_l + \mathfrak{n}_r \subseteqq \mathfrak{n}_o.$$

We leave out a detailed justification of these relations since they are mostly irrelevant for our purposes.

Corollary 2,4. *Let the semigroup S with the kernel $\mathfrak{n}$ be the direct sum of two-sided ideals. Then to every $a \in S - \mathfrak{n}_o$ there exist two elements e, f, $e \in S - \mathfrak{n}_l$, $f \in S - \mathfrak{n}_r$ such that $a = eaf$.*

Proof. According to Theorem 2,4 the two-sided ideal $M_a = (a, Sa, aS, SaS)$ is simple. Since a does not belong to $\mathfrak{n}_o$ the ideal SaS is a proper overset of $\mathfrak{n}$. Hence — with respect to the simplicity of M_a —

$$(a, Sa, aS, SaS) = SaS.$$

The relation $a \in SaS$ implies the exitence of e, f with $a = eaf$. The element e does not belong to $\mathfrak{n}_l$ since otherwise there would hold $a = eaf \subseteqq \subseteqq \mathfrak{n}_l af \subseteqq \mathfrak{n}_l S = \mathfrak{n}$, contrary to the supposition. Analogously $f \in S - \mathfrak{n}_r$. This proves Corollary 2,4.

Examples. In theorems of sections 1 and 2 we spoke about simple left (right, two-sided) ideals assuming tacitly that such ideals really exist. We show on simple examples that this assumption needs not always be satisfied.

1. The simplest commutative example is the semigroup of non — negative integers $S = \{0, 1, 2, 3, \ldots\}$ the multiplication being defined as the ordinary multiplication of numbers. Every ideal contains the kernel $\mathfrak{n} = \{0\}$, but there are no simple (two-sided) ideals.

2. An example of a non-commutative semigroup with a kernel different from a zero element, without simple left and without simple right ideals, is the following.

Let a, b, c, d be non-negative integers. Consider the set of all linear polynomials and constants $ax + b$ with the following definition of multiplication[2]) $\odot$

[2]) We admit a, c to be zero.

$$(ax + b) \odot (cx + d) = a(cx + d) + b = acx + ad + b.$$

Minimal right ideals of S are the sets $\{0\}, \{1\}, \{2\}, \ldots$ There exists only one minimal left ideal: $\{0, 1, 2, \ldots\}$. This is at the same time the kernel $\mathfrak{n}$ of S.

Every left ideal containing $\mathfrak{n}$ and different from $\mathfrak{n}$ must have at least one element $rx + s$ with $r \neq 0$. Any left ideal containing the element $rx + s$ (r, s fixed) contains also all elements $S \odot (rx + s)$, i. e. the totality of all polynomials of the form $arx + (as + b)$ with $a = 0, 1, 2, \ldots$, $b = 0, 1, 2, \ldots$ variable. Since we can put especially $a = 0$, $b = 0, 1, 2, \ldots$, it is clear that $\mathfrak{n}$ is contained in every such left ideal. But it is easily seen that every such left ideal contains an infinity of proper subideals all containing $\mathfrak{n}$. It is for instance

$$S \odot (rx + s) \supset S \odot (2rx + s) \supset S \odot (4rx + s) \supset \ldots \supset \mathfrak{n}.$$

Therefore there are no simple left ideals in S.

A right ideal R containing $\mathfrak{n}$ and different from $\mathfrak{n}$ must have again at least one element $rx + s$ with $r \neq 0$. If it contains $rx + s$, it contains also the sum of two sets $(rx + s) \odot S$ and $\mathfrak{n}$. The first set is the set of all elements of the form $arx + (br + s)$ (a, b variable). But R contains clearly again an infinite number of proper subideals containing $\mathfrak{n}$. It is e. g.

$$R \supseteqq \{(rx + s) \odot S, \mathfrak{n}\} \supset \{(2rx + s) \odot S, \mathfrak{n}\} \supset \{(4rx + s) \odot S, \mathfrak{n}\} \supset \ldots$$

Hence there are no simple right ideals.

To assure in the sequel the existence of simple ideals we shall impose to our semigroups occasionally one of the following two conditions:

Condition A. *S is a semigroup having a kernel and at least one simple left ideal.*

Condition B. *S is a semigroup having a kernel, at least one simple left ideal and at least one simple right ideal.*

This notation will be kept throughout the paper. Our main goal is to answer the question: what can be said about various types of semigroups satisfying one of these conditions.

3. The notion of $\mathfrak{n}$-potency.

Let S be a semigroup with the kernel $\mathfrak{n}$. Consider the sequence

$$S \supseteqq S^2 \supseteqq S^3 \supseteqq \ldots \tag{8}$$

Each member of this sequence is a two-sided ideal of S. Since $\mathfrak{n}$ is a subset of every member in (8), there are two possibilities:

a) either $S^\varrho \supset \mathfrak{n}$ for every $\varrho \geqq 1$,

b) or there exists a least $\varrho \geqq 1$ with $S^\varrho = \mathfrak{n}$.

In this second case it is also $\mathfrak{n} = S^{\varrho+1} = S^{\varrho+2} = \ldots$

Definition 3,1. *Let S be a semigroup with the kernel $\mathfrak{n}$. Let S' be any subset of S, $S' \subseteqq S$. We shall say that S' is $\mathfrak{n}$-potent if for some integer $\varrho \geqq 1$ $S'^\varrho \subseteqq \mathfrak{n}$ holds. The least such ϱ is called the index of $\mathfrak{n}$-potency.*

Especially: An element $x \in S$ is called $\mathfrak{n}$-potent if $x^\varrho \in \mathfrak{n}$; a left ideal L is $\mathfrak{n}$-potent if $L^\varrho \subseteqq \mathfrak{n}$. A two-sided ideal M is $\mathfrak{n}$-potent if $M^\varrho = \mathfrak{n}$ holds. (In the last case the sign of equality holds since it is $\mathfrak{n} \subseteqq M^\varrho$ for every ϱ.) The semigroup S is itself $\mathfrak{n}$-potent if $S^\varrho = \mathfrak{n}$ for some $\varrho \geqq 1$. (For a semigroup having a proper kernel it is $\varrho \geqq 2$ if such an index ϱ really exists.)

We prove first two simple theorems:

Theorem 3,1. *The sum of a finite number of $\mathfrak{n}$-potent left (right) ideals is a $\mathfrak{n}$-potent left (right) ideal.*

Proof. It is sufficient to prove it for two ideals. Let L_1, L_2 be two $\mathfrak{n}$-potent left ideals, $L_1^{\varrho_1} \subseteqq \mathfrak{n}$, $L_2^{\varrho_2} \subseteqq \mathfrak{n}$. We show that $(L_1, L_2)^{\varrho_1+\varrho_2} \subseteqq \mathfrak{n}$. In fact, every summand of the product at the left hand side of this relation contains at least ϱ_1 factors of L_1 or ϱ_2 factors of L_2. In the first case the summand belongs to $L_1^{\varrho_1}$, in the second to $L_2^{\varrho_2}$; hence always to $\mathfrak{n}$.

Theorem 3,2. *Every $\mathfrak{n}$-potent left (right) ideal is contained in some two-sided $\mathfrak{n}$-potent ideal.*

Proof. Let L be a left $\mathfrak{n}$-potent ideal of S, $L^\varrho \subseteqq \mathfrak{n}$. The ideal (L, LS) is a two-sided ideal of S. With respect to Theorem 3,1 our theorem will be proved if we show that LS is $\mathfrak{n}$-potent. This is true, since

$$(LS)^\varrho = LS \, . \, LS \ldots LS \subseteqq L^\varrho S \subseteqq \mathfrak{n}S = \mathfrak{n}.$$

We introduce a further definition which we shall use later:

Definition 3,2. *The sum of all two-sided $\mathfrak{n}$-potent ideals of S is called the radical of S.*

The radical $\mathfrak{r}$ is always a two-sided ideal satisfying $\mathfrak{n} \subseteqq \mathfrak{r} \subseteqq S$. Moreover, since the annihilators $\mathfrak{n}_o$, $\mathfrak{n}_l$, $\mathfrak{n}_r$ satisfy $\mathfrak{n}_l^2 = \mathfrak{n}_r^2 = \mathfrak{n}$, $\mathfrak{n}_0^3 = \mathfrak{n}$, it is also $\mathfrak{n} \subseteqq \mathfrak{n}_i \subseteqq \mathfrak{r} \subseteqq S$, where i denotes o or l or r.

We shall study in the following mainly semigroups with $\mathfrak{r} \subset S$ (i. e. non-$\mathfrak{n}$-potent semigroups), especially the case $\mathfrak{r} = \mathfrak{n}$ (semigroups without a proper radical). We shall need often the existence of simple left (or right) ideals. Therefore we shall often replace Conditions A and B by the following conditions:

Condition A_1. *S is a semigroup with the kernel $\mathfrak{n}$ having at least one non-$\mathfrak{n}$-potent simple left ideal.*

Condition B_1. *S is a semigroup with the kernel $\mathfrak{n}$ having at least one non-$\mathfrak{n}$-potent simple left and at least one non-$\mathfrak{n}$-potent simple right ideal.*

4. Simple semigroups satisfying Condition A_1.

In this and the following section we shall study simple semi-groups. We define again simplicity otherwise as it is done in Clifford [3] and Rees [1], [2] but analogously as in Clifford [4].

Definition 4,1. *Let S be a semigroup with the kernel $\mathfrak{n}$. The semigroup is called simple if it contains no two-sided ideal different from $\mathfrak{n}$ and S itself.*

Remark. If S has a zero element our definition coincides again with that of Clifford and Rees.

The relation between simple semigroups and simple two-sided ideals of a general semigroup will be given in Theorem 8,2.

With respect to the relation $\mathfrak{n} \subseteqq S^2 \subseteqq S$ there exist for a simple semigroup only two possibilities: a) either $S^2 = S$, b) or $S^2 = \mathfrak{n}$. We shall study only the case $S^2 = S$. The case $S^2 = \mathfrak{n}$ is not interesting. For, in a general semigroup satisfying $S^2 = \mathfrak{n}$ every subset containing $\mathfrak{n}$ is clearly a two-sided ideal of S. Hence an $\mathfrak{n}$-potent simple semigroup is the sum of the kernel and of a single ($\mathfrak{n}$-potent) element.

Theorem 4,1. *Let S be a simple semigroup having at least one non-$\mathfrak{n}$-potent left ideal of S which itself contains $\mathfrak{n}$. Then*

a) $S^2 = S$,

b) *every left ideal $L \supset \mathfrak{n}$ is non-$\mathfrak{n}$-potent.*

Proof. a) Let $L^* \supset \mathfrak{n}$ be the non-$\mathfrak{n}$-potent left ideal of S. The relation $S^2 = \mathfrak{n}$ would imply $L^{*2} \subseteqq S^2 = \mathfrak{n}$, i. e. $L^{*2} \subseteqq \mathfrak{n}$, which contradicts the supposition.

b) The proof of the second statement follows indirectly. Suppose that some left ideal $L \supset \mathfrak{n}$ of S is $\mathfrak{n}$-potent, i. e. $L^\varrho \subseteqq \mathfrak{n}$ for some $\varrho > 0$. The set (L, LS) is a two-sided ideal of S and $\supset \mathfrak{n}$, hence equal to S. Both ideals L and LS are $\mathfrak{n}$-potent. (See Theorem 3,2.) S being sum of two $\mathfrak{n}$-potent ideals would be $\mathfrak{n}$-potent, i. e. there would hold $S^2 = \mathfrak{n}$. This contradicts a).

The converse of Theorem 4,1 is

Theorem 4,2. *Let S be a simple semigroup. If S has at least one $\mathfrak{n}$-potent left ideal $L \supset \mathfrak{n}$, then S is $\mathfrak{n}$-potent and hence all left ideals of S are $\mathfrak{n}$-potent.*

Proof. It follows, as in Theorem 3,2, that (under our conditions) L and LS are $\mathfrak{n}$-potent ideals. Further $(L, LS) \supset \mathfrak{n}$ is a two-sided ideal. Hence $S = (L, LS)$. But S, as a sum of two $\mathfrak{n}$-potent ideals, is $\mathfrak{n}$-potent.

Remark. Observe that in Theorems 4,1 and 4,2 there was no mention about simple left ideals.

Corollary 4,2a. *Let S be a simple semigroup satisfying Condition A. Then* *a) either all simple left ideals of S are $\mathfrak{n}$-potent and S itself is $\mathfrak{n}$-potent (both with the index of $\mathfrak{n}$-potency $\varrho = 2$),*

b) or none of the simple left ideals L is $\mathfrak{n}$-potent (satisfying $L^2 = L$) and S itself is non-$\mathfrak{n}$-potent (satisfying $S^2 = S$).

Corollary 4,2b. *In a simple semigroup the Condition A_1 is equivalent to the Condition A with the additional requirement $S^2 \neq \mathfrak{n}$ (or $S^2 = S$).*

Corollary 4,2c. *In a simple semigroup satisfying Condition A_1 the radical $\mathfrak{r}$ and the annihilators $\mathfrak{n}_l$, $\mathfrak{n}_r$, $\mathfrak{n}_o$ are all equal to $\mathfrak{n}$.*

Proof. Follows from $\mathfrak{n}_i \subseteqq \mathfrak{r}$ $(i = o, l, r)$ and the fact that $\mathfrak{r} = S$ contradicts $S^2 \neq \mathfrak{n}$.

Theorem 4,3. *Let S be a simple semigroup satisfying Condition A_1. Then*

a) S is the class sum of its simple left ideals.

b) If L^* is one fixed simple left ideal of S, then every simple left ideal of S is of the form $L = \mathfrak{n} + L^*c$ with $c \in L$.

Proof. The set L^*S is a two-sided ideal of S. There cannot hold $L^*S = \mathfrak{n}$ since we would have $L^{*2} \subseteqq L^*S = \mathfrak{n}$, which contradicts Theorem 4,1. Hence $L^*S = S$. Let us write $S = \sum_{c_\varkappa \in S} c_\varkappa$.

$$S = L^*S = L^* \sum_\varkappa c_\varkappa = \sum_\varkappa (L^*c_\varkappa) = \mathfrak{n} + \sum_\varkappa (L^*c_\varkappa) = \sum_\varkappa (\mathfrak{n} + L^*c_\varkappa). \quad (9)$$

Every summand on the right hand side of (9) is either $\mathfrak{n}$ (and can be then released) or is a simple left ideal of S. Hence S is the class sum of its simple left ideals.

Since this decomposition is uniquely determined (see Theorem 2,1) we have: every simple left ideal of S is of the form $L = \mathfrak{n} + L^*c_\varkappa$, $c_\varkappa \in S$. The fact that $c_\varkappa$ belongs to L can be shown as follows. Let be $c_\varkappa \in L_\varkappa$ ($L_\varkappa$ simple left ideal). Then it is

$$L = \mathfrak{n} + L^*c_\varkappa \subseteqq \mathfrak{n} + L^*L_\varkappa \subseteqq \mathfrak{n} + L_\varkappa = L_\varkappa,$$

hence $L_\varkappa = L$, q. e. d.

Theorem 4,4. *Let S be a simple semigroup satisfying Condition A_1. Then for every two simple left ideals L_α, L_β of S there holds always: $L_\alpha L_\beta = L_\beta$.*

Proof. According to the proof of Theorem 4,3 S can be written as the class sum of its simple left ideals $S = \sum_\varkappa L_\varkappa$. We can assume thereby that for every couple $L_{\varkappa_1}$, $L_{\varkappa_2}$ $(\varkappa_1 \neq \varkappa_2)$ $L_{\varkappa_1} \cap L_{\varkappa_2} = \mathfrak{n}$ holds. According to the proof of Theorem 4,3 we have for every L_α: $L_\alpha S = S$, i. e. $L_\alpha \sum_\varkappa L_\varkappa = \sum L_\varkappa$,

$$\sum_\varkappa L_\alpha L_\varkappa = \sum_\varkappa L_\varkappa. \quad (10)$$

Now, according to Theorem 1,3, the product $L_\alpha L_\varkappa$ is either $L_\varkappa$ or $\mathfrak{n}$. If for some $\varkappa = \beta$ there would be $L_\alpha L_\beta = \mathfrak{n}$, the elements of $L_\beta - \mathfrak{n}$ would not be contained in the set on the left hand side of (10) and there would hold $\sum_\varkappa L_\alpha L_\varkappa \subset \Sigma_\varkappa$. This contradicts (10). Hence for every $\varkappa$ $L_\alpha L_\varkappa = L_\varkappa$.

Theorem 4,5. *Let S be a simple semigroup satisfying Condition A_1. Then to every $a \in S - \mathfrak{n}$ there exists an element $e \in S - \mathfrak{n}$ with $a = ea$. Moreover the following implication holds:*

$$a = xb \Rightarrow \bar{x}a = b \ \ (a, b, x \in S - \mathfrak{n} \text{ and } \bar{x} \in S - \mathfrak{n}).$$

Proof. Follows immediately from Theorem 4,3 by means of Theorem 2,2 and Corollary 2,2 using the fact that we have for the annihilators $\mathfrak{n}_l = \mathfrak{n}_r = \mathfrak{n}$.

Remark. Theorems 4,3—4,5 give in some sense the maximum what can be said about simple semigroups satisfying only Condition A_1. To obtain more precise results we must introduce supplementary conditions. This will be done in section 5 by introducing Condition B_1 and in section 7 by introducing the requirement of the existence of at least one idempotent $\in S - \mathfrak{n}$.

5. Simple semigroups satisfying Condition B_1.

In this section we shall show that assuming Condition B_1 it is easy to prove the existence of idempotents in $S - \mathfrak{n}$ and to prove the so called „complete simplicity" of S.

Theorem 5,1. *Let S be a simple semigroup satisfying Condition B_1. Then to every $a \in S - \mathfrak{n}$ there exist two elements $e, f \in S - \mathfrak{n}$ with*

$$a = ea, \ a = af. \tag{11}$$

Moreover: for every $a, b, x, y \in S - \mathfrak{n}$ the following implications hold:

$$a = xb \Rightarrow b = \bar{x}a, \tag{12}$$

$$a = by \Rightarrow b = a\bar{y}, \tag{13}$$

with $\bar{x}, \bar{y} \in S - \mathfrak{n}$.

Proof. Follows from Theorem 4,5 and its right dual Theorem.

Now we prove the most important assertion:

Theorem 5,2. *The elements e and f of Theorem 5,1 are idempotents.*

Proof. a) We show first that e is an idempotent. The relation $a = ea$ implies according to (13) the existence of an element $\bar{a}$ with $a\bar{a} = e$. Now it is successively

$$e = a\bar{a} = (ea)\bar{a} = e(a\bar{a}) = e \,.\, e = e^2.$$

Hence e is an idempotent. If e would belong to $\mathfrak{n}$ there would be $a = ea \in \mathfrak{n}a \subseteq \mathfrak{n}$, contrary to the hypothesis. Hence $e \in S - \mathfrak{n}$.

b) Analogously the relation $a = af$ implies according to (12) the existence of an $\bar{a}$ with $\bar{a}a = f$. It is again $f = \bar{a}a = \bar{a}(af) = (\bar{a}a)f = f \cdot f = f^2$. There follows analogously that $f \in S - \mathfrak{n}$, which completes our proof.

Observe carefully: the relation (13) is a consequence of the decomposition of S into a sum of simple right ideals. This relation alone implies that every element e which reproduces on the left any element $a \in S - \mathfrak{n}$ is an idempotent.

Theorem 5,3. *Let S be a simple semigroup satisfying Condition B_1. Then every simple left ideal L of S is of the form $L = \mathfrak{n} + Se$, where e is an idempotent $\in L - \mathfrak{n}$. Analogously, every right simple ideal R of S has the form $R = \mathfrak{n} + fS$, where f is an idempotent $\in R - \mathfrak{n}$.*

Proof. Let be $c \in L$. According to Theorem 1,4, it is either $\mathfrak{n} + Lc = L$ or $\mathfrak{n} + Lc = \mathfrak{n}$. It follows immediately that there must be at least one element $c \in L - \mathfrak{n}$ with $\mathfrak{n} + Lc = L$. If namely for every $c \in L$ there would be $Lc \subseteq \mathfrak{n}$ we would have $L^2 = L \sum_{c \in L} c \subseteq \mathfrak{n}$, contrary to Theorem 4,1.

The equation $\mathfrak{n} + Lc = L$ implies: there exists an element $e \in L$ with $ec = c$. According to Theorem 5,2 e is an idempotent. It does not belong to $\mathfrak{n}$ since otherwise c would be in $\mathfrak{n}$, contrary to the choice of c.

Consider now the left ideal $\mathfrak{n} + Se \subseteq \mathfrak{n} + SL \subseteq L$. Se is a left ideal of S. It contains the element $e \cdot e = e$, hence $\mathfrak{n} + Se \supset \mathfrak{n}$, i. e. $\mathfrak{n} + Se = L$.

The proof for the simple right ideal is similar.

Remark. We can also write $L = \mathfrak{n} + Le$, $e \in L$. For $\mathfrak{n} + Le$ is a simple left ideal containing the element e, hence $\supset \mathfrak{n}$. The intersection of the two simple left ideals L and $\mathfrak{n} + Le$ is $L \cap (\mathfrak{n} + Le) \supset \mathfrak{n}$. Hence, according to Theorem 1,1, $L = \mathfrak{n} + Le$.

Theorem 5,4. *Let S be a simple semigroup satisfying Condition B_1. Let L be a simple left ideal of S. Then L is a semigroup having the following properties*:

a) *It has at least one right identity $e \in L - \mathfrak{n}$ for every element $\in L - \mathfrak{n}$.*

b) *Every idempotent $e^* \in L - \mathfrak{n}$ is a right identity for every element $\in L - \mathfrak{n}$.*

c) *In L it is possible to cancel on the right with every element c for which $e \cdot c$ non $\in \mathfrak{n}$ holds.*

Proof. a) Since $L = \mathfrak{n} + Se$ (where $e \in L - \mathfrak{n}$ is an idempotent), every $a \in L - \mathfrak{n}$ is of the form $a = u \cdot e$ $(u \in S)$. Therefore $a \cdot e = (ue) \cdot e = ue^2 = ue = a$.

b) Follows from the fact that for every idempotent $e^* \in L - \mathfrak{n}$ the relation $L = \mathfrak{n} + Se^*$ holds.

c) Let c be any element of L for which ec non $\in \mathfrak{n}$ holds.[3]) Since $e \in L - \mathfrak{n}$, $c \in L - \mathfrak{n}$, the set $\mathfrak{n} + Lc$ contains at least one element (namely ec) not belonging to $\mathfrak{n}$. Hence $\mathfrak{n} + Lc \supset \mathfrak{n}$ and $L = \mathfrak{n} + Lc$. This implies: to every $d \in L - \mathfrak{n}$ there exists an $x \in L - \mathfrak{n}$ with $xc = d$. In other words: the equation $xc = d$ has always a solution for such elements $c \in L - \mathfrak{n}$ for which ec non $\in \mathfrak{n}$ holds.

Let now be

$$ac = bc, \quad ec \text{ non} \in \mathfrak{n}. \tag{14}$$

Find first an $\bar{c} \in L - \mathfrak{n}$ with $c\bar{c} = e$. Then $(\bar{c}c)^2 = \bar{c}(c\bar{c})c = (\bar{c}e)c$...

Hence $\bar{c}c = e^*$ is an idempotent belonging to $L - \mathfrak{n}$. It does not belong to $\mathfrak{n}$, for $\bar{c}c \in \mathfrak{n}$ would imply $c\bar{c}c \in \mathfrak{n}$, i. e. $ce \in \mathfrak{n}$, $c \in \mathfrak{n}$, contrary to the supposition.

Multiply now (14) by $\bar{c}$ on the right. We have $ac\bar{c} = bc\bar{c}$, $ae^* = be^*$, $a = b$. This completes the proof of Theorem 5,4.

As generalization of two notions, first given by Rees [1], we introduce the following two definitions:

Definition 5,1. *An idempotent $e \in S - \mathfrak{n}$ is called primitive if the only idempotent $x \in S - \mathfrak{n}$ satisfying the relation*

$$ex = xe = x$$

is $x = e$.

Definition 5,2. *A simple semigroup having a kernel $\mathfrak{n}$ is called completely simple if*

a) to every $a \in S - \mathfrak{n}$ there exist idempotents $e, f \in S - \mathfrak{n}$ such that $ea = af = a$,

b) every idempotent $\in S - \mathfrak{n}$ is primitive.

The structure of completely simple semigroups having a zero element (i. e. the case $\mathfrak{n} = \{z\}$) was studied in detail by Rees [1], who proved that it can be realised as a type of matrix semigroups. (See a further development in Clifford [6].)

We show easily:

Theorem 5,5. *A simple semigroup satisfying Condition B_1 is completely simple.*

Proof. a) Condition a) of the Definition 5,2 is simply the statement of Theorems 5,1 and 5,2.

b) Let now be e and x any idempotents $\in S - \mathfrak{n}$. We show that the relation $ex = xe = x$ implies $x = e$.

Since S is the sum of simple left ideals, we can find a simple left ideal L such that $x \in L$. Further there holds $\mathfrak{n} + Sx \subseteqq \mathfrak{n} + SL \subseteqq L$.

3) If ec non $\in \mathfrak{n}$ holds for one idempotent $e \in L - \mathfrak{n}$, it is for every idempotent $e^* \in L - \mathfrak{n}$, e^*c non $\in \mathfrak{n}$. For, according to b), it is $e = ee^*$ and therefore $ec = ee^*c$. The relation $e^*c \in \mathfrak{n}$ would imply $ec \in e\mathfrak{n} \subseteqq \mathfrak{n}$, contrary to the supposition.

Hence either $\mathfrak{n} + Lx = L$ or $\mathfrak{n} + Lx = \mathfrak{n}$. The second alternative is impossible, since there would hold $x = x^2 \in Lx \subseteqq \mathfrak{n}$, which contradicts the hypothesis. Hence it is $\mathfrak{n} + Sx = L$.

The relation $xe = x$ implies, according to Theorem 5,1, the existence of an $\bar{x}$ with $e = \bar{x}x$.[4]) Therefore it is $e \in Sx$, $e \in L$.

The elements e, x (and therefore xe, ex) belong to L. It holds $x = x^2 = ex$. Further the element x satisfies ex non $\in \mathfrak{n}$ since otherwise x would be contained in $\mathfrak{n}$, contrary to the hypothesis. According to Theorem 5,4 we can cancel on the right with x and we get $x = e$. This completes the proof of Theorem 5,5.

6. The structure of simple left ideals.

In this section we shall study the detailed structure of simple left ideals of a general semigroup having a kernel.

Theorem 6,1. *Let S be a semigroup satisfying Condition A.*[5]) *Let L be a simple left ideal of S having at least one idempotent $e \in L - \mathfrak{n}$. Then*

a) e and every idempotent $\in L - \mathfrak{n}$ is a right identity for every element $\in L - \mathfrak{n}$.

b) In L it is possible to cancel on the right with every element c for which ec non $\in \mathfrak{n}$ *holds.*

Proof. It is the same as in Theorem 5,4, since we did not use there explicitely the existence of simple right ideals. (We used it naturally implicitely to prove the existence of at least one idempotent.)

Remark. There are examples showing the existence of simple left non-$\mathfrak{n}$-potent ideals without an idempotent. For such simple left ideals Theorem 6,1 does not hold.

Theorem 6,2. *Let S be a semigroup satisfying Condition A. Let L be a simple left ideal of S. Let e_α be any idempotent $\in L - \mathfrak{n}$ and $\mathfrak{g}_\alpha$ the set of all elements of $e_\alpha L$ which do not belong to $\mathfrak{n}$. Then $\mathfrak{g}_\alpha$ is a group.*

Proof. 1. We prove first that $\mathfrak{g}_\alpha$ is a semigroup. Since

$$e_\alpha L \,.\, e_\alpha L = e_\alpha(Le_\alpha L) \subseteqq e_\alpha SL \subseteqq e_\alpha L$$

holds, it is only necessary to show: for any two $x, y \in e_\alpha L$ the relation $xy \in \mathfrak{n}$ is satisfied if and only if at least one of the elements x, y belongs to $\mathfrak{n}$.

Remember: the elements of $\mathfrak{g}_\alpha$ are exactly those elements $e_\alpha c$, $c \in L$ for which $e_\alpha c$ non $\in \mathfrak{n}$ holds. In Theorem 5,4 (resp. Theorem 6,1) we proved: for every such $c \in L$ and every $d \in L - \mathfrak{n}$ the equation $xc = d$ has a solution with $x \in L - \mathfrak{n}$.

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[4]) We note that this is a consequence of Condition A_1 only. (We shall use this remark in section 7.)

[5]) L having the idempotent $e \in L - \mathfrak{n}$ is certainly non-$\mathfrak{n}$-potent, so that it is sufficient to suppose Condition A.

Let now be (in the way to get a contradiction) $x = e_\alpha c_1$, $y = e_\alpha c_2$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\alpha$, but $e_\alpha c_1 \cdot e_\alpha c_2 = e_\alpha c_1 c_2 \in \mathfrak{n}$. Find an element $\bar{x}$ such that $\bar{x}(e_\alpha c_1) = e_\alpha$. As we have just remarked this is always possible. The relation $e_\alpha c_1 c_2 \in \mathfrak{n}$ implies $\bar{x} e_\alpha c_1 c_2 \in \mathfrak{n}$ and therefore $e_\alpha c_2 \in \mathfrak{n}$. This is a contradiction. The set $\mathfrak{g}_\alpha$ is a semigroup.

2. We prove that $\mathfrak{g}_\alpha$ contains the only idempotent e_α. Suppose that there exist two different idempotents $e_\alpha, e_\beta \in \mathfrak{g}_\alpha$. It is $e_\beta^2 = e_\beta$ and $e_\alpha e_\beta = e_\beta$. (This last relation holds, since e_α is clearly a two-sided unity element for every element $\in \mathfrak{g}_\alpha$.) Hence $e_\alpha e_\beta = e_\beta^2$. Since $e_\alpha e_\beta = e_\beta$ non $\in \mathfrak{n}$, it is possible to cancel with e_β. We get $e_\alpha = e_\beta$, which is a contradiction.

3. We next prove that for every $a \in \mathfrak{g}_\alpha$ the equations

$$xa = e_\alpha, \tag{15}$$

$$ay = e_\alpha, \tag{16}$$

have solutions with $x, y \in \mathfrak{g}_\alpha$ and that these solutions are uniquely determined. It is well-known from the elementary theory of group-axioms that these properties are sufficient to prove that $\mathfrak{g}_\alpha$ is a group.

a) Let be $a \in \mathfrak{g}_\alpha$ (i. e. $a = e_\alpha a$ non $\in \mathfrak{n}$). Then $\mathfrak{n} + La = L$, $e_\alpha \mathfrak{n} + e_\alpha La = e_\alpha L$. The element e_α is contained in $e_\alpha L$ and not in $\mathfrak{n}$. Hence it is contained also in $e_\alpha La$. Therefore there exists always an $x \in e_\alpha L - \mathfrak{n} = \mathfrak{g}_\alpha$ such that $xa = e_\alpha$ holds.

b) Now we prove the existence of the solution of (16). Find first an $\bar{a} \in \mathfrak{g}_\alpha$ with $\bar{a}a = e_\alpha$. (This is possible with respect to (15).) We show that $\bar{a}$ satisfies also the relation $a\bar{a} = e_\alpha$. The element $a\bar{a}$ belongs to $\mathfrak{g}_\alpha$. It is an idempotent, since $(a\bar{a})^2 = a(\bar{a}a)\bar{a} = ae_\alpha\bar{a} = a\bar{a}$. With respect to the fact proved in 2 above, it is $a\bar{a} = e_\alpha$. This proves the solvability of (16).

c) We prove at last the uniqueness of the solutions of (15) and (16). For the equation (15), this is a consequence of Theorem 6,1b, since $x_1 a = x_2 a$ and $e_\alpha a = a$ non $\in \mathfrak{n}$ imply $x_1 = x_2$. For the equation (16) it follows in the following manner. Let be $ay_1 = ay_2 = e_\alpha$ $(a, y_1, y_2 \in \mathfrak{g}_\alpha)$. Find an element $\bar{x}$ with $\bar{x}a = e_\alpha$.[6]) Multiplying on the left we get $\bar{x}ay_1 = \bar{x}ay_2$, $e_\alpha y_1 = e_\alpha y_2$, hence $y_1 = y_2$. This completes the proof of Theorem 6,2.

Theorem 6,3. *Let the suppositions of Theorem 6,2 be satisfied. Let* e_α, e_β *be two different idempotents* $\in L - \mathfrak{n}$. *Then the groups* $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ *are disjoint.*

Proof. Suppose that $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ have at least one element $a \in L - \mathfrak{n}$ in common. This element can be written in both forms $a = e_\alpha a = e_\beta a$. Since $e_\alpha a = a$ non $\in \mathfrak{n}$ holds, it is possible to cancel with a. We get $e_\alpha = e_\beta$, contrary to the supposition.

Theorem 6,4. *Let S be a semigroup satisfying Condition A. Let L be a simple left ideal of S having at least one idempotent* non $\in \mathfrak{n}$. *Then L is*

[6]) This is possible with respect to (15).

a sum of disjoint groups $\sum_{\alpha} \mathfrak{g}_\alpha$ and of one $\mathfrak{n}$-potent semigroup P with the index of $\mathfrak{n}$-potency $\varrho = 2$. In formulae

$$L = \sum_{\alpha} \mathfrak{g}_\alpha + P, \quad P^2 = \mathfrak{n}.$$

Proof. The set E of all idempotents $\epsilon L - \mathfrak{n}$ is non-vacuous. Construct to every e_α the group $\mathfrak{g}_\alpha$. Put $P = L - \sum_{\alpha} \mathfrak{g}_\alpha$.

a) We show first: for every $c \epsilon P$ there holds $Lc \subseteqq \mathfrak{n}$. This follows indirectly. Let be $c \epsilon P$, but $\mathfrak{n} + Lc \supseteqq \mathfrak{n}$, hence $\mathfrak{n} + Lc = L$. Since $\mathfrak{n} + Lc = \mathfrak{n} + (\mathfrak{n} + Le_\alpha) c = \mathfrak{n} + Le_\alpha c$ our assumption[7]) clearly implies $e_\alpha c$ non $\epsilon\ \mathfrak{n}$. The relation $\mathfrak{n} + Lc = L$ implies further the existence of an e^* with $e^*c = c$. The element e^* is an idempotent. For it is $c = e^*c = e^{*2}c$ and since it holds $e_\alpha c$ non $\epsilon\ \mathfrak{n}$, it is possible to cancel with c what gives $e^* = e^{*2}$. Now the group constituted by all elements of $e^*L - \mathfrak{n}$ contains $e^*c = c$. But this is a contradiction to the assumption $c \epsilon P$.

The proof just given shows that P is exactly the set of all $c \epsilon L$ with $Lc \subseteqq \mathfrak{n}$.

b) P is a semigroup. For, let be $c_1 \epsilon P$, $c_2 \epsilon P$, i. e. $Lc_1 \subseteqq \mathfrak{n}$, $Lc_2 \subseteqq \mathfrak{n}$. Then it is also $Lc_1c_2 \subseteqq \mathfrak{n}c_2 \subset \mathfrak{n}$, i. e. $c_1c_2 \epsilon P$.

c) P is $\mathfrak{n}$-potent, more precisely $P^2 = \mathfrak{n}$. This follows immediately. Since for every $c \epsilon P$ the relation $Lc \subseteqq \mathfrak{n}$ holds, it is also $L \sum_{c \epsilon P} c \subseteqq \mathfrak{n}$, $LP \subseteqq \mathfrak{n}$. But $P^2 \subseteqq LP \subseteqq \mathfrak{n}$ and since $\mathfrak{n}^2 = \mathfrak{n} \subseteqq P^2$, we get $P^2 = \mathfrak{n}$, which completes the proof.

Notation. In what follows we shall call the groups $\mathfrak{g}_\alpha$ simply „group-components of L".

Corollary 6,4. *Let L be a simple left ideal of a semigroup S having a kernel $\mathfrak{n}$. Let L have at least one idempotent $\epsilon L - \mathfrak{n}$. Then the necessary and sufficient condition that $L - \mathfrak{n}$ can be written as a sum of disjoint groups is: $L - \mathfrak{n}$ contains no $\mathfrak{n}$-potent element.*

Proof. a) The condition is necessary. For if $S = \mathfrak{n} + \sum_{\alpha} \mathfrak{g}_\alpha$ ($\mathfrak{g}_\alpha$ groups), then every potency of an element $c \epsilon \mathfrak{g}_\nu$ belongs to $\mathfrak{g}_\nu$ and cannot belong to $\mathfrak{n}$.

b) The condition is sufficient. If namely for every $c \epsilon L - \mathfrak{n}$ the relation c^2 non $\epsilon\ \mathfrak{n}$ holds, we have for every $c \epsilon L - \mathfrak{n}$ $\mathfrak{n} + Lc = L$.[8]) Hence — as in the proof of Theorem 6,3 — every $c \epsilon L - \mathfrak{n}$ belongs to some group. This proves our Corollary.

Theorem 6,5. *The group — components of L in Theorem 6,4 are all isomorphic together.*

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[7]) The existence of at least one idempotent e_α is essentially.

[8]) For $\mathfrak{n} + Lc = \mathfrak{n}$ would imply $c^2 \epsilon\ \mathfrak{n}$, contrary to the supposition.

Proof. Let e_α, e_β be two idempotents $\epsilon\, L - P$. Let X be the totality of all elements $\epsilon\, L - P$ such that $e_\alpha X$ equals a fixed element $x \epsilon\, \mathfrak{g}_\alpha$.[9]) (The set is non-vacuous, since $e_\alpha \xi = x$ has a solution in $\mathfrak{g}_\alpha \subset L$.) Then $e_\beta X = (e_\beta e_\alpha) X = e_\beta(e_\alpha X) = e_\beta x$ is again a single element $e_\beta x$.[10]) This element belongs clearly to $e_\beta L$. More precisely, it belongs to $\mathfrak{g}_\beta$. For $e_\beta x \epsilon\, \mathfrak{n}$ would imply $e_\alpha e_\beta x \epsilon\, \mathfrak{n}$, i. e. $e_\alpha x \epsilon\, \mathfrak{n}$ and since it is $x \epsilon\, \mathfrak{g}_\alpha$, $e_\alpha x = x$ we would have $x \epsilon\, \mathfrak{n}$, contrary to the supposition $x \epsilon\, \mathfrak{g}_\alpha$. Conversely, let X' be the totality of all elements $\epsilon\, L - P$ for which $e_\beta X' = e_\beta x$ holds. Clearly $X' \supseteqq X$. But $e_\alpha X' = (e_\alpha e_\beta) X' = e_\alpha(e_\beta X') = e_\alpha e_\beta x = e_\alpha x = x$, therefore $X' \subseteqq X$, whence $X' = X$.

As a consequence of the state just proved we can decompose $L - P$ in a sum of disjoint sets $X, Y, Z, \ldots$, which, multiplied on the left by any idempotent $\epsilon\, L - P$, give always a single element $\epsilon\, L - P$ (to different idempotents correspond naturally different elements $\epsilon\, L - P$).

Choose from each of the sets $X, Y, Z, \ldots$ a single element $\xi, \eta, \zeta, \ldots$ (representants of the sets $X, Y, Z, \ldots$). Then clearly

$$\mathfrak{g}_\alpha = \{e_\alpha \xi, e_\alpha \eta, e_\alpha \zeta, \ldots\},$$
$$\mathfrak{g}_\beta = \{e_\beta \xi, e_\beta \eta, e_\beta \zeta, \ldots\},$$
$$\mathfrak{g}_\gamma = \{e_\gamma \xi, e_\gamma \eta, e_\gamma \zeta, \ldots\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

We now prove that $e_\alpha \xi \to e_\beta \xi$ is an isomorphic mapping of the group $\mathfrak{g}_\alpha$ to the group $\mathfrak{g}_\beta$.

a) To the product $e_\alpha \xi \,.\, e_\alpha \eta = e_\alpha\, \xi\eta$ correspond in this mapping the element $e_\beta \xi \eta$. But this element is just the product $e_\beta \xi \,.\, e_\beta \eta$. (For — with respect to Theorem 6,1a — there holds $e_\beta(\xi e_\beta)\eta = e_\beta \xi \eta$.)

b) To two different elements $e_\alpha \xi \neq e_\alpha \eta$ correspond two different elements $e_\beta \xi \neq e_\beta \eta$. For $e_\beta \xi = e_\beta \eta$ would imply $e_\alpha e_\beta \xi = e_\alpha e_\beta \eta$, i. e. $e_\alpha \xi = e_\alpha \eta$, contrary to the supposition. (We use again Theorem 6,1a.) This proves our Theorem.

Theorem 6,6. *Let S be a semigroup having a kernel $\mathfrak{n}$. Let L_i and L_j be any two simple left ideals of S, each having at least one idempotent non $\epsilon\, \mathfrak{n}$ and satisfying the relation $L_j L_i \neq \mathfrak{n}$. Then the group-components of L_i and L_j are isomorphic to one another.*

Proof. Let e_i be an idempotent $\epsilon\, L_i$ and $\mathfrak{g}_i = e_i L_i - \mathfrak{n}$ the group-component of L_i corresponding to this idempotent. With respect to Theorem 6,5 it is clearly sufficient to show that we can find in L_j a group-component $\mathfrak{g}_j$ (with identity element e_j) such that $\mathfrak{g}_j$ is isomorphic with $\mathfrak{g}_i$. Since $L_j L_i = L_i$, there exist two elements $a \epsilon\, L_j - \mathfrak{n}$, $b \epsilon\, L_i - \mathfrak{n}$ such that $ab = e_i$. Consider the element $ba \epsilon\, bL_j \subseteqq L_j$. Since $(ba)^2 = b(ab)a =$

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[9]) For every $c \epsilon P$ it is $e_\alpha\, c \epsilon\, e_\alpha P \subseteqq LP \subseteqq \mathfrak{n}$, hence $e_\alpha c$ cannot belong to the groups $\sum_\alpha \mathfrak{g}_\alpha$.

[10]) We use here (and in the following) several times Theorem 6,1a.

$= (be_i)a = ba$, the element ba is an idempotent $\in L_j$. It is not contained in $\mathfrak{n}$, since $ba \in \mathfrak{n}$ implies $bab = be_i = b \in \mathfrak{n}$, contrary to the assumption.

Denote $ba = e_j$, $\mathfrak{g}_j = e_j L_j - \mathfrak{n}$. We prove that the correspondence

$$x_i \rightarrow bx_i a \qquad (*)$$

is an isomorphic mapping of the group $\mathfrak{g}_i$ to the group $\mathfrak{g}_j$. The element $x_j = bx_i a$ is an element of $\mathfrak{g}_j$. For it is $e_j b = bab = be_i = b$, hence $x_j = bx_i a \subseteqq e_j b x_i L_j \subseteqq e_j L_j$ and $bx_i a \in \mathfrak{n}$ would imply $abx_i ab = e_i x_i e_i = x_i \in \mathfrak{n}$, contrary to the hypothesis. The correspondence (*) is a homomorphism since if $x_i \rightarrow bx_i a$, $x_i' \rightarrow bx_i' a$, we have $x_i x_i' \rightarrow b(x_i x_i')a = (bx_i a)(bx_i' a)$. This homomorphism is even an isomorphism, for $x_j = bx_i a$ implies $ax_j b = abx_i ab = x_i$, i. e. $x_j \rightarrow ax_j b$ is the inverse mapping to (*), so that the mapping (*) is a one-to-one correspondence. This completes the proof of Theorem 6,6.

7. Simple semigroups satisfying Condition A and having an idempotent.

This section is — roughly to say — devoted to the proof that the existence of a simple right ideal in Theorem 5,5 was needed only to insure the existence of idempotent elements and only this really enables us to prove Theorem 5,5.

Theorem 7,1. *Let S be a semigroup*[11]) *with kernel, which is a class sum of its simple left ideals. If $S - \mathfrak{n}$ has idempotents, then every idempotent $\in S - \mathfrak{n}$ is primitive.*

Proof. The proof is similar to that of Theorem 5,5. Let $e \in S - \mathfrak{n}$ be an idempotent. We have to show that the only idempotent $x \in S - \mathfrak{n}$ satisfying $ex = xe = x$ is $x = e$.

Since S is a sum of simple left ideals we can find a simple left ideal L such that $x \in L$ holds. Then — as in Theorem 5,5 — $L = \mathfrak{n} + Sx$.

a) We use first the relation $xe = x$. With respect to Theorem 2,3 it implies the existence of an $\bar{x}$ with $e = \bar{x}x$. Hence $e \in Sx$, $e \in L$.

b) We use the relation $ex = x$. The elements x and e belong to L and satisfy $ex = x^2$. According to Theorem 6,1b it is possible to cancel with x and we get $e = x$. This proves our theorem.

Theorem 7,2. *Let S be a simple semigroup satisfying Condition A. Let S have an idempotent $e \in S - \mathfrak{n}$. Then S is completely simple.*

First proof of Theorem 7,2. The theorem will be proved if we show that S has at least one non-$\mathfrak{n}$-potent simple right ideal. For the suppositions of Theorem 5,5 are then satisfied.

We show that $\mathfrak{n} + eS$ is a simple right ideal of S. Suppose that there

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[11]) Not necessarily simple.

exists a right ideal R of S with $\mathfrak{n} \subset R \subset \mathfrak{n} + eS$. Let a be any element of $R - \mathfrak{n}$. Then

$$\mathfrak{n} \subset \mathfrak{n} + (a, aS) \subseteq R \subset \mathfrak{n} + eS. \tag{17}$$

We prove that this is impossible.

Since S is simple and a does not belong to the annihilator $\mathfrak{n}_0 = \mathfrak{n}$, it is $SaS = S$. Hence there exist elements $x, y \in S - \mathfrak{n}$ with $xay = e$. Put $x^* = exe$, $y^* = ye$, $f = ay^*x^*$.

Since $x^*ay^* = exeaye = e(xay)e = e^3 = e$ the elements x^*, y^* do not belong to $\mathfrak{n}$. It is

$ef = f$ (since $ea = a$ holds for every $a \in eS$),

$fe = ay^*x^*e = ay^*x^* = f$ (since $x^*e = x^*$),

$f^2 = ay^*(x^*ay^*)x^* = ay^*ex^* = f$ (since as we just proved $x^*ay^* = e$).

The idempotent f satisfying $ef = fe = f$ does not belong to $\mathfrak{n}$. For $x^* = ex^* = x^*(ay^*x^*) = x^*f$ and $f \in \mathfrak{n}$ would imply $x^* \in \mathfrak{n}$, which is not true.

According to Theorem 7,1 e is primitive, hence it is $f = e$. Therefore $e = ay^*x^*$, $e \in aS$, $eS \subseteq aS$.

$$\mathfrak{n} + eS \subseteq \mathfrak{n} + (a, aS). \tag{18}$$

The relations (17) and (18) give together $\mathfrak{n} + (a, aS) = R = \mathfrak{n} + eS$, contrary to the supposition. This proves our theorem.

Second proof of Theorem 7,2. We give another proof of our theorem by means of Theorem 6,4, without using Theorem 7,1. As above, it is sufficient to prove that $\mathfrak{n} + eS$ is a simple right ideal of S. Suppose again, contrary to this statement, that there exists an right ideal R of S with $\mathfrak{n} \subset R \subset \mathfrak{n} + eS$. Let be $a \in R - \mathfrak{n}$. Then $\mathfrak{n} \subset \mathfrak{n} + (a, aS) \subseteq R \subset \mathfrak{n} + eS$. It is again $SaS = S$. Hence there exist two elements x, y non $\in \mathfrak{n}$ such that $xay = e$ (*).

Since S is a sum of simple left ideals, there exist a simple left ideal L with $e \in L - \mathfrak{n}$, whence $L = \mathfrak{n} + Se$. With respect to Theorem 4,5 the relation (*) implies the existence of an $\bar{x}$ with $ay = \bar{x}e$. Since $\bar{x}e$ belongs to L, so does ay. Moreover, the element ay does not belong to $\mathfrak{n}$, since this would be contrary to (*). Since further $ay \in aS \subseteq R$, it is clearly $ay \in L \cap R$.

a) The relation $ay \in eS$ implies $eay = ay$.

b) The element ay belongs to some group-component $\mathfrak{g}_\alpha$ of L. If e_α is the unity element of this group there holds $e_\alpha ay = ay$. The elements e, e_α, ay belong to L. It is $eay = e_\alpha ay$. Now it is possible to cancel on the right with ay, so that we get $e = e_\alpha$. Find now to ay an element $\bar{a}$ with

$ay \cdot \bar{a} = e$. Then it is $eS = ay\bar{a}S \subseteqq aS \subseteqq R$. Therefore $\mathfrak{n} + eS \subseteqq R$. This contradicts our assumption.

Using the proof of Theorem 7,2 and Theorem 5,3 we get the following

Corollary 7,2. *Let S be a simple semigroup satisfying Condition A. If one of the simple left ideals of S has at least one idempotent $\epsilon S - \mathfrak{n}$, then all simple left ideals of S have idempotents $\epsilon S - \mathfrak{n}$. Moreover, there exist in S simple right ideals each of which has idempotent elements. Further, every simple left and simple right ideal of S is generated by idempotents (i. e. is of the form mentioned in Theorem 5,3).*

Now we use our results of section 6 and 7 to get a clear insight in the structure of simple semigroups having a kernel.

Theorem 7,3. *Let S be a simple semigroup satisfying Condition A. Let S have at least one idempotent $\epsilon S - \mathfrak{n}$. Then S can be written as a sum of two disjoint sets $S = \mathfrak{G} + \mathfrak{P}$. The set $\mathfrak{G}$ is a sum of disjoint isomorphic groups. The set $\mathfrak{P}$ is a sum of $\mathfrak{n}$-potent semigroups with the index of $\mathfrak{n}$-potency equal to 2. The intersection of any two of these semigroups is $\mathfrak{n}$.*

Proof. It follows from our previous results (see Theorem 4,3) that under our assumptions S is a sum of simple left ideals. Decompose every simple left ideal in accordance with Theorem 6,4. Theorem 7,3 follows then with the aid of Theorems 6,5 and 6,6.

Remark. It is easily seen that $\mathfrak{G}$ and $\mathfrak{P}$ are uniquely determined. $\mathfrak{P}$ is the totality of all $\mathfrak{n}$-potent elements of S, $\mathfrak{G}$ the totality of all non-$\mathfrak{n}$-potent elements of S. The sets $\mathfrak{G}$ and $\mathfrak{P}$ are not in general semigroups. It can be shown on simple examples that the decomposition of $\mathfrak{P}$ into $\mathfrak{n}$-potent semigroups is not uniquely determined. On the other hand we prove at once that the decomposition of $\mathfrak{G}$ into disjoint groups is unique. Let us suppose (in the way of an indirect proof) that there exist an element a belonging (in two different decompositions of $\mathfrak{G}$) to two different groups $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$ (with unity elements e_α, e_β). Then it is

$$a = e_\alpha a = e_\beta a. \tag{19}$$

The element a belongs to some simple left ideal L of S. With a all elements Sa belong to L, especially also all elements of the group $\mathfrak{g}_\alpha$ and similarly all elements of the group $\mathfrak{g}_\beta$. Hence a, e_α, e_β are elements of L. Since it is $e_\alpha a$ non $\epsilon\, \mathfrak{n}$, it is possible in (19) to cancel with a on the right. We get so $e_\alpha = e_\beta$, contrary to the supposition.

Theorem 7,3 and Corollary 6,4 give:

Corollary 7,3. *Let S be a simple semigroup satisfying Condition A. The necessary and sufficient condition that $S - \mathfrak{n}$ should be a sum of disjoint isomorphic groups is: $S - \mathfrak{n}$ contains at least one idempotent and does not contain any $\mathfrak{n}$-potent element.*

8. Simple two-sided ideals of a semigroup having a kernel.

Theorem 8,1. *Let S be a semigroup*[12]) *with kernel $\mathfrak{n}$. Let M be any two-sided ideal*[12]) *of S. Then M has a kernel and this equals exactly $\mathfrak{n}$.*

Proof. Since $M\mathfrak{n} = \mathfrak{n}$ and $\mathfrak{n}M = \mathfrak{n}$, the set $\mathfrak{n}$ is clearly a two-sided ideal of M. We show that $\mathfrak{n}$ is even the minimal two-sided ideal of M. This follows indirectly. Let $\mathfrak{n}' \subseteqq \mathfrak{n}$ be a two-sided ideal of M. Then there must hold $M\mathfrak{n}' \subseteqq \mathfrak{n}'$, $\mathfrak{n}'M \subseteqq \mathfrak{n}'$, i. e. $M\mathfrak{n}'M \subseteqq \mathfrak{n}'$. But $M\mathfrak{n}'M$ is a two-sided ideal of S. Hence it is certainly $M\mathfrak{n}'M \supseteqq \mathfrak{n}$. The relations $\mathfrak{n} \subseteqq M\mathfrak{n}'M \subseteqq \mathfrak{n}'$ and $\mathfrak{n}' \subseteqq \mathfrak{n}$ imply $\mathfrak{n}' = \mathfrak{n}$. This proves our theorem.

Remark 1. If S has a kernel, a subsemigroup $S' \subset S$ (which is not a two-sided ideal of S) needs not have a kernel. The simplest example: $S = \{0, 1, 2, \ldots\}$ with the usual multiplication of numbers has the kernel $\mathfrak{n} = \{0\}$. But the subsemigroup $S' = \{1, 2, 3, \ldots\}$ has no kernel.

Remark 2. Theorem 8,1 is of greatest importance for our investigations. For, according to our definition, a simple (left, right, two-sided) ideal of S contains the whole set $\mathfrak{n}$. Let M be a twosided ideal of S. Then — as we just proved — M contains $\mathfrak{n}$ as its kernel. Hence every simple (left, right, two-sided) ideal of M contains again all elements of $\mathfrak{n}$.

Theorem 8,2. *Let S have the kernel $\mathfrak{n}$. Let M be a simple two-sided ideal of S satisfying $M^2 \neq \mathfrak{n}$. Then M is a simple semigroup (with kernel $\mathfrak{n}$).*

Before proving this theorem we make some remarks.

Remark 1. In the case $M^2 = \mathfrak{n}$ the theorem needs not be valid. For every subset of such an M containing $\mathfrak{n}$ is clearly a two-sided ideal of M. But a simple $\mathfrak{n}$-potent semigroup is the sum of $\mathfrak{n}$ and a single ($\mathfrak{n}$-potent) element. (See the remark made before Theorem 4,1.) Therefore Theorem 8,2 cannot hold, if $M^2 = \mathfrak{n}$ and $M^2 - \mathfrak{n}$ contains more than one element.

Remark 2. Let us show on an example that a semigroup can contain at the same time $\mathfrak{n}$-potent and non-$\mathfrak{n}$-potent simple two-sided ideals. The commutative semigroup $S = \{0, a_1, a_2, a_3\}$ with the kernel $\mathfrak{n} = \{0\}$ and the Cayley-table

	0	a_1	a_2	a_3
0	0	0	0	0
a_1	0	0	0	0
a_2	0	0	a_2	a_2
a_3	0	0	a_2	a_2

has two simple ideals $M_2 = \{0, a_2\}$, $M_1 = \{0, a_1\}$. It is $M_2^2 = M_2$, but $M_1^2 = \mathfrak{n}$.

Remark 3. Note that under the assumptions of our Theorem 8,2 M cannot contain a left ideal L of S not entirely in $\mathfrak{n}$ with $L^2 \subseteqq \mathfrak{n}$. For the

[12]) Not necessarily simple.

existence of such a left ideal would imply (analogously as in Theorem 4,1 and 4,2) the existence of a two-sided ideal $\bar{M} = (L, LS) \supset \mathfrak{n}$ of S, for which $\bar{M}^2 = \mathfrak{n}$. But it holds $\mathfrak{n} \subset \bar{M} = (L, LS) \subseteqq (L, MS) \subseteqq M$. Hence $\bar{M} = M$, $M^2 = \mathfrak{n}$, contrary to the assumptions. We shall use this argument in the following several times.

Proof of Theorem 8,2. We prove it indirectly. Suppose that M contains a two-sided ideal B of M with $\mathfrak{n} \subset B \subset M$. The set MBM satisfies $MBM \subseteqq B \subset M$. But MBM is also a two-sided ideal of S, hence $MBM = \mathfrak{n}$.

We show first that it holds: $MB \subseteqq \mathfrak{n}$, $BM \subseteqq \mathfrak{n}$. The set MB is a left ideal of S contained in M. It is $(MB)^2 = MBMB = (MBM)B = \mathfrak{n}B \subseteqq \mathfrak{n}$; hence — with respect to the Remark 3 above — $MB \subseteqq \mathfrak{n}$. Analogously $BM \subseteqq \mathfrak{n}$.

We show secondly that it is even $SB \subseteqq \mathfrak{n} \subseteqq B$, $BS \subseteqq \mathfrak{n} \subseteqq B$. This will prove our theorem, for these relations are contrary to the assumption that M is a simple ideal of S. Since $SBS \subseteqq SMS \subseteqq M$ is a two-sided ideal of S, it is a) either $SBS = \mathfrak{n}$, b) or $SBS = M$. a) $SBS = \mathfrak{n}$ implies $(SB)^2 = SBSB = \mathfrak{n}B \subseteqq \mathfrak{n}$, $(BS)^2 = BSBS = B\mathfrak{n} \subseteqq \mathfrak{n}$. With respect to the Remark 3 above we get $SB \subseteqq \mathfrak{n}$, $BS \subseteqq \mathfrak{n}$. b) The second alternative $SBS = M$ implies $(SB)^2 = SBSB = MB \subseteqq \mathfrak{n}$, $(BS)^2 = BSBS = BM \subseteqq \mathfrak{n}$ and hence (again with respect to the Remark 3) $SB \subseteqq \mathfrak{n}$, $BS \subseteqq \mathfrak{n}$. This proves our theorem.

Theorem 8,3. *Let S be a semigroup with the kernel $\mathfrak{n}$. Let M be a two-sided ideal*[13]) *of S satisfying $M^2 \neq \mathfrak{n}$. Then*

a) Every simple left ideal $L^{(M)}$ of M is a (simple) left ideal of S.

b) Conversely: Every simple left ideal $L^{(S)}$ of S satisfying $L^{(S)} \cap M \supset \mathfrak{n}$ is a simple left ideal of M. (Hence $L^{(S)} \subseteqq M$.)

Proof. a) It is sufficient to show that $L^{(M)}$ is a left ideal of S. For $L^{(M)}$ — being a simple left ideal of M — is the more simple in S.

The set $ML^{(M)}$ is a left ideal of M contained in $L^{(M)}$. Hence either $ML^{(M)} = \mathfrak{n}$ or $ML^{(M)} = L^{(M)}$. The first alternative would imply $L^{(M)2} = L^{(M)} . L^{(M)} \subseteqq ML^{(M)} \subseteqq \mathfrak{n}$, which is not possible according to the Remark 3 above. Hence $ML^{(M)} = L^{(M)}$. But then $SL^{(M)} = SML^{(M)} \subseteqq ML^{(M)} = L^{(M)}$, i. e. $SL^{(M)} \subseteqq L^{(M)}$. This says: $L^{(M)}$ is a left ideal of S.

b) Let $L^{(S)}$ be a left simple ideal of S. The set $L^{(S)} \cap M$ is a left ideal of S and $\supset \mathfrak{n}$. It is contained in $L^{(S)}$, hence $L^{(S)} \cap M = L^{(S)}$. The set $L^{(S)}$, being a left ideal of S, is clearly the more a left ideal of M. It is therefore sufficient to show that $L^{(S)}$ is a simple left ideal of M. We prove it indirectly (by an analogous argument as in Theorem 8,2).

[13]) Not necessarily simple.

Suppose there exists a left ideal $B^{(M)}$ of M with $\mathfrak{n} \subset B^{(M)} \subset L^{(S)}$. The set $MB^{(M)}$ is a left ideal of S satisfying $MB^{(M)} \subseteq B^{(M)} \subset L^{(S)}$. Hence $MB^{(M)} = \mathfrak{n}$. The set $SB^{(M)}$ is a left ideal of S satisfying $SB^{(M)} \subseteq SL^{(S)} \subseteq L^{(S)}$. With respect to the simplicity of $L^{(S)}$ it is either $SB^{(M)} = \mathfrak{n}$ or $SB^{(M)} = L^{(S)}$. α) if $SB^{(M)} = \mathfrak{n}$, we have $\mathfrak{n} = SB^{(M)} \subset B^{(M)}$, i. e. $B^{(M)}$ is a left ideal of S, which contradicts the assumption of the simplicity of $L^{(S)}$. β) If $SB^{(M)} = L^{(S)}$, it is $L^{(S)2} = SB^{(M)} . SB^{(M)} \subseteq SMSB^{(M)} \subseteq MB^{(M)} = \mathfrak{n}$. Hence $L^{(S)2} \subseteq \mathfrak{n}$. But this is again impossible according to the Remark 3 above. This completes the proof of Theorem 8,3.

Corollary 8,3. *Let S be a semigroup with the kernel $\mathfrak{n}$. Let M be a simple two-sided ideal of S containing at least one non-$\mathfrak{n}$-potent simple left ideal of M (or — what is the same — of S). Then M is the class sum all simple left ideals of M (or — what is the same — of all simple left ideals of S contained in M).*

The proof is an immediate consequence of Theorems 8,2, 8,3 and 4,3.

9. Semigroups without a proper radical.

In this section we shall study the properties of semigroups without a proper radical (see Definition 3,2). Under this condition every left ideal $L \supset \mathfrak{n}$ of S is non-$\mathfrak{n}$-potent. (See Theorem 3,2.)

Theorem 9,1. *Let S be a semigroup with kernel $\mathfrak{n}$ and without a proper radical. Then every simple left ideal is contained in some simple two-sided ideal.*

Proof. Let L_α be a simple left ideal of S. Then $M = (L_\alpha, L_\alpha S)$ is a two-sided-ideal of S. We prove that M is simple. Suppose — in the way of contradiction — that there exist a two-sided ideal M' of S with $\mathfrak{n} \subset M' \subset M$. The set $M'L_\alpha$ is a left ideal of S contained in L_α, hence either $M'L_\alpha = L_\alpha$ or $M'L_\alpha \subseteq \mathfrak{n}$.

The second alternative would imply

$$M'^2 \subseteq M'M = M'(L_\alpha, L_\alpha S) = (M'L_\alpha, M'L_\alpha S) \subseteq \mathfrak{n},$$

contrary to the hypothesis. Therefore $M'L_\alpha = L_\alpha$. But then it is $M = (L_\alpha, L_\alpha S) = (M'L_\alpha, M'L_\alpha S)$ and (since M' is a two-sided ideal of S) $M \subseteq (M', M') = M'$. The relation $M \subseteq M'$ is a contradiction with $M' \subset M$. This proves our theorem.

Remark 1. Let us remark that a simple left ideal needs not belong necessarily to a two-sided simple ideal (if the conditions of Theorem 9,1 are not satisfied). The semigroup $S = \{0, a_1, a_2, a_3\}$ with the kernel $\mathfrak{n} = \{0\}$ and the table

	0	a_1	a_2	a_3
0	0	0	0	0
a_1	0	0	a_3	0
a_2	0	0	0	0
a_3	0	0	0	0

has a simple left ideal $L = \{0, a_1\}$. This semigroup has only one simple two-sided ideal $M = \{0, a_3\}$. The left ideal $\{0, a_1\}$ is contained in the two-sided ideal $(L, LS) = \{0, a_1, a_3\}$. But this ideal is not simple. Of course here the conditions of Theorem 9,1 are not satisfied. The radical of our semigroup S equals directly S, since it is evidently $S^3 = \{0\}$.

Remark 2. Note that we did not succeed to prove that the existence of a non-$\mathfrak{n}$-potent simple left ideal implies the existence of at least one simple non-$\mathfrak{n}$-potent two-sided ideal of S. To get a result, we had to suppose that none of the two-sided ideals of S is $\mathfrak{n}$-potent.

Corollary 9,1. *Let S be a semigroup without a proper radical satisfying Condition A. Then the class sum $\mathfrak{L}$ of all simple left ideals of S is a semigroup, which is a direct sum of two-sided ideals.*

Proof. Let be $\mathfrak{L} = \sum_{\alpha} L_{\alpha}$, L_{α} running through all simple left ideals of S. We know that $\mathfrak{L}$ is a two-sided ideal of S (see Theorem 1,5). Every L_{α} is contained in some simple two-sided ideal M_{α}. Hence $\mathfrak{L}$ is overlaped by a sum $\sum_{\alpha} M_{\alpha}$ of two-sided ideals of S. The intersection $\mathfrak{L} \cap M_{\alpha}$ is a two-sided ideal of S contained in M_{α} and $\supset \mathfrak{n}$ (containing itself L_{α}). Hence $\mathfrak{L} \cap M_{\alpha} = M_{\alpha}$. Therefore every M_{α} belongs to $\mathfrak{L}$ and $\mathfrak{L}$ is a sum of simple two-sided ideals of S : $\mathfrak{L} = \sum_{\alpha} M_{\alpha}$. Every M_{α} is the more a two-sided ideal of $\mathfrak{L}$ and, being (according to Theorem 8,2) a simple semigroup, it is also a simple two-sided ideal of $\mathfrak{L}$. This proves our corollary.

Remark. The example of Remark 1 above shows that the condition that S has no proper radical is not necessary for the validity of Corollary 9,1. For in this example the only left ideals are $\{0, a_1\}$ and $\{0, a_3\}$. It is $\mathfrak{L} = \{0, a_1, a_3\}$ and this is a direct sum of simple two-sided ideals of $\mathfrak{L}$, namely $\{0, a_1\}$, $\{0, a_3\}$ (which are of course not two-sided in S).

Let now S be a semigroup satisfying Condition B. We can construct analogously the sum of all simple right ideals $\mathfrak{R} = \sum_{\alpha} R_{\alpha}$. Again $\mathfrak{R}$ is a two-sided ideal of S. If S has no proper radical, $\mathfrak{R}$ is a direct sum of two-sided ideals.

The question arises whether and when the sets $\mathfrak{R}$ and $\mathfrak{L}$ are identical.

The answer is given by

Theorem 9,2. *Let S be a semigroup with a kernel and without a proper radical. Let $\mathfrak{L}$ and $\mathfrak{R}$ be the class sum of all simple left and simple right* *ideals respectively. The necessary and sufficient condition that* $\mathfrak{R} = \mathfrak{L}$ *is: every simple two-sided ideal of* S *containing a simple left ideal contains also a simple right ideal and conversely.*

Proof. According to Corollary 9,1 $\mathfrak{L} = \sum_{\alpha} \oplus M'_{\alpha}$, $\mathfrak{R} = \sum_{\alpha} \oplus M''_{\alpha}$, where M'_{α}, M''_{α} are two-sided ideals of S. a) The condition is necessary. For if one M'_{α} does not contain a simple right ideal, the set M'_{α} cannot be a sum of simple right ideals and hence M'_{α} cannot be contained in the set $\mathfrak{R}$. b) The condition is sufficient. For if M'_{α} (which is a sum of simple left ideals) contains a simple right ideal, then M'_{α} is a sum of simple right ideals (Theorem 4,3). Hence M'_{α} is contained in $\mathfrak{R}$.

Remark. A semigroup S with kernel $\mathfrak{n}$ can contain also simple two-sided ideals without simple left and simple right ideals, so that the sum $\mathfrak{M} = \sum_{\alpha} M_{\alpha}$ of all simple two-sided ideals of S needs not be equal to $\mathfrak{L}$ or $\mathfrak{R}$ even in the case when these two sets are equal. Clearly it is always:

$$\mathfrak{n} \subseteqq \mathfrak{L} \cap \mathfrak{R} \subseteqq \begin{matrix}\mathfrak{L}\\ \mathfrak{R}\end{matrix} \subseteqq \mathfrak{L} + \mathfrak{R} \subseteqq \mathfrak{M} \subseteqq S.$$

When do the relations $\mathfrak{L} = \mathfrak{M}$, $\mathfrak{R} = \mathfrak{M}$, $\mathfrak{L} = \mathfrak{R} = \mathfrak{M}$ hold?

Theorem 9,3. *Let* S *be a semigroup having a kernel and without a proper radical. Let the sets* $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{M}$ *have the meaning introduced above. Then*

a) $\mathfrak{L} = \mathfrak{M}$ *holds if and only if every simple two-sided ideal of* S *contains at least one simple left ideal.*

b) $\mathfrak{R} = \mathfrak{M}$ *holds if and only if every simple two-sided ideal of* S *contains at least one simple right ideal.*

c) $\mathfrak{L} = \mathfrak{R} = \mathfrak{M}$ *holds if and only if every simple two-sided ideal of* S *contains at least one simple left and one simple right ideal.*

The proof is analogous to that of Theorem 9,2 and its explicit formulation can be clearly omitted.

Remark. Theorems 9,2 and 9,3 show that especially in every finite semigroup without radical the relations $\mathfrak{L} = \mathfrak{R} = \mathfrak{M}$ hold.

Now let us study briefly the question under what conditions a semigroup without a proper radical is a direct sum of two-sided ideals. With respect to the sharper suppositions about S we shall obtain naturally less general conditions as those given in Theorem 2,4.

The first theorem of this kind follows immediately from Corollary 9,1. There holds:

Theorem 9,4. *Let* S *be a semigroup with kernel and without a proper radical. Let* S *be the class sum of its simple left ideals. Then* S *is a direct sum of two-sided ideals.*

By means of Theorem 9,3 this result can be formulated also in the following manner.

Theorem 9,5. *Let S be a semigroup with kernel and without a proper radical. Let every simple two-sided ideal of S contain at least one simple left ideal. Then S is a direct sum of two-sided ideals if and only if it is the class sum of its simple left ideals.*

Remark. Especially: A finite semigroup S without a radical is a direct sum of two-sided ideals if and only if it is the sum of all simple left ideals of S.

The right dual theorems to Theorems 9,4 and 9,5 are obvious.

Theorem 9,5 and Theorem 2,2 together give the the following interesting

Corollary 9,5. *Let S be a semigroup without a proper radical. Let every simple two-sided ideal of S contain at least one simple left ideal. Then S is a direct sum of two-sided ideals if and only if every relation $a = xb$, a, b non $\in \mathfrak{n}$ implies a relation $\bar{x}a = b$ with $\bar{x}$ non $\in \mathfrak{n}$.*

Theorem 9,4 combined with our previous results gives the further

Theorem 9,6. *Let S be a semigroup without a proper radical. Let S be the class sum of its simple left ideals. Let, moreover, every simple two-sided ideal of S contain at least one simple right ideal of S. Then*

a) S is also the class sum of all simple right ideals of S,

b) S is a direct sum of two-sided ideals each of which is a completely simple semigroup.

Theorem 9,6 with the right dual of Corollary 9,5 has again the following very interesting corollary:

Corollary 9,6. *Let S be a semigroup without a proper radical. Let every simple two-sided ideal of S contain at least one simple left and at least one simple right ideal of S. Then, if for every relation of the form $a = xb$ $(a, b, x \in S - \mathfrak{n})$ the implication*

$$a = xb \Rightarrow \bar{x}a = b \text{ with some } \bar{x} \text{ non} \in \mathfrak{n} \tag{19}$$

holds, so holds also, for every relation of the form $a = by$ $(a, b, y \in S - \mathfrak{n})$, the implication

$$a = by \Rightarrow a\bar{y} = b \text{ with some } \bar{y} \text{ non} \in \mathfrak{n}. \tag{20}$$

Conversely: If (20) holds, so holds also (19).

Proof. If (19) holds, then (with respect to Theorem 2,2) S is the sum of its simple left ideals. Further — according to Theorem 9,4 — S is a direct sum of two-sided ideals. Since now every simple two-sided ideal contains at least one simple right ideal, S is also the sum of its simple right ideals. Hence (with respect to Theorem 2,3) the implication (20) holds.

The converse part follows analogously.

10. Semigroups with a proper radical.

Let S have a kernel and a proper radical $\mathfrak{r}$. Let us put the question: what can be said about the totality of all simple $\mathfrak{n}$-potent and non-$\mathfrak{n}$-potent left or right ideals?

In this section we introduce the following notations:

$\mathfrak{L}$ ($\mathfrak{R}$, $\mathfrak{M}$) will denote the class sum of all simple left (right, two-sided) ideals of S,

$\mathfrak{L}^{(0)}$ ($\mathfrak{R}^{(0)}$, $\mathfrak{M}^{(0)}$) the class sum of all $\mathfrak{n}$-potent simple left (right, two-sided) ideals of S,

$\mathfrak{L}^{(1)}$ ($\mathfrak{R}^{(1)}$, $\mathfrak{M}^{(1)}$) the class sum of all non-$\mathfrak{n}$-potent simple left (right, two-sided) ideals of S.

It is

$$\mathfrak{L}^{(0)} \subseteqq \mathfrak{r},\quad \mathfrak{R}^{(0)} \subseteqq \mathfrak{r},\quad \mathfrak{M}^{(0)} \subseteqq \mathfrak{r}.$$

Further:

$$\mathfrak{L} = \mathfrak{L}^{(0)} + \mathfrak{L}^{(1)},\quad \mathfrak{R} = \mathfrak{R}^{(0)} + \mathfrak{R}^{(1)},\quad \mathfrak{M} = \mathfrak{M}^{(0)} + \mathfrak{M}.^{(1)}$$

We cannot say that

$$\mathfrak{n} \subseteqq \begin{matrix}\mathfrak{L}^{(1)}\\ \mathfrak{R}^{(1)}\end{matrix} \subseteqq \mathfrak{M}^{(1)}$$

holds, since in general we did not prove that every simple left (right) non-$\mathfrak{n}$-potent ideal is contained in a simple two-sided (non-$\mathfrak{n}$-potent) ideal.

According to Theorem 1,5 the sets $\mathfrak{L}$ and $\mathfrak{R}$ are two-sided ideals of S. We show further:

Theorem 10,1. *The sets $\mathfrak{L}^{(0)}$ and $\mathfrak{R}^{(0)}$ are two-sided ideals of S.*

Proof. Let $L^{(0)}$ be any simple left ideal belonging to $\mathfrak{L}^{(0)}$. We know (see Theorem 1,4) that for every $c \in S$ the set $\mathfrak{n} + L^{(0)}c$ is either $\mathfrak{n}$ or a simple left ideal of S. Since $L^{(0)}$ is $\mathfrak{n}$-potent, there exists a $\varrho \geqq 1$ with $(L^{(0)})^\varrho \subseteqq \mathfrak{n}$. Hence

$$(L^{(0)}c)^\varrho = L^{(0)} \,.\, (cL^{(0)})^{\varrho-1}\, c \subseteqq L^{(0)} \,.\, L^{(0)\varrho-1} \,.\, c = L^{(0)\varrho} \,.\, c \subseteqq \mathfrak{n}.$$

Therefore the ideal $\mathfrak{n} + L^{(0)}c$ belongs to $\mathfrak{L}^{(0)}$. Now let be $\mathfrak{L}^{(0)} = \sum_\alpha L_\alpha^{(0)}$. This is evidently a left ideal of S. To prove our theorem it is sufficient to prove that $\mathfrak{L}^{(0)}$ is also a right ideal of S. Let c be any element, $c \in S$. Then $\mathfrak{L}^{(0)}c = \sum_\alpha L_\alpha^{(0)}c \subseteqq \sum_\alpha(\mathfrak{n} + L_\alpha^{(0)}c)$. But, since for every α $\mathfrak{n} + L_\alpha^{(0)}c$ is yet contained in $\mathfrak{L}^{(0)}$, we get $\mathfrak{L}^{(0)}c \subseteqq \mathfrak{L}^{(0)}$. This proves our theorem.

The radical $\mathfrak{r}$ of our semigroup was defined as the sum of all $\mathfrak{n}$-potent two-sided ideals of S. It need not be itself $\mathfrak{n}$-potent. In this section we shall, however, impose the condition that $\mathfrak{r}$ is $\mathfrak{n}$-potent. This is equivalent to requiring that S contains a maximal $\mathfrak{n}$-potent two-sided ideal which is then of course $\mathfrak{r}$. To avoid repetitions I introduce the following

Condition C. *S is a semigroup having a kernel $\mathfrak{n}$ and a maximal $\mathfrak{n}$-potent ideal $\mathfrak{r}$.*

Theorem 10,2. *Let S satisfy Condition C. Then $\mathfrak{L}$ is a semigroup having a kernel and the radical $\mathfrak{L}^{(0)}$.*

Proof. a) That (the two-sided ideal) $\mathfrak{L}$ has the kernel $\mathfrak{n}$ follows immediately from Theorem 8,1.

b) Note first that every simple left ideal $L_\alpha^{(1)}$ of S belonging to $\mathfrak{L}^{(1)}$ (which is naturally supposed to be non-vacuous) is also a simple left ideal of $\mathfrak{L}$. This follows from the second part of Theorem 8,3. The suppositions of Theorem 8,3 are satisfied since $\mathfrak{L}$ is a two-sided ideal and $\mathfrak{L}^2 \neq \mathfrak{n}$. (The relation $\mathfrak{L}^2 = \mathfrak{n}$ would imply $[L_\alpha^{(1)}]^2 \subseteq \mathfrak{n}$, contrary to the fact that $L_\alpha^{(1)}$ is non-$\mathfrak{n}$-potent.)

Now let $\mathfrak{r}^*$ denote the radical of $\mathfrak{L}$. Clearly $\mathfrak{L}^{(0)} \subseteq \mathfrak{r}^* \subseteq \mathfrak{r}$. To show $\mathfrak{L}^{(0)} = \mathfrak{r}^*$ it is sufficient to prove that none of the simple ideals contained in $\mathfrak{L}^{(1)}$ belongs to $\mathfrak{r}^*$. This follows indirectly. If such a simple ideal $L_\alpha^{(1)}$ would satisfy $L_\alpha^{(1)} \subseteq \mathfrak{r}^* \subseteq \mathfrak{r}$, there would be for every $\varrho \geqq 1$ $[L_\alpha^{(1)}]^\varrho \subseteq \mathfrak{r}^{*\varrho} \subseteq \mathfrak{r}^\varrho$. But (with respect to the Condition C) $\mathfrak{r}^\varrho = \mathfrak{n}$ for some $\varrho \geqq 1$. Hence $[L_\alpha^{(1)}]^\varrho \subseteq \mathfrak{n}$. This is contrary to the assumption that $L_\alpha^{(1)}$ is non-$\mathfrak{n}$-potent. Therefore $\mathfrak{r}^*$ is just the class sum of all $\mathfrak{n}$-potent simple left ideals of S (namely $\mathfrak{L}^{(0)}$), which proves Theorem 10,2.

Analogous results hold for $\mathfrak{R}$ and $\mathfrak{M}$.

Theorem 10,3. *Let S satisfy Condition C. Let the set $\mathfrak{L}^{(1)}$ be a two-sided ideal of S. Then the semigroup $\mathfrak{L}$ can be written as the sum of two summands: the radical $\mathfrak{L}^{(0)}$ and the semigroup $\mathfrak{L}^{(1)}$, which is itself a direct sum of simple two-sided ideals of S.*[14]) *In formulae:*

$$\mathfrak{L} = \mathfrak{L}^{(0)} + \mathfrak{L}^{(1)}, \quad \mathfrak{L}^{(0)} \cap \mathfrak{L}^{(1)} = \mathfrak{n},$$

where

$$\mathfrak{L}^{(1)} = \sum_\alpha \oplus M_\alpha^{(1)}.$$

Proof. $\mathfrak{L}^{(1)}$, being a two-sided ideal of S, is itself a semigroup. According to Theorem 8,1 it has the kernel $\mathfrak{n}$. In Theorem 10,2 we proved that every simple left ideal of S belonging to $\mathfrak{L}^{(1)}$ is also a simple left ideal of $\mathfrak{L}$. It follows by the same argument that such an ideal is, moreover, a simple left ideal of $\mathfrak{L}^{(1)}$. Hence $\mathfrak{L}^{(1)}$ is the sum of simple left ideals of $\mathfrak{L}^{(1)}$ from which none is $\mathfrak{n}$-potent. Therefore $\mathfrak{L}^{(1)}$ is a semigroup with kernel and without a proper radical. Hence we can apply Corollary 9,1 according to which $\mathfrak{L}^{(1)}$ is a direct sum of two-sided ideals of $\mathfrak{L}^{(1)}$: $\mathfrak{L}^{(1)} = \sum_\alpha \oplus M_\alpha^{(1)}$.

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[14]) Or (what will be shown to be the same) of $\mathfrak{L}$ or $\mathfrak{L}^{(1)}$.

To complete the proof we must show that every $M_\alpha^{(1)}$ is even a (simple) two-sided ideal of S. First, since $M_\alpha^{(1)}$ is a non-$\mathfrak{n}$-potent simple two-sided ideal of $\mathfrak{L}^{(1)}$, it is $\mathfrak{L}^{(1)}M_\alpha^{(1)} = M_\alpha^{(1)}$, $M_\alpha^{(1)}\mathfrak{L}^{(1)} = M_\alpha^{(1)}$. Further it is (according to the supposition) $S\mathfrak{L}^{(1)} \subseteqq \mathfrak{L}^{(1)}$, $\mathfrak{L}^{(1)}S \subseteqq \mathfrak{L}^{(1)}$. Therefore we have

$$SM_\alpha^{(1)} = S\mathfrak{L}^{(1)}M_\alpha^{(1)} \subseteqq \mathfrak{L}^{(1)} . M_\alpha^{(1)} = M_\alpha^{(1)},$$
$$M_\alpha^{(1)}S = M_\alpha^{(1)}\mathfrak{L}^{(1)}S \subseteqq M_\alpha^{(1)}\mathfrak{L}^{(1)} = M_\alpha^{(1)},$$

which completes the proof.

Theorem 10,4. *Let S be a semigroup satisfying Condition C. Then every non-$\mathfrak{n}$-potent simple left ideal of S is contained in a simple two-sided ideal of S if and only if the sum of all simple non-$\mathfrak{n}$-potent left ideals of S is a two-sided ideal of S.*

Proof. a) According to Theorem 10,3 the condition is sufficient.

b) It is also necessary. For let every $L_\alpha^{(1)} \in \mathfrak{L}^{(1)}$ be contained in some simple two-sided ideal $M_\alpha^{(1)}$ of S. The set $\sum_\alpha M_\alpha^{(1)}$ overlaps $\mathfrak{L}^{(1)}$, i. e. $\mathfrak{L}^{(1)} \subseteqq \sum_\alpha M_\alpha^{(1)}$. Every summand in $\sum_\alpha M_\alpha^{(1)}$ is clearly a simple (non-$\mathfrak{n}$-potent) semigroup having at least one simple left ideal. According to Theorem 4,3 every $M_\alpha^{(1)}$ is a sum of simple left non-$\mathfrak{n}$-potent ideals of $M_\alpha^{(1)}$, which are also simple left in S (see Theorem 8,3). Therefore $\sum_\alpha M_\alpha^{(1)} \subseteqq \mathfrak{L}^{(1)}$. So we have at last $\sum_\alpha M_\alpha^{(1)} = \mathfrak{L}^{(1)}$. Now $\mathfrak{L}^{(1)}$, being a sum of two-sided ideals of S, is itself a two-sided ideal of S. This proves our theorem.

An analogous theorem holds for simple right non-$\mathfrak{n}$-potent ideals.

Remark. The condition of Theorem 10,4 is especially satisfied if S has no proper radical. For then the set $\mathfrak{L}$ of all simple left ideals concides with the set $\mathfrak{L}^{(1)}$ of all non-$\mathfrak{n}$-potent simple left ideals. But, according to Theorem 1,5, $\mathfrak{L}$ is a two-sided ideal of S. We get so again Theorem 9,1.

A special case in which Theorem 10,3 holds is the case $\mathfrak{R} = \mathfrak{L}$. Then there holds also $\mathfrak{R}^{(0)} = \mathfrak{L}^{(0)}$. For (according to Theorem 10,2) the radical of $\mathfrak{L}$ is $\mathfrak{L}^{(0)}$, the radical of $\mathfrak{R}$ is $\mathfrak{R}^{(0)}$. But since $\mathfrak{R} = \mathfrak{L}$, it is $\mathfrak{R}^{(0)} = \mathfrak{L}^{(0)}$. Since $\mathfrak{L}^{(0)} \cap \mathfrak{L}^{(1)} = \mathfrak{n}$, $\mathfrak{R}^{(0)} \cap \mathfrak{R}^{(1)} = \mathfrak{n}$, the last equalities imply $\mathfrak{R}^{(1)} = \mathfrak{L}^{(1)}$, which says that both $\mathfrak{R}^{(1)}$ and $\mathfrak{L}^{(1)}$ are two-sided ideals of S.

If moreover $\mathfrak{R} = \mathfrak{L} = S$ holds, we get the following

Corollary 10,3. *Let S satisfy Condition C. Let S be the sum of its simple left ideals and at the same time of its simple right ideals. Then S can be written as the sum of the radical and of a direct sum of two-sided ideals.*

Remark. The weaker assumption that S can be written as the sum of its simple left ideals, i. e. $S = \mathfrak{L}$, is not sufficient for the validitity of this corollary. For the semigroup $\mathfrak{L}^{(1)}$ will be a two-sided ideal of S if and only if $\mathfrak{L}^{(1)}S = \mathfrak{L}^{(1)}(\mathfrak{L}^{(0)} + \mathfrak{L}^{(1)}) \subseteqq \mathfrak{L}^{(1)}$ holds. Since $\mathfrak{L}^{(1)}\mathfrak{L}^{(0)} + \mathfrak{L}^{(1)}\mathfrak{L}^{(1)} \subseteqq \subseteqq \mathfrak{L}^{(1)}\mathfrak{L}^{(0)} + \mathfrak{L}^{(1)}$ and $\mathfrak{L}^{(1)}\mathfrak{L}^{(0)} \subseteqq \mathfrak{L}^{(0)}$, this is clearly equivalent to the requirement $\mathfrak{L}^{(1)}\mathfrak{L}^{(0)} = \mathfrak{n}$. Thus the validity of the relation $\mathfrak{L}^{(1)}\mathfrak{L}^{(0)} = \mathfrak{n}$ is the necessary and sufficient condition that under the assumption $\mathfrak{L} = S$ the semigroup should be the sum of the radical and of a direct sum of two-sided ideals.

This relation does not hold in general if only $\mathfrak{L} = S$ is satisfied. But if, however, $\mathfrak{R} = \mathfrak{L} = S$ (i. e. $\mathfrak{L}^{(0)} = \mathfrak{R}^{(0)}$, $\mathfrak{L}^{(1)} = \mathfrak{R}^{(1)}$) holds, we get: $\mathfrak{L}^{(1)}\mathfrak{L}^{(0)} = \mathfrak{R}^{(1)}\mathfrak{L}^{(0)} \subseteqq \mathfrak{R}^{(1)} \cap \mathfrak{L}^{(0)} = \mathfrak{L}^{(1)} \cap \mathfrak{L}^{(0)} = \mathfrak{n}$, which gives another proof of Corollary 10,3.

The question arises: when does the relation $\mathfrak{L}^{(1)} = \mathfrak{R}^{(1)}$ hold? The answer is given by

Theorem 10,5. *Let S be a semigroup satisfying Condition C. Then $\mathfrak{R}^{(1)} = \mathfrak{L}^{(1)}$ holds if and only if:*

a) $\mathfrak{R}^{(1)}$ and $\mathfrak{L}^{(1)}$ are two-sided ideals,

b) every simple non-$\mathfrak{n}$-potent two-sided ideal containing a simple left ideal contains also a simple right ideal and conversely.

Proof. It is analogous to that of Theorem 9,2.

a) The condition is necessary. For if $\mathfrak{R}^{(1)} = \mathfrak{L}^{(1)}$, then $\mathfrak{R}^{(1)}$ and $\mathfrak{L}^{(1)}$ are two-sided ideals of S. Hence, according to Theorem 10,4, every non-$\mathfrak{n}$-potent simple left (right) ideal is contained in a simple two-sided ideal of S. Moreover, every simple two-sided ideal containing a simple non-$\mathfrak{n}$-potent left ideal is a sum of simple left ideals and belongs therefore to $\mathfrak{L}^{(1)}$. But since $\mathfrak{L}^{(1)} = \mathfrak{R}^{(1)}$, it belongs also to $\mathfrak{R}^{(1)}$, hence it contains at least one simple non-$\mathfrak{n}$-potent right ideal. The converse part follows analogously.

b) The condition is sufficient. We have to show that if the conditions a) and b) are satisfied, every non-$\mathfrak{n}$-potent left ideal is contained in $\mathfrak{R}^{(1)}$ and every non-$\mathfrak{n}$-potent right ideal of S in $\mathfrak{L}^{(1)}$. Let $L_\alpha^{(1)}$ be a simple non-$\mathfrak{n}$-potent left ideal of S. Since, according to a), $\mathfrak{L}^{(1)}$ is a two-sided ideal of S, $L_\alpha^{(1)}$ is contained in some simple non-$\mathfrak{n}$-potent two-sided ideal $M_\alpha^{(1)}$ of S. According to b) this ideal contains at least one simple non-$\mathfrak{n}$-potent right ideal. Hence $M_\alpha^{(1)}$ is also a sum of simple right non-$\mathfrak{n}$-potent ideals of S. Therefore $L_\alpha^{(1)}$ belongs to $\mathfrak{R}^{(1)}$. The dual part that every simple non-$\mathfrak{n}$-potent right ideal of S belongs to $\mathfrak{L}^{(1)}$ follows analogously.

Now we introduce the sets $\mathfrak{M}$, $\mathfrak{M}^{(0)}$, $\mathfrak{M}^{(1)}$. In general none of the relations

$$\mathfrak{n} \subseteqq \begin{matrix}\mathfrak{L}^{(0)}\\ \mathfrak{R}^{(0)}\end{matrix} \subseteqq \mathfrak{M}^{(0)}, \qquad (*)$$

$$\mathfrak{n} \subseteqq \begin{matrix}\mathfrak{L}^{(1)}\\ \mathfrak{R}^{(1)}\end{matrix} \subseteqq \mathfrak{M}^{(1)} \qquad (**)$$

is true.

According to Theorem 10,4, (**) holds if and only if $\mathfrak{L}^{(1)}$ and $\mathfrak{R}^{(1)}$ are two-sided ideals of S. It is easy to find necessary and sufficient conditions for the equalities $\mathfrak{L}^{(1)} = \mathfrak{M}^{(1)}$, $\mathfrak{R}^{(1)} = \mathfrak{M}^{(1)}$, $\mathfrak{L}^{(1)} = \mathfrak{R}^{(1)} = \mathfrak{M}^{(1)}$.

There holds:

Theorem 10,6. *Let S be a semigroup satisfying Condition C. Let $\mathfrak{L}^{(1)}$, $\mathfrak{R}^{(1)}$, $\mathfrak{M}^{(1)}$ have the meaning introduced above.*

1. $\mathfrak{L}^{(1)} = \mathfrak{M}^{(1)}$ *holds if and only if*

a) $\mathfrak{L}^{(1)}$ *is a two-sided ideal of* S,

b) every simple non-n-potent two-sided ideal of S contains at least one simple left ideal.

2. $\mathfrak{R}^{(1)} = \mathfrak{M}^{(1)}$ *holds if and only if*

a) $\mathfrak{R}^{(1)}$ *is a two-sided ideal of* S,

b) every simple non-n-potent two-sided ideal of S contains at least one simple right ideal.

3. $\mathfrak{L}^{(1)} = \mathfrak{R}^{(1)} = \mathfrak{M}^{(1)}$ *holds if and only if*

a) $\mathfrak{L}^{(1)}$ *and* $\mathfrak{R}^{(1)}$ *are two-sided ideals of* S,

b) every simple non-n-potent two-sided ideal of S contains at least one simple left and at least one simple right ideal of S.

We omit the explicit proof of this theorem, since it is analogous to that of Theorem 10,5.

From these theorems we can obtain again a number of others. We note the following one, the proof of which follows from Theorem 10,6 and Corollary 10,3.

Corollary 10,6. *Let S satisfy Condition C. Let every simple non-n-potent two-sided ideal of S contain at least one simple left and at least one simple right ideal of S. Then S is the sum of its radical and of a direct sum of two-sided simple ideals if and only if it can be written in both forms: a) as the sum of the radical and of all simple non-n-potent left ideals, b) as the sum of the radical and of all simple non-n-potent right ideals.*

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