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PRODUCTIVITY OF COREFLECTIVE SUBCATEGORIES OF UNIFORM SPACES

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During the course of the seminar, the authors throughly investigated the conditions under which a full coreflective subcategory of uniform spaces is closed under the formation of finite and infinite products. This note summarizes the main results and consequences of the investigation. The complete details and proofs will appear in [HR] .

The most interesting new basic result to emerge from the work was the following:

Theorem 1: Let $\{f_i : X_i \to Y_i\}$ be an arbitrary family of uniform quotient mappings. Then the product mapping $\Pi f_i : \Pi X_i \to \Pi Y_i$ defined by $(x_i) \to (f_i(x_i))$ is a uniform quotient mapping.

Corollary 1: Each product of proximal quotient mappings is a proximal quotient mapping.

Corollary 2: Each product of complete metric spaces (and hence each injective uniform space) is a uniform quotient of a space of the form $\mathbb{D}^m$, where $\mathbb{D}$ is uniformly discrete.

Corollary 2 follows from the fact that each complete metric space is a uniform quotient of a Baire space, i.e. a space of the form $\mathbb{D}^{\aleph_0}$, where $\mathbb{D}$ is uniformly discrete.

Theorem 2 (Finite Products): Let $\mathcal{C}$ be a full coreflective subcategory of uniform spaces. Then $\mathcal{C}$ is finitely productive if and only if $X \times \mathbb{D} \in \mathcal{C}$ for each $X \in \mathcal{C}$ and uniformly discrete space $\mathbb{D}$. Furthermore, if $\mathcal{C}$ = coreflective hull $(\mathcal{A})$, then $\mathcal{C}$ is finitely productive if and only if $A \times \mathbb{D} \in \mathcal{C}$ for each $A \in \mathcal{A}$ and uniformly discrete space $\mathbb{D}$.

Theorem 3 (Infinite Products): Let $\mathcal{C}$ be a full coreflective subcategory of uniform spaces and $m \geq \aleph_0$. Then $\mathcal{C}$ is m-productive (each product of m members from $\mathscr{C}$ belongs to $\mathscr{C}$) if and only if $\mathscr{C}$ is finitely productive and $\mathbb{D}^m \in \mathscr{C}$ for each uniformly discrete space $\mathbb{D}$.

Corollary 3: A full coreflective subcategory of uniform spaces is productive if and only if it is finitely productive and contains all powers of uniformly discrete spaces.

Theorem 2 also provides a uniform method for constructing the smallest (respectively largest) finitely productive coreflective subcategory $\mathscr{C}_h$ (respectively $\mathscr{C}_k$) containing (respectively contained in) a given coreflective subcategory $\mathscr{C}$ (the dual of the construction used in $[R]_2$):

(kernel) $\mathscr{C}_k = \{X : X \times \mathbb{D} \in \mathscr{C}$ for each uniformly discrete space $\mathbb{D}\}$.

(hull) $\mathscr{C}_h$ = coreflective hull $\{X \times \mathbb{D} : X \in \mathscr{C}, \mathbb{D}$ uniformly discrete$\}$.

In connection with these ideas, the following interesting results were also obtained. Each uniform space is the quotient of a space of the form $X \times \mathbb{D}$, where $\mathbb{D}$ is uniformly discrete and X may be chosen to be either a fine or proximally discrete space.

The classes of measurable spaces, metric-fine spaces, and spaces whose family of uniformly continuous real-valued mappings form a ring under pointwise operations, have the same kernel - the uniform spaces that admit $\aleph_0$.

With regard to productive coreflective subcategories, we note that Corollary 3 establishes the existence of a smallest non-trivial class $\mathscr{D}$ of this type - the coreflective hull of all powers of uniformly discrete spaces. In $[R]_1$, the second author constructed the first example (based on work in $[Hu]_2$) of a non-trivial productive coreflective subcategory - the class $\mathcal{E}$ of equi-proximally fine spaces (in the present context $\mathcal{E}$ may be described as the kernel of the proximally fine spaces). Here we will outline the construction of a productive coreflective subcategory $\mathscr{L}$, based on well-ordered adjacent nets, which lies strictly between $\mathcal{D}$ and $\mathcal{E}$. We say that two well-ordered nets $(x_\beta)_{\beta<\alpha}$ and $(y_\beta)_{\beta<\alpha}$ (α limit ordinal) in a uniform space X are adjacent if for each uniform cover $\mathcal{U}$ there exists $\alpha_0<\alpha$ such that $\{x_\beta, y_\beta\}$ lies in some member of $\mathcal{U}$, for each $\beta \geq \alpha_0$. A mapping $f : X \to Y$ is uniformly sequentially continuous of type α if f preserves the adjacency of well-ordered nets of type γ, for each $\gamma \leq \alpha$. Define $\mathcal{L}_\alpha$ (α infinite cardinal) = $\{X : f : X \to Y$ uniformly sequentially continuous of type α $\Longrightarrow f$ uniformly continuous$\}$. Then $\mathcal{L}_{\omega_0} \subset \mathcal{L}_{\omega_1}, \subset \ldots$ is an ascending class of distinct coreflective subcategories and one may establish the following result.

<u>Theorem 4:</u> (i) $\mathcal{L}_{\omega_0}$ = coreflective hull (metric spaces) = coreflective hull $(2^{\aleph} \times \mathbb{N})$.

(ii) Each uniform space with a linearly ordered base belongs to some $\mathcal{L}_\alpha$.

(iii) Each $\mathcal{L}_\alpha$ is α-productive.

<u>Corollary 4:</u> $\mathcal{L} = \bigcup_{\alpha \geq \aleph_0} \mathcal{L}_\alpha$ is a productive coreflective subcategory and $\mathcal{L}_{\omega_0}$ is the smallest countably productive coreflective subcategory of uniform spaces.

We remark that the α-productivity of $\mathcal{L}_\alpha$ is based on the followinf fact: the adjacency concept is suitable for the use of an induction argument for it allows the factorization of uniformly sequentially continuous mappings through a smaller number of co-ordinates.

We now have the following situation:

$$\mathcal{L}_{\omega_0} \subset \mathcal{D} \subset \mathcal{L} \subset \mathcal{E} .$$

Since $\mathcal{E}$ contains all compact spaces and $\mathcal{L}$ contains non-discrete spaces which admit $\aleph_0$, the following result shows that $\mathcal{D} \neq \mathcal{L}$ and $\mathcal{E} \neq \mathcal{L}$.

<u>Theorem 5:</u> (i) $\mathcal{D} \cap$ spaces admitting $\aleph_0$ = uniformly discrete spaces.

(ii) If X is not pseudocompact, then βX (the Čech-

-Stone compactification of X) is not a member of $\mathcal{L}$.

(iii) $\mathcal{L}$ contains no infinite compact F-space.

We note that $\mathcal{D}$ contains all injective spaces (Corollary 2), and all dyadic spaces, so that Theorem 5 generalizes well-known statements about dyadic spaces. We have no description of the compact spaces which belong to $\mathcal{D}$ or $\mathcal{L}$, but $\mathcal{L}_{\omega_0}$ contains a compact first-countable space that is not dyadic and the one-point compactification of the space of countable ordinals is a member of $\mathcal{L}$ which does not belong to $\mathcal{D}$. We also note that the family of compact members from $\mathcal{L}$ has properties similar to the properties of De Groot's class of supercompact spaces, but we do not know if each supercompact space is a member $\mathcal{L}_{\omega_0}$. $\mathcal{L}_{\omega_0}$ does contain all fine spaces having a sequential topology.

Notice also that the above comments do not distinguish $\mathcal{L}_{\omega_0}$ from $\mathcal{D}$; in fact the authors have discovered that to distinguish the two classes, one must assume the existence of a (first) uniformly sequential cardinal u. u is the first cardinal α for which there exists a real-valued uniformly sequentially continuous mapping of type ω_0 on the Cantor space 2^{α}. If s denotes the first sequential cardinal in the sense of Mazur and m_R denotes the first real-valued measurable cardinal, one can show that $s < u \leq m_R$; hence by recent work of Čudnovskii, Martin's Axiom implies that $s = u = m_R$.

<u>Theorem 6:</u> The following conditions are equivalent.

(i) $\mathcal{L}_{\omega_0} = \mathcal{D}$. ($\mathcal{L}_{\omega_0}$ is productive).

(ii) Each countably productive coreflective subcategory of uniform spaces is productive.

(iii) There exists no uniformly sequential cardinal.

Thus under the set-theoretic assumption (iii) stated above, one can (in view of Theorems 3 and 4) establish that a coreflective subcategory is productive by showing that it is finitely productive and contains $2^{\aleph_0}$. Hence under assumption (iii) it follows from the work in [Hu]$_1$ that a coreflective subcategory must contain all uniform spaces if it is closed-hereditary, finitely productive, and contains the Cantor set.

Finally, we note that if u exists, then the Cantor space 2^u is a member of $\mathcal{D}$ that does not belong to $\mathcal{L}_{\omega_0}$.

References

[Hu]$_1$ M.Hušek, Reflective and coreflective subcategories of Unif (and Top), Seminar Uniform Spaces, ČSAV, Prague, 1973-74, 113-126

[Hu]$_2$ M.Hušek, Factorization of mappings (products of proximally fine spaces), above volume, 173-190

[HR] M.Hušek and M.D.Rice, Productivity of coreflective subcategories of uniform spaces, to appear in Gen.Top. and Appl.

[R]$_1$ M.D.Rice, A strengthening of the proximally fine condition, Seminar Uniform Spaces, ČSAV, Prague, 1975-76, 125-134

[R]$_2$ M.D.Rice, Equi-reflective subcategories of uniform spaces, this volume