Miroslav Hušek

Products of uniform spaces (Summary)

In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1978. pp. 101–103.

Persistent URL: http://dml.cz/dmlcz/703171

PRODUCTS OF UNIFORM SPACES (SUMMARY)

M. Hušek

This is a summary of results from the paper with the same title which is submitted for publication in Czech.Math.J. It deals with the productivity of coreflective subcategories of uniform spaces. The results generalize those published in the earlier Seminar Uniform Spaces.

Theorem 1. The following conditions for a coreflective subcategory $\mathcal{C}$ of Unif are equivalent:

(a) $\mathcal{C}$ contains all powers of uniformly discrete spaces;

(b) A product belongs to $\mathcal{C}$ if all finite subproducts belong to $\mathcal{C}$.

Corollary. A coreflective subcategory of Unif is productive iff it is finitely productive and contains all powers of uniformly discrete spaces.

The next two results are not stated here in full generality because this would require the definition of special uniform spaces depending on relatively sequential cardinals.

Theorem 2. If a coreflective subcategory $\mathcal{C}$ of Unif contains all fine spaces, then a product belongs to $\mathcal{C}$ if any countable subproduct belongs to $\mathcal{C}$.

Theorem 3. The following conditions are equivalent for a coreflective subcategory $\mathcal{C}$ of Unif containing all fine spaces:

(a) $\mathcal{C}$ contains all countable powers of uniformly discrete spaces;

(b) $\mathcal{C}$ contains all metrizable spaces;

(c) A product belongs to $\mathcal{C}$ if all finite subproducts belong to $\mathcal{C}$.

Corollary 1. If F is a concrete functor on Unif that preserves proximity, then a product is F-fine iff all finite subproducts are F-fine.

Corollary 2 (MA). If there is no (two-valued) measurable cardinal, then a coreflective subcategory $\mathcal{C}$ of Unif is productive iff it is countably productive (or iff it is finitely productive and contains the Cantor space or a converging sequence).

It follows from the proof of Theorem 2 that any product of uniform spaces is inductively generated by canonical embeddings of countable subproducts and by fine spaces, or that $\prod_I X_i$ is inductively generated by the topological modification of $\prod_I DX_i$ (DX_i is the uniformly discrete modification of X_i) and by a (Corson) $\sum$-product of X_i's.

Theorem 4. For any infinite cardinal κ there are coreflective subcategories $\mathcal{C}$ in Unif and spaces $X_\alpha \in \mathcal{C}$, $\alpha < \kappa$, such that $\prod_{\alpha<\kappa} X_\alpha \notin \mathcal{C}$, but $\prod_{\alpha<\beta} X_\alpha \in \mathcal{C}$ for any $\beta < \kappa$.

Theorem 5. Each proximally continuous and separately uniformly continuous map on the product of a uniform space and precompact space is uniformly continuous.

The proof differs from the earlier one; it uses the relations (Y precompact):

$$U(X\times Y, pZ) = U(Y, U(X, pZ)) \supset U(Y, U(X, Z)) = U(X\times Y, Z) .$$

Theorem 6. If F is a concrete functor on Unif preserving proximity, each X_i is F-fine, and all but at most one X_i is precompact, then $\prod X_i$ is F-fine.

An example is provided that in Theorem 6, F-fine spaces cannot be replaced by a coreflective subcategory $\mathcal{C}$ of Unif containing all proximally fine spaces ($\mathcal{C}$ is the coreflective hull of all proximally fine spaces and of a finest precompact infinite space).

Theorem 7. Let $\mathcal{C}$ be a coreflective subcategory of Unif containing all fine spaces. If $X\times D \in \mathcal{C}$ for a uniformly discrete space D, then $X\times Y \in \mathcal{C}$ for each $Y \in \mathcal{C}$ admitting dX (density character) with card $Y \leq$ card D.

By a similar method, one can prove that if $\mathcal{C}$ contains all proximally fine spaces and $X \times D \in \mathcal{C}$, then $X \times \prod_I X_i \in \mathcal{C}$ provided each X_i has a linearly ordered base and card $\prod X_i \leq$ card D.

Theorem 8. No proper coreflective subcategory of Unif containing all fine spaces is finitely productive.

Theorem 9. Let F be an upper modification in Unif and D a uniformly discrete subspace of X. If $D \times P$ is not F-fine, then $X \times P$ is not F-fine.

Corollary. A proximally fine space X is precompact iff $X \times Y$ is proximally fine for each proximally fine space Y.

Theorem 9 can be extended to other coreflective subcategories, but I do not know their nice description. E.g. if $\mathcal{C}$ is the class of coz-fine or fine spaces, than for each non-precompact $X \in \mathcal{C}$ there is a fine Y such that $X \times Y \notin \mathcal{C}$.

There are examples showing that Theorem 9 is not valid in general (e.g. a coreflective subcategory $\mathcal{C}$ of Unif is constructed such that $\mathcal{C}$ contains all fine spaces and there is a countable 0-dimensional P such that $\kappa \times P \notin \mathcal{C}$, but Cone $\kappa \times X \in \mathcal{C}$ for all $X \in \mathcal{C}$).